Lebesgue measure on $\mathbb{R}$ (summary of last lesson)

External measure:

$$m^*(E) = \inf\left\{ \sum_{n\in\mathbb{N}} \ell(I_n) : I_n \text{ open intervals}, \ E \subseteq \bigcup_{n\in\mathbb{N}} I_n \right\}$$

It does not work well on all subsets. In particular, it is false that

$$m^*(E\cup F) = m^*(E) + m^*(F) \quad \text{if } E\cap F = \emptyset.$$

So we restrict the family of sets which we consider.

DEF. We say that $E \subseteq \mathbb{R}$ is measurable if

$$m^*(F) = m^*(F\cap E) + m^*(F\setminus E) \qquad \forall F \subseteq \mathbb{R}$$

We define $\mathcal{M} = \{\text{measurable sets}\}$.

Then:

$\mathcal{M}$ is stable with respect to the passage to the complement, and w.r. to countable unions and intersections.

We define $\quad m(E) := m^*(E) \qquad \forall E$ measurable.

Lebesgue measure

Then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n) \quad \text{if } E_n \text{ are measurable and disjoint}$$

and in general $\qquad$ m is countably subadditive.

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m(E_n) \quad \text{if } E_n \text{ are measurable}$$

Measurable functions

$E \subseteq \mathbb{R}$ measurable, $f: E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

f is <u>measurable</u> if $\{x \in E : f(x) > \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

$< \; \geqslant \; \leqslant$

The set of measurable functions is stable with respect to the arithmetic operations, to $\sup\limits_{n\in\mathbb{N}} f_n(x)$, $\inf\limits_{n\in\mathbb{N}} f_n(x)$,

$\lim\limits_{n\to+\infty} f_n(x) = f(x)$.

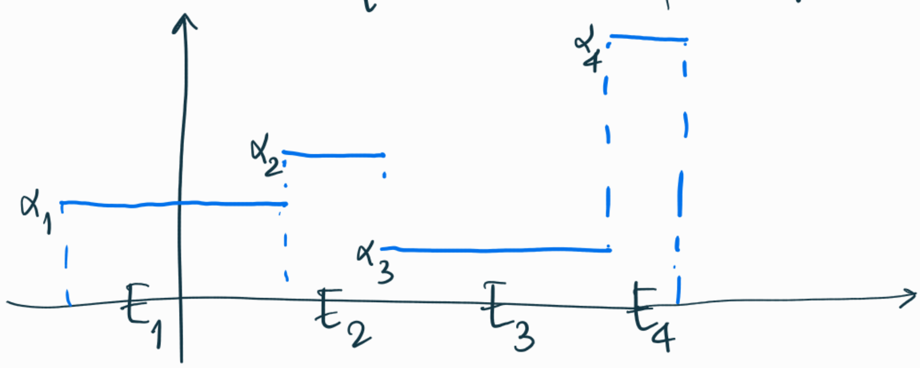
Lebesgue integral on $\mathbb{R}$.

1) For simple functions:

If $\varphi(x) = \sum\limits_{i=1}^{k} \alpha_i \chi_{E_i}(x)$

if $E_i \subseteq E$ are measurable and with finite measure, $\alpha_i \in \mathbb{R}$

$$\chi_{E_i}(x) = \begin{cases} 1 & \text{if } x \in E_i \\ 0 & \text{if } x \notin E_i \end{cases}$$



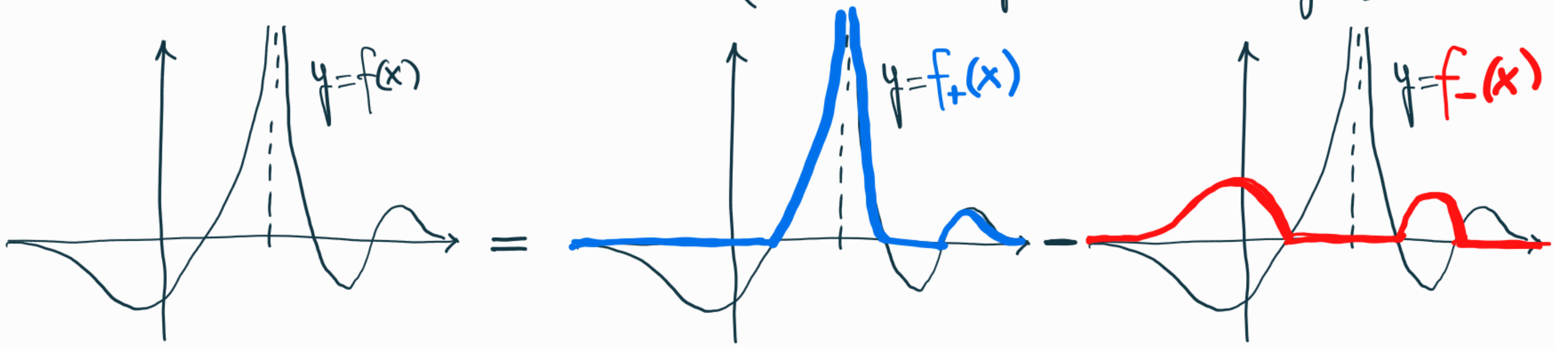
$$\int_E \varphi(x)\,dx = \sum_{i=1}^{k} \alpha_i\, m(E_i)$$

2) Non negative, measurable functions.

$f: E \longrightarrow [0, +\infty]$

$$\int_E f(x)\,dx = \sup\left\{ \int_E \varphi(x)\,dx,\ \varphi(x) \text{ simple function}, \ \varphi(x) \le f(x)\ \forall x \in E \right\}$$

3) measurable functions (no assumption on sign):



where $f_+(x) = \max\{f(x), 0\}$, $f_-(x) = \max\{-f(x), 0\}$

Note that $f = f_+ - f_-$, $|f| = f_+ + f_-$

We define

$$\int_E f(x)\,dx = \int_E f_+(x)\,dx - \int_E f_-(x)\,dx$$

if this is not $(+\infty - \infty)$.

f is <u>integrable</u> in E if the integrals of $f_+(x)$ and $f_-(x)$ are both finite, that is, if $|f|$ has finite integral.

In simbols, we write

$f \in L^1(E)$ to say that f is integrable on E.

(End of summary)

Properties of Lebesgue integral

We assume $E \subseteq \mathbb{R}$ measurable, $f(x), g(x) \in L^1(E)$, $c \in \mathbb{R}$. Then

1) $c f(x) \in L^1(E)$, and $\displaystyle\int_E c f(x)\,dx = c \int_E f(x)\,dx$

2) $f(x) + g(x) \in L^1(E)$, and $\displaystyle\int_E \big(f(x) + g(x)\big)\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx$

(Linearity of the integral.)

3) $f(x) \leq g(x)$ a.e. on $E \Rightarrow \displaystyle\int_E f(x)\,dx \leq \int_E g(x)\,dx$

(monotonicity of the integral.)

4) If A, B are measurable disjoint subsets of E, then

$$\int_{A \cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx$$

(additivity w.r. to the integration set.)

5) $$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$$

6) <u>Comparison with Riemann integral.</u>

If $E = [a,b]$ and $f(x)$ is an integrable function according to Riemann, then it is also integrable according to Lebesgue, and the two integrals are the same.

So, the Lebesgue integral is an extension of the Riemann integral.

Moreover: a function $f:[a,b]\to\mathbb{R}$ bounded is Riemann-integrable $\Leftrightarrow$ the set of its discontinuity points has Lebesgue measure zero.

(Vitali-Lebesgue theorem).

7) If $f(x)=g(x)$ a.e. on E, then

$$\int_E f(x)\,dx = \int_E g(x)\,dx$$

A natural question: assume that

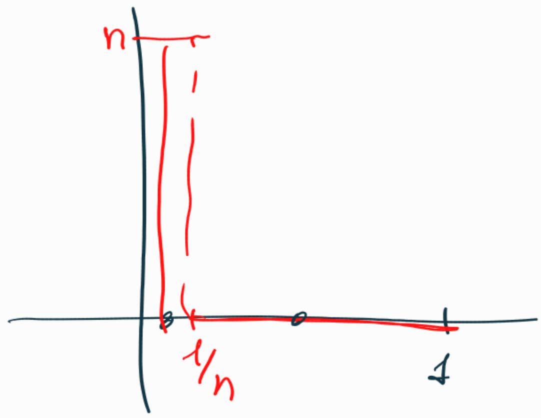
$$f_n(x) \xrightarrow{n\to+\infty} f(x) \qquad \forall x\in[a,b].$$

Can we say that

$$\int_a^b f_n(x)\,dx \xrightarrow{?} \int_a^b f(x)\,dx \; ? \qquad \underline{\text{NO}}$$

Consider

$$f_n(x)=\begin{cases} n & \left(0,\frac{1}{n}\right) \\ 0 & \left[\frac{1}{n},1\right) \end{cases}$$



$$f_n(x)\longrightarrow f(x)\equiv 0 \qquad \forall x\in(0,1)$$

$$\int_0^1 f_n(x)\,dx = 1 \quad \overset{n\to+\infty}{\not\longrightarrow} \quad \int_0^1 f(x)\,dx = 0$$

Theorems for passing to the limit under the integral:

1) <u>Monotone convergence Theorem</u> (Beppo Levi)

E measurable $\subseteq \mathbb{R}$

Le $\{f_n\}$ is a sequence of measurable functions on E s.t.

$$0 \le f_1(x) \le f_2(x) \le \dots \le f_n(x) \le \dots \quad \text{a.e } x \in E$$

Let $f(x) = \lim_{n\to+\infty} f_n(x)$. Then

$$\int_E \underbrace{f(x)}_{\lim_{n\to+\infty} f_n(x)} dx = \lim_{n\to+\infty} \int f_n(x)\,dx$$

2) <u>Lebesgue's dominated convergence theorem</u>

Assume that $\{f_n(x)\}$ measurable in E s.t.

$\lim_{n\to+\infty} f_n(x) = f(x)$ for a.e. $x \in E$,

and that there exists $g(x) \in L^1(E)$ s.t.

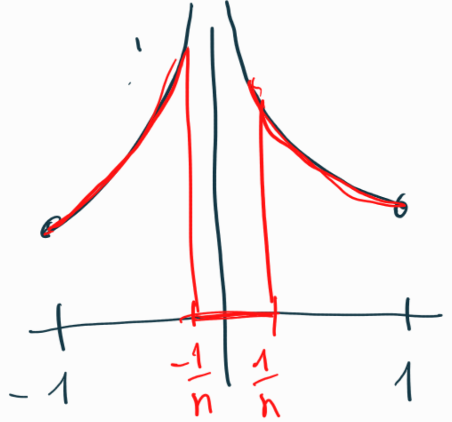
$$|f_n(x)| \le g(x) \quad \text{a.e. in } E.$$

Then
$$\int_E f(x)\,dx = \lim_{n\to+\infty} \int f_n(x)\,dx$$

Note that the previous example does not comply with the assumptions of these theorem.

Some important examples of integrals:

1. $f(x) = \dfrac{1}{|x|^\alpha}$ in $(-1,1)$ $\qquad \alpha > 0$



We consider

$$f_n(x) = f(x)\,\chi_{(-1,-\frac{1}{n})\cup(\frac{1}{n},1)}$$

$$f_n(x) \uparrow f(x) \quad \text{as } n \to +\infty.$$

So, by the monotone convergence theorem we have

$$\int_{-1}^{1} f(x)\,dx = \lim_{n\to+\infty}\int_{-1}^{1} f_n(x)\,dx = \begin{cases} +\infty & \text{if } \alpha \geq 1 \\ \frac{2}{1-\alpha} & \text{if } 0<\alpha<1 \end{cases}$$

$$\int_{-1}^{1} f_n(x)\,dx = 2\int_{1/n}^{1}\frac{1}{x^\alpha}\,dx = \begin{cases} \frac{2}{1-\alpha}\, x^{1-\alpha}\Big|_{1/n}^{1} & \text{if } \alpha\neq 1 \\ 2\log x\Big|_{1/n}^{1} & \alpha = 1 \end{cases}$$

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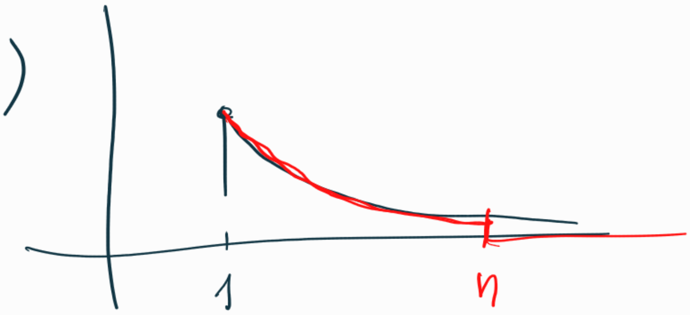
2) $f(x) = \frac{1}{x^\alpha}$ in $(1,+\infty)$, $\alpha>0$.



is integrable $\Leftrightarrow$ $\alpha > 1$

$$f_n(x) = f(x)\,\chi_{(1,n)}(x)$$

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3) Integrable functions can be very irregular

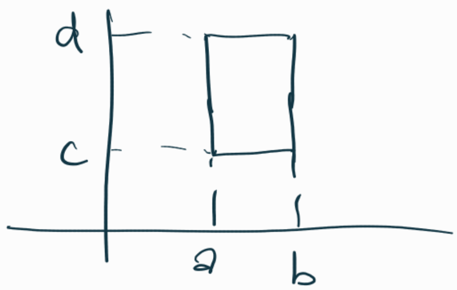
$E = (0,1)$. Let $\{q_n\}$ be a sequence of all rationals in $(0,1)$

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\frac{1}{\sqrt{|x-q_n|}}$$

is an integrable function, though it has a singularity at <u>all</u> rational numbers

Lebesgue measure and integration in $\mathbb{R}^2$ $(\mathbb{R}^3, \mathbb{R}^n)$

Same scheme.
Instead of intervals we use rectangles $Q = (a,b) \times (c,d)$



$$a(Q) = (b-a)(d-c)$$

if $E \subseteq \mathbb{R}^2$.

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} a(Q_n) : \ Q_n \text{ open rectangles } E \subset \bigcup_{n=1}^{\infty} Q_n \right\}$$

E is measurable (as before) if

$$m^*(F) = m^*(F \cap E) + m^*(F \setminus E) \qquad \forall F \subseteq \mathbb{R}^2.$$

the measurable sets form a class $\mathcal{M}$, which is stable w.r. to passage to the complement, countable unions and intersections.

$$m(E) = m^*(E) \qquad \forall E \in \mathcal{M}$$

enjoys the properties as in dimension 1.

$f(x,y) : E \to \overline{\mathbb{R}}$ is a measurable function if the set

$$\{(x,y) \in E : \ f(x,y) > \alpha\} \text{ is measurable set } \forall \alpha \in \mathbb{R}.$$

The definition of integral is also the same.

1) for simple functions, the same as before

2) for nonnegative measurable functions

$$\iint_E f(x,y)\,dx\,dy = \sup\Big\{ \iint_E \varphi(x,y)\,dx\,dy,\ \varphi \text{ simple}$$
$$-\varphi(x,y) \le f(x,y) \text{ for every } (x,y)\in E \Big\}$$

3) for measurable functions

$$f(x,y) = f_+(x,y) - f_-(x,y)$$

Properties are the same as in the 1-d case

Reduction formulas for double integrals (to reduce double integrals to a sequence of two integrals on $\mathbb{R}$)

Oss

$$\int_E f(x,y)\,dx\,dy = \int_{\mathbb{R}^2} \tilde{f}(x,y)\,dx\,dy.$$

$$\tilde{f}(x,y) = f(x,y)\cdot \chi_E(x,y)$$

Therefore we will state the following theorems in $\mathbb{R}^2$

Theorem (Fubini)

Assume that $f(x,y)$ is integrable in $\mathbb{R}^2$. Then

i) for a.e. $x\in\mathbb{R}$, the function $y \longmapsto f(x,y)$ is integrable in $\mathbb{R}$;

ii) the function $g(x) = \int_{\mathbb{R}} f(x,y)\,dy$, which is defined for a.e. $x\in\mathbb{R}$, is integrable in $\mathbb{R}$

iii) $$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}}\Big[\int_{\mathbb{R}} f(x,y)\,dy\Big]\,dx$$

Problem in order to apply Fubini's theorem, how do we check that f is integrable in $\mathbb{R}^2$? SIMPLE: for nonnegative functions we can ALWAYS apply the reduction formulas.

Theorem (Tonelli)

Assume that $f: \mathbb{R}^2 \to [0,+\infty]$ is measurable in $\mathbb{R}^2$. Then

i) for a.e. $x \in \mathbb{R}$, the function $y \longmapsto \overset{(\text{1}^{\circ})}{f(x,y)}$ is measurable in $\mathbb{R}$;

ii) the function $g(x) = \int_{\mathbb{R}} f(x,y)\,dy$, which is defined for a.e. $x \in \mathbb{R}$, is measurable in $\mathbb{R}$, and $g(x) \geq 0$.

iii) $$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}} g(x)\,dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\,dy \right] dx \in [0,+\infty]$$

Therefore, in practice:

First one checks that $|f(x,y)|$ is integrable, by calculating the integral $\iint_{\mathbb{R}^2} |f(x,y)|\,dx\,dy$. If this is finite, one can calculate the integral of $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy$ via Fubini's theorem.

Formulas for change of variables (in $\mathbb{R}^2$, but in $\mathbb{R}^3$ is similar)

$A, B \subseteq \mathbb{R}^2$ open set;

a function $\underline{\phi}: A \to B$

$$(u,v) \longmapsto \underline{\phi}(u,v) = (x(u,v), y(u,v))$$

is called a C^1 diffeomorphism if:

i) it is bijective;

ii) ϕ and its inverse ϕ^{-1} are of class C^1

(i.e., continuous and with continuous partial derivatives)

THEOREM Let $\underline{\Phi}: A \to B$ be a C^1 diffeomorphism.

$(x,y) = \underline{\phi}(u,v)$.

If $f: B \to \mathbb{R}$ is integrable, then the function

$$(f \circ \phi)(u,v) \underbrace{|J_\phi(u,v)|}_{\left| \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \right|}$$ is integrable in A

Jacobian determinant

Moreover

$$\iint_B f(x,y)\,dx\,dy = \iint_A \underbrace{f(\underline{\phi}(u,v))}_{= f(x(u,v),\,y(u,v))} |J_\phi(u,v)|\,du\,dv$$

Example
$$\iint_{\mathbb{R}^2} e^{-\underbrace{(x^2+y^2)}_{\rho^2}}\,dx\,dy =$$

Polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad |J_\phi(\rho,\theta)| = \rho$$

$$= \int_0^{+\infty} d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi}\, e^{-\rho^2}\rho = 2\pi \int_0^{+\infty} \rho\, e^{-\rho^2}\,d\rho = \pi \int_0^{+\infty} e^{-t}\,dt =$$

sost. $\rho^2 = t$

$dt = 2\rho\,d\rho$

$$= \pi\left(-e^{-t}\right)_0^{+\infty} = \pi.$$

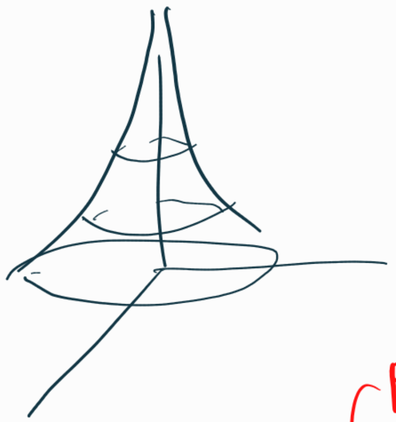
On the other hand, $\pi = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy \overset{\text{reduction formulas}}{=} \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-y^2}\,dy =$

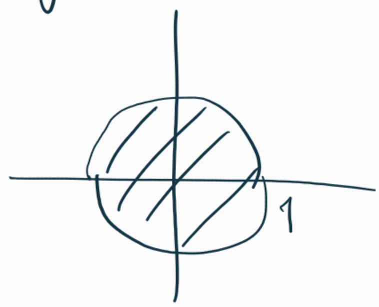
$$= \left(\int_{\mathbb{R}} e^{-x^2}dx\right)^2 .$$

Therefore we have found that $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$

Example $\quad f(x,y) = \dfrac{1}{|(x,y)|^\alpha} = \dfrac{1}{(\sqrt{x^2+y^2})^\alpha} \qquad \alpha>0$

in $B_1(0)$





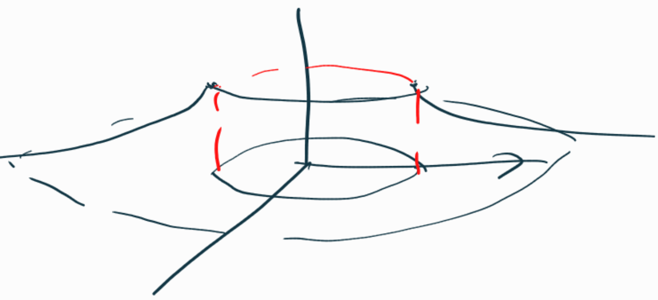
$$\iint_{B_1} f(x,y)\,dx\,dy = \iint_{B_1} \frac{dx\,dy}{(x^2+y^2)^{\alpha/2}} \overset{\text{polar coordinates}}{=} \underbrace{\int_0^{2\pi} d\theta}_{=2\pi} \int_0^1 d\rho\, \frac{\rho}{\rho^\alpha} =$$

$$= 2\pi \int_0^1 \frac{d\rho}{\rho^{\alpha-1}} \quad \text{converges} \iff \boxed{\alpha<2}$$

On the other hand,

$$\iint_{\mathbb{R}^2\setminus B_1} \frac{dx\,dy}{|(x,y)|^\alpha} \quad \text{converges} \iff \alpha>2$$

outside of a circle



Metric spaces

X generic set

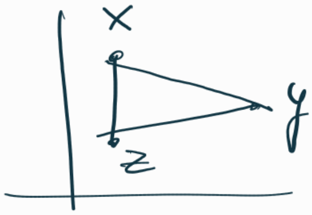
$$d(x,y): X\times X \to [0,+\infty) \quad (\text{distance from } x \text{ to } y).$$
$$(x,y) \mapsto d(x,y)$$

d must satisfy the following properties:

1) $d(x,y)=0 \iff x=y$

2) $d(x,y) = d(y,x)$

3) $d(x,y) \le d(x,z) + d(z,y)$ (triangle) $\forall x,y,z \in X$.



Prototype $X = \mathbb{R}^N$

$$d(\underline{x},\underline{y}) = |\underline{x}-\underline{y}| = \left[\sum_{i=1}^{N}(x_i-y_i)^2\right]^{1/2}$$

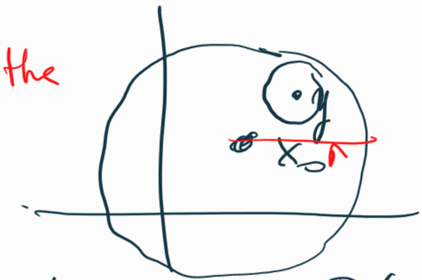
The pair (X,d) is called a metric space.

On a metric space, one can do limits, therefore Mathematical Analysis

$\boxed{\text{Spherical neighborhood}}$ of center $x_0 \in X$ and radius $\underline{r>0}$

$$B_r(x_0) = \{x \in X : d(x,x_0) < r\}$$

Properties of neighborhoods: (they follow from the definition)



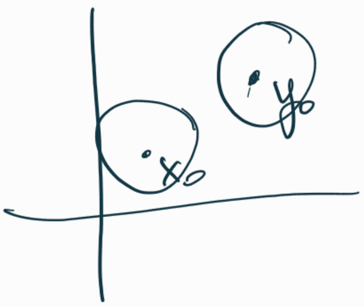
i) $x_0 \in B_r(x_0)$

ii) $y \in B_r(x_0) \Rightarrow \exists\ B_\sigma(y)$ all contained in $B_r(x_0)$

Remark. It is enough to choose $\sigma = r - d(x_0,y) > 0$

iii) $B_r(x_0) \cap B_\sigma(x_0) = B_{\min\{r,\sigma\}}(x_0)$ is still a neighborhood of x_0

iv) If $x_0 \ne y_0$, then $\exists\ B_r(x_0),\ B_\sigma(y_0)$ such that $B_r(x_0) \cap B_\sigma(y_0) = \emptyset$



Limit of sequences in a metric space

Let $\{x_n\}$ be a sequence of elements of X, $x \in X$. Then

$$\lim_{n\to+\infty} x_n = x \overset{\text{def.}}{\iff} \lim_{n\to+\infty} d(x_n, x) = 0 \iff$$

$$\forall \varepsilon > 0 \quad \exists\, \bar{n}_\varepsilon \text{ t.c. } \forall n \geq \bar{n} \quad d(x_n, x) < \varepsilon$$

(X, d), $(Y, \tilde{d})$ two metric spaces.

Let F be a function from X to Y. Then we will say that F is continuous in $\bar{x} \in X$ if

$\forall \varepsilon > 0 \quad \exists\, \delta > 0$ s.t. $\forall x \in X$ satisfying

$0 < d(x, \bar{x}) < \delta$ one has $\tilde{d}(F(x), F(\bar{x})) < \varepsilon$.

or, equivalently, if:

$\forall$ sequence $\{x_n\}$ of elements of X s.t. $x_n \xrightarrow{n\to+\infty} \bar{x}$.

one has $F(x_n) \xrightarrow{n\to+\infty} F(\bar{x})$

Normed vector space. Let X be a vector space.

A norm on X is a function $\|\cdot\| : X \to [0, +\infty)$

such that:

(prototype: $|\underline{x}| = \sqrt{x_1^2 + x_2^2}$)

1) $\|x\| = 0 \iff x = \underline{0}$

2) $\|\lambda x\| = |\lambda|\,\|x\| \quad \forall x \in X \ \forall \lambda \in \mathbb{R}$

3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a normed space.

A normed space is also a metric space, setting

$$d(x,y) = \| x-y \|$$

The triangle inequality is true, since

$$d(x,y) = \| x-y \| = \| (x-z)+(z-y) \| \leq \| x-z \| + \| z-y \|$$
$$= d(x,z) + d(z,y)$$

Examples of normed spaces:

1) $(\mathbb{R}, |x|)$

2) $(\mathbb{C}, |z|)$

3) $(\mathbb{R}^2, |\underline{x}|)$ $\qquad |\underline{x}| = \sqrt{x_1^2 + x_2^2}$ (euclidean norm)

4) $(\mathbb{R}^2, \|\underline{x}\|_1)$ $\qquad \|\underline{x}\| = |x_1| + |x_2|$

5) $(\mathbb{R}^2, \|\underline{x}\|_p)$ $\qquad \|\underline{x}\|_p = \left[|x_1|^p + |x_2|^p \right]^{1/p}$

$1 < p < \infty$ $\qquad$ the triangle inequality is complicated.

6) $(\mathbb{R}^2, \|\underline{x}\|_\infty)$ $\qquad \|\underline{x}\|_\infty = \max \{ |x_1|, |x_2| \}$

What is the form of the neighborhood of $(0,0)$ in the norms 3) – 6)?

Balls of center $(0,0)$ and radius 1 in the norms $\|\cdot\|_p$