Corollary 4 *The following sets are countably infinite:*

(i) *For each natural number n, the Cartesian product* $\overbrace{\mathbf{N}\times\cdots\times\mathbf{N}}^{n \text{ times}}$.

(ii) *The set of rational numbers* $\mathbf{Q}$.

Proof We prove (i) for $n = 2$ and leave the general case as an exercise in induction. Define the mapping g from $\mathbf{N}\times\mathbf{N}$ to $\mathbf{N}$ by $g(m, n) = (m+n)^2 + n$. The mapping g is one-to-one. Indeed, if $g(m, n) = g(m', n')$, then $(m+n)^2 - (m'+n')^2 = n' - n$ and hence

$$|m+n+m'+n'|\cdot|m+n-m'-n'| = |n'-n|.$$

If $n \neq n'$, then the natural number $m+n+m'+n'$ both divides and is greater than the natural number $|n'-n|$, which is impossible. Thus $n = n'$, and hence $m = m'$. Therefore $\mathbf{N}\times\mathbf{N}$ is equipotent to $g(\mathbf{N}\times\mathbf{N})$, a subset of the countable set $\mathbf{N}$. We infer from the preceding theorem that $\mathbf{N}\times\mathbf{N}$ is countable. To verify the countability of $\mathbf{Q}$, we first infer from the prime factorization theorem that each positive rational number x may be written uniquely as $x = p/q$, where p and q are relatively prime natural numbers. Define the mapping g from $\mathbf{Q}$ to $\mathbf{N}$ by $g(x) = 2((p+q)^2 + q)$ for $x = p/q > 0$ with p and q relatively prime natural numbers, $g(0) = 1$, and $g(x) = g(-x) + 1$ for $x < 0$. We leave it as an exercise to show that g is one-to-one. Thus $\mathbf{Q}$ is equipotent to a subset of $\mathbf{N}$ and hence, by the preceding theorem, is countable. We leave it as an exercise to use the pigeonhole principle to show that neither $\mathbf{N}\times\mathbf{N}$ nor $\mathbf{Q}$ is finite. □

For a countably infinite set X, we say that $\{x_n \mid n \in \mathbf{N}\}$ is an **enumeration** of X provided

$$X = \{x_n \mid n \in \mathbf{N}\} \text{ and } x_n \neq x_m \text{ if } n \neq m.$$

Theorem 5 *A non-empty set is countable if and only if it is the image of a function whose domain is a non-empty countable set.*

Proof Let A be a non-empty countable set and f be mapping of A onto B. We suppose that A is countably infinite and leave the finite case as an exercise. By composing with a one-to-one correspondence between A and $\mathbf{N}$, we may suppose that $A = \mathbf{N}$. Define two points x, x' in A to be equivalent provided $f(x) = f(x')$. This is an equivalence relation, that is, it is reflexive, symmetric, and transitive. Let E be a subset of A consisting of one member of each equivalence class. Then the restriction of f to E is a one-to-one correspondence between E and B. But E is a subset of $\mathbf{N}$ and therefore, by Theorem 3, is countable. The set B is equipotent to E and therefore B is countable. The converse assertion is clear; if B is a non-empty countable set, then it is equipotent either to an initial segment of natural numbers or to the natural numbers. □

Corollary 6 *The union of a countable collection of countable sets is countable.*

Proof Let Λ be a countable set and for each $\lambda \in \Lambda$, let E_λ be a countable set. We will show that the union $E = \bigcup_{\lambda\in\Lambda} E_\lambda$ is countable. If E is empty, then it is countable. So we assume $E \neq \emptyset$. We consider the case that Λ is countably infinite and leave the finite case as an exercise. Let $\{\lambda_n \,|\, n \in \mathbf{N}\}$ be an enumeration of Λ. Fix $n \in \mathbf{N}$. If E_{λ_n} is finite and non-empty, choose a natural number $N(n)$ and a one-to-one mapping f_n of $\{1, \ldots, N(n)\}$ onto E_{λ_n}; if E_{λ_n} is countably infinite, choose a one-to-one mapping f_n of $\mathbf{N}$ onto E_{λ_n}. Define

$$E' = \big\{(n,\, k) \in \mathbf{N} \times \mathbf{N} | E_{\lambda_n} \text{is non-empty, and } 1 \le k \le N(n) \text{ if } E_{\lambda_n} \text{is also finite}\big\}.$$

Define the mapping f of E' to E by $f(n,\, k) = f_n(k)$. Then f is a mapping of E' onto E. However, E' is a subset of the countable set $\mathbf{N} \times \mathbf{N}$ and hence, by Theorem 3, is countable. Theorem 5 tells us that E also is countable. □

We call an interval of real numbers degenerate if it is empty or contains a single member.

Theorem 7 *A non-degenerate interval of real numbers is uncountable.*

Proof Let I be a non-degenerate interval of real numbers. Clearly I is not finite. We argue by contradiction to show that I is uncountable. Suppose I is countably infinite. Let $\{x_n \,|\, n \in \mathbf{N}\}$ be an enumeration of I. Let $[a_1,\, b_1]$ be a non-degenerate closed, bounded subinterval of I which fails to contain x_1. Then let $[a_2,\, b_2]$ is a non-degenerate closed, bounded subinterval of $[a_1,\, b_1]$, which fails to contain x_2. We inductively choose a countable collection $\{[a_n,\, b_n]\}_{n=1}^{\infty}$ of non-degenerate closed, bounded intervals, which is descending in the sense that, for each n, $[a_{n+1},\, b_{n+1}] \subseteq [a_n,\, b_n]$ and such that for each n, $x_n \notin [a_n,\, b_n]$. The non-empty set $E = \{a_n \,|\, n \in \mathbf{N}\}$ is bounded above by b_1. The Completeness Axiom tells us that E has a supremum. Define $x^* = \sup E$. Since x^* is an upper bound for E, $a_n \le x^*$ for all n. On the other hand, since $\{[a_n,\, b_n]\}_{n=1}^{\infty}$ is descending, for each n, b_n is an upper bound for E. Hence, for each n, $x^* \le b_n$. Therefore, x^* belongs to $[a_n,\, b_n]$ for each n. But x^* belongs to $[a_1,\, b_1] \subseteq I$ and therefore there is a natural number n_0 for which $x^* = x_{n_0}$. We have a contradiction since $x^* = x_{n_0}$ does not belong to $[a_{n_0},\, b_{n_0}]$. Therefore I is uncountable. □

PROBLEMS

16. Show that the set $\mathbf{Z}$ of integers is countable.
17. Show that a set A is countable if and only if there is a one-to-one mapping of A to $\mathbf{N}$.
18. Use an induction argument to complete the proof of part (i) of Corollary 4.
19. Prove Corollary 6 in the case of a finite family of countable sets.
20. Let both $f\colon A \to B$ and $g\colon B \to C$ be one-to-one and onto. Show that the composition $g \circ f\colon A \to B$ and the inverse $f^{-1}\colon B \to A$ are also one-to-one and onto.
21. Use an induction argument to establish the pigeonhole principle.
22. Show that $2^{\mathbf{N}}$, the collection of all sets of natural numbers, is uncountable.
23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding problem to show that $\mathbf{N}^{\mathbf{N}}$, the collection of all mappings of $\mathbf{N}$ into $\mathbf{N}$, is not countable.
24. Show that a non-degenerate interval of real numbers fails to be finite.

25. Show that any two non-degenerate intervals of real numbers are equipotent.
26. Is the set $\mathbf{R} \times \mathbf{R}$ equipotent to $\mathbf{R}$?

1.4 OPEN SETS, CLOSED SETS, AND BOREL SETS OF REAL NUMBERS

Definition *A set $\mathcal{O}$ of real numbers is called* **open** *provided for each $x \in \mathcal{O}$, there is a $r > 0$ for which the interval $(x - r,\, x + r)$ is contained in $\mathcal{O}$.*

For $a < b$, the interval (a, b) is an open set. Indeed, let x belong to (a, b). Define $r = \min\{b - x,\, x - a\}$. Observe that $(x - r,\, x + r)$ is contained in (a, b). Thus (a, b) is an open bounded interval and each bounded open interval is of this form. For $a, b \in \mathbf{R}$, we defined

$$(a, \infty) = \{x \in \mathbf{R} \mid\ a < x\}\,, (-\infty, b) = \{x \in \mathbf{R} \mid\ x < b\} \text{ and } (-\infty, \infty) = \mathbf{R}.$$

Observe that each of these sets is an open interval. Moreover, it is not difficult to see that since each set of real numbers has an infimum and supremum in the set of extended real numbers, each unbounded open interval is of the above form.

Proposition 8 *The set of real numbers $\mathbf{R}$ and the empty-set $\emptyset$ are open; the intersection of any finite collection of open sets is open; and the union of any collection of open sets is open.*

Proof It is clear that $\mathbf{R}$ and $\emptyset$ are open and the union of any collection of open sets is open. Let $\{\mathcal{O}_k\}_{k=1}^n$ be a finite collection of open subsets of $\mathbf{R}$. If the intersection of this collection is empty, then the intersection is the empty-set and therefore is open. Otherwise, let x belong to $\cap_{k=1}^n \mathcal{O}_k$. For $1 \leq k \leq n$, choose $r_k > 0$ for which $(x - r_k,\, x + r_k) \subseteq \mathcal{O}_k$. Define $r = \min\{r_1, \ldots, r_n\}$. Then $r > 0$ and $(x - r,\, x + r) \subseteq \cap_{k=1}^n \mathcal{O}_k$. Therefore $\cap_{k=1}^n \mathcal{O}_k$. is open. □

It is not true, however, that the intersection of any collection of open sets is open. For example, for each natural number n, let $\mathcal{O}_n$ be the open interval $(-1/n, 1/n)$. Then, by the Archimedean Property of $\mathbf{R}$, $\bigcap_{n=1}^\infty \mathcal{O}_n = \{0\}$, and $\{0\}$ is not an open set.

Proposition 9 *Every non-empty open set is the union of a countable, disjoint collection of open intervals.*

Proof Let $\mathcal{O}$ be a non-empty open subset of $\mathbf{R}$. Let x belong to $\mathcal{O}$. There is a $y > x$ for which $(x,\, y) \subseteq \mathcal{O}$ and a $z < x$ for which $(z,\, x) \subseteq \mathcal{O}$. Define the extended real numbers a_x and b_x by

$$a_x = \inf\{z \mid\ (z,\, x) \subseteq \mathcal{O}\} \text{ and } b_x = \sup\{y \mid\ (x,\, y) \subseteq \mathcal{O}\}.$$

Then $I_x = (a_x,\, b_x)$ is an open interval that contains x. We claim that

$$I_x \subseteq \mathcal{O} \text{ but } a_x \notin \mathcal{O}, b_x \notin \mathcal{O}. \tag{2}$$

Indeed, let w belong to I_x, say $x < w < b_x$. By the definition of b_x, there is a number $y > w$ such that $(x,\, y) \subseteq \mathcal{O}$, and so $w \in \mathcal{O}$. Moreover, $b_x \notin \mathcal{O}$, for if $b_x \in \mathcal{O}$, then for some $r > 0$ we have $(b_x - r,\, b_x + r) \subseteq \mathcal{O}$. Thus $(x,\, b_x + r) \subseteq \mathcal{O}$, contradicting the definition of b_x.

Similarly, $a_x \notin \mathcal{O}$. Consider the collection of open intervals $\{I_x\}_{x\in\mathcal{O}}$. Since each x in $\mathcal{O}$ is a member of I_x, and each I_x is contained in $\mathcal{O}$, we have $\mathcal{O} = \bigcup_{x\in\mathcal{O}} I_x$. We infer from (2) that $\{I_x\}_{x\in\mathcal{O}}$ is disjoint. Thus $\mathcal{O}$ is the union of a disjoint collection of open intervals. It remains to show that this collection is countable. By the density of the rationals, Theorem 2, each of these open intervals contains a rational number. This establishes a one-to-one correspondence between the collection of open intervals and a subset of the rational numbers. We infer from Theorem 3 and Corollary 4 that any set of rational numbers is countable. Therefore, $\mathcal{O}$ is the union of a countable disjoint collection of open intervals. □

Definition *For a set E of real numbers, a real number x is called a* **point of closure** *of E provided every open interval that contains x also contains a point in E. The collection of points of closure of E is called the* **closure** *of E and denoted by $\overline{E}$.*

It is clear that we always have $E \subseteq \overline{E}$. If E contains all of its points of closure, that is, $E = \overline{E}$, then the set E is said to be **closed.**

Proposition 10 *For a set of real numbers E, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed set that contains E, in the sense that if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.*

Proof The set $\overline{E}$ is closed provided it contains all its points of closure. Let x be a point of closure of $\overline{E}$. Consider an open interval I_x which contains x. There is a point $x' \in \overline{E} \cap I_x$. Since x' is a point of closure of E and the open interval I_x contains x', there is a point $x'' \in E \cap I_x$. Therefore, every open interval that contains x also contains a point of E and hence $x \in \overline{E}$. So the set $\overline{E}$ is closed. It is clear that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$, and hence if F is closed and contains E, then $\overline{E} \subseteq \overline{F} = F$. □

Proposition 11 *A set of real numbers is open if and only if its complement in* $\mathbf{R}$ *is closed.*

Proof First suppose E is an open subset of $\mathbf{R}$. Let x be a point of closure of $\mathbf{R} \sim E$. Then x cannot belong to E because otherwise there would be an open interval that contains x and is contained in E and thus is disjoint from $\mathbf{R} \sim E$. Therefore x belongs to $\mathbf{R} \sim E$ and hence $\mathbf{R} \sim E$ is closed. Now suppose $\mathbf{R} \sim E$ is closed. Let x belong to E. Then there must be an open interval that contains x that is contained in E, for otherwise every open interval that contains x contains points in $\mathbf{R} \sim E$ and therefore x is a point of closure of $\mathbf{R} \sim E$. Since $\mathbf{R} \sim E$ is closed, x also belongs to $\mathbf{R} \sim E$. This is a contradiction. □

Since $\mathbf{R} \sim [\mathbf{R} \sim E] = E$, it follows from the preceding proposition that *a set is closed if and only if its complement is open.* Therefore, by De Morgan's Identities, Proposition 8 may be reformulated in terms of closed sets as follows.

Proposition 12 *The empty-set $\emptyset$ and* $\mathbf{R}$ *are closed; the union of any finite collection of closed sets is closed; and the intersection of any collection of closed sets is closed.*

A collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ is said to be a **cover** of a set E provided $E \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda$. By a subcover of a cover of E, we mean a subcollection of the cover that itself also is a cover of E. If each set E_λ in a cover is open, we call $\{E_\lambda\}_{\lambda\in\Lambda}$ an **open cover** of F. If the cover

$\{E_\lambda\}_{\lambda \in \Lambda}$ contains only a finite number of sets, we call it a **finite cover**. This terminology is inconsistent: In "open cover," the adjective "open" refers to the sets in the cover; in "finite cover," the adjective "finite" refers to the collection and does not imply that the sets in the collection are finite sets. Thus the term "open cover" is an abuse of language and should properly be "cover by open sets." Unfortunately, the former terminology is well established in mathematics.

A set F of real numbers is said to be compact provided that every open cover of F has a finite subcover.

The Heine-Borel Theorem *A set of real numbers is compact if and only if it is closed and bounded.*

Proof We leave as an exercise the proofs, by contradiction, that a compact set is closed and is bounded. To prove the converse, let us first consider the case that F is the closed, bounded interval $[a, b]$. Let $\mathcal{F}$ be an open cover of $[a, b]$. Define E to be the set of numbers $x \in [a, b]$ with the property that the interval $[a, x]$ can be covered by a finite number of the sets of $\mathcal{F}$. Since $a \in E$, E is non-empty. Since E is bounded above by b, by the completeness of $\mathbf{R}$, E has a supremum; define $c = \sup E$. Since c belongs to $[a, b]$, there is an $\mathcal{O} \in \mathcal{F}$ that contains c. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$, such that the interval $(c - \epsilon, c + \epsilon)$ is contained in $\mathcal{O}$. Now $c - \epsilon$ is not an upper bound for E, and so there must be an $x \in E$ with $x > c - \epsilon$. Since $x \in E$, there is a finite collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ of sets in $\mathcal{F}$ that covers $[a, x]$. Consequently, the finite collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_k, \mathcal{O}\}$ covers the interval $[a, c + \epsilon)$. Thus $c = b$, for otherwise $c < b$ and c is not an upper bound for E. Thus, $[a, b]$ can be covered by a finite number of sets from $\mathcal{F}$, proving our special case.

Now let F be any closed and bounded set and $\mathcal{F}$ an open cover of F. Since F is bounded, it is contained in some closed, bounded interval $[a, b]$. The preceding proposition tells us that the set $\mathcal{O} = \mathbf{R} \sim F$ is open since F is closed. Let $\mathcal{F}^*$ be the collection of open sets obtained by adding $\mathcal{O}$ to $\mathcal{F}$, that is, $\mathcal{F}^* = \mathcal{F} \cup \mathcal{O}$. Since $\mathcal{F}$ covers F, $\mathcal{F}^*$ covers $[a, b]$. By the case just considered, there is a finite subcollection of $\mathcal{F}^*$ that covers $[a, b]$ and hence F. By removing $\mathcal{O}$ from this finite subcover of F if $\mathcal{O}$ belongs to the finite subcover, we have a finite collection of sets in $\mathcal{F}$ that covers F. □

We say that a countable collection of sets $\{E_n\}_{n=1}^\infty$ is **descending** provided $E_{n+1} \subseteq E_n$ for every natural number n. It is said to be **ascending** provided each $E_n \subseteq E_{n+1}$.

The Nested Set Theorem *Let $\{F_n\}_{n=1}^\infty$ be a descending countable collection of non-empty closed sets of real numbers and F_1 be bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof We argue by contradiction. Suppose the intersection is empty. Then for each real number x, there is a natural number n for which $x \notin F_n$, that is, $x \in \mathcal{O}_n = \mathbf{R} \sim F_n$. Therefore $\bigcup_{n=1}^\infty \mathcal{O}_n = \mathbf{R}$. According to Proposition 11, since each F_n is closed, each $\mathcal{O}_n$ is open. Therefore $\{\mathcal{O}_n\}_{n=1}^\infty$ is an open cover of $\mathbf{R}$ and hence also of F_1. The Heine-Borel Theorem tells us that there is a natural number N for which $F_1 \subseteq \bigcup_{n=1}^N \mathcal{O}_n$. Since $\{F_n\}_{n=1}^\infty$ is descending, the collection of complements $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is ascending. Therefore $\bigcup_{n=1}^{N} \mathcal{O}_n = \mathcal{O}_N = \mathbf{R} \sim F_N$. Hence $F_1 \subseteq \mathbf{R} \sim F_N$, which contradicts the assumption that F_N is a non-empty subset of F_1. □

Definition *Given a set X, a collection $\mathcal{A}$ of subsets of X is called a σ-algebra (of subsets of X) provided (i) the empty-set, $\emptyset$, belongs to $\mathcal{A}$; (ii) the complement in X of a set in $\mathcal{A}$ also belongs to $\mathcal{A}$; (iii) the union of a countable collection of sets in $\mathcal{A}$ also belongs to $\mathcal{A}$.*

Given a set X, the collection $\{\emptyset, X\}$ is a σ-algebra which has two members and is contained in every σ-algebra of subsets of X. At the other extreme is the collection of sets 2^X which consists of all subsets of X and contains every σ-algebra of subsets of X. For any σ-algebra $\mathcal{A}$, we infer from De Morgan's Identities that $\mathcal{A}$ is closed with respect to the formation of intersections of countable collections of sets that belong to $\mathcal{A}$; moreover, since the empty-set belongs to $\mathcal{A}$, $\mathcal{A}$ is closed with respect to the formation of finite unions and finite intersections of sets that belong to $\mathcal{A}$. We also observe that a σ-algebra is closed with respect to relative complements since if A_1 and A_2 belong to $\mathcal{A}$, so does $A_1 \sim A_2 = A_1 \cap [X \sim A_2]$. The proof of the following proposition follows directly from the definition of σ-algebra.

Proposition 13 *Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{F}$ is a σ-algebra that contains $\mathcal{F}$. Moreover, it is the smallest σ-algebra of subsets of X that contains $\mathcal{F}$, in the sense that any σ-algebra that contains $\mathcal{F}$ also contains $\mathcal{A}$.*

Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets that belong to a σ-algebra $\mathcal{A}$. Since $\mathcal{A}$ is closed with respect to the formation of countable intersections and unions, the following two sets belong to $\mathcal{A}$:

$$\limsup\{A_n\}_{n=1}^{\infty} = \bigcap_{k=1}^{\infty} \left[\bigcup_{n=k}^{\infty} A_n \right] \text{ and } \liminf\{A_n\}_{n=1}^{\infty} = \bigcup_{k=1}^{\infty} \left[\bigcap_{n=k}^{\infty} A_n \right].$$

The set $\limsup\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n for countably infinitely many indices n while the set $\liminf\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n except for at most finitely many indices n.

Although the union of any collection of open sets is open and the intersection of any finite collection of open sets is open, as we have seen, the intersection of a *countable* collection of open sets need not be open. In our development of Lebesgue measure and integration on the real line, we will see that the smallest σ-algebra of sets of real numbers that contains the open sets is a natural object of study.

Definition *The collection $\mathcal{B}$ of Borel sets of real numbers is the smallest σ-algebra of sets of real numbers that contains all of the open sets of real numbers.*

Every open set is a Borel set and since a σ-algebra is closed with respect to the formation of complements, we infer from Proposition 11 that every closed set is a Borel set. Therefore, since each singleton set is closed, every countable set is a Borel set. A countable intersection of open sets is called a G_δ set. A countable union of closed sets is called an F_σ set.

Since a σ-algebra is closed with respect to the formation of countable unions and countable intersections, each G_δ set and each F_σ set is a Borel set. Moreover, both the lim inf and lim sup of a countable collection of sets of real numbers, each of which is either open or closed, are a Borel set.

PROBLEMS

27. Is the set of rational numbers open or closed?
28. What are the sets of real numbers that are both open and closed?
29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.
30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \sim \{x\}$.
    (i) Show that the set E' of accumulation points of E is a closed set.
    (ii) Show that $\overline{E} = E \cup E'$.
31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x-r, x+r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.
32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x-r, x+r)$ is contained in E. The set of interior points of E is called the **interior** of E denoted by $\text{int}\, E$. Show that
    (i) E is open if and only if $E = \text{int}\, E$.
    (ii) E is dense if and only if $\text{int}(\mathbf{R} \sim E) = \emptyset$.
33. Show that the Nested Set Theorem is false if F_1 is unbounded.
34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.
35. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.
36. Show that the collection of Borel sets is the smallest σ-algebra that contains intervals of the form $[a, b)$, where $a < b$.
37. Show that each open set is an F_σ set.

1.5 SEQUENCES OF REAL NUMBERS

A **sequence** of real numbers is a real-valued function whose domain is the set of natural numbers. Rather than denoting a sequence with standard functional notation such as $f: \mathbf{N} \rightarrow \mathbf{R}$, it is customary to use subscripts, replace $f(n)$ with a_n, and denote a sequence by $\{a_n\}$. A natural number n is called an **index** for the sequence, and the number a_n corresponding to the index n is called the nth **term** of the sequence. Just as we say that a real-valued function is bounded provided its image is a bounded set of real numbers, we say a sequence $\{a_n\}$ is **bounded** provided there is some $c \geq 0$ such that $|a_n| \leq c$ for all n. A sequence is said to be **increasing** provided $a_n \leq a_{n+1}$ for all n, is said to be **decreasing** provided $\{-a_n\}$ is increasing, and is said to be **monotone** provided it is either increasing or decreasing.

Definition *A sequence $\{a_n\}$ is said to* **converge** *to the number a, provided for every $\epsilon > 0$ there is an index N for which*

$$\text{if } n \geq N, \text{ then } |a - a_n| < \epsilon. \tag{3}$$

We call a the **limit** *of the sequence and denote the convergence of $\{a_n\}$ by writing*

$$\{a_n\} \to a \text{ or } \lim_{n \to \infty} a_n = a.$$

We leave the proof of the following proposition as an exercise.

Proposition 14 *Let the sequence of real numbers $\{a_n\}$ converge to the real number a. Then the limit is unique, the sequence is bounded, and, for a real number c,*

$$\text{if } a_n \leq c \text{ for all } n, \text{ then } a \leq c.$$

Theorem 15 (**the Monotone Convergence Criterion for Real Sequences**) *A monotone sequence of real numbers converges if and only if it is bounded.*

Proof Let $\{a_n\}$ be an increasing sequence. If this sequence converges, then, by the preceding proposition, it is bounded. Now assume that $\{a_n\}$ is bounded. By the Completeness Axiom, the set $S = \{a_n \mid n \in N\}$ has a supremum: define $a = \sup S$. We claim that $\{a_n\} \to a$. Indeed, let $\epsilon > 0$. Since s is an upper bound for S, $a_n \leq a$ for all n. Since $a - \epsilon$ is not an upper bound for S, there is an index N for which $a_N > a - \epsilon$. Since the sequence is increasing, $a_n > a - \epsilon$ for all $n \geq N$. Thus if $n \geq N$, then $|a - a_n| < \epsilon$. Therefore $\{a_n\} \to a$. The proof for the case when the sequence is decreasing is the same. □

For a sequence $\{a_n\}$ and a strictly increasing sequence of natural numbers $\{n_k\}$, we call the sequence $\{a_{n_k}\}$ whose kth term is a_{n_k} a **subsequence** of $\{a_n\}$.

Theorem 16 (**the Bolzano-Weierstrass Theorem**) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $\{a_n\}$ be a bounded sequence of real numbers. Choose $M \geq 0$ such that $|a_n| \leq M$ for all n. Let n be a natural number. Define $E_n = \overline{\{a_j \mid j \geq n\}}$. Then $E_n \subseteq [-M, M]$ and E_n is closed since it is the closure of a set. Therefore, $\{E_n\}$ is a descending sequence of nonempty closed bounded subsets of $\mathbf{R}$. The Nested Set Theorem tells us that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$; choose $a \in \bigcap_{n=1}^{\infty} E_n$. For each natural number k, a is a point of closure of $\{a_j \mid j \geq k\}$. Hence, for infinitely many indices $j \geq n$, a_j belongs to $(a - 1/k, a + 1/k)$. By induction, choose a strictly increasing sequence of natural numbers $\{n_k\}$ such that $|a - a_{n_k}| < 1/k$ for all k. We infer from the Archimedean Property of $\mathbf{R}$ that the subsequence $\{a_{n_k}\}$ converges to a. □

Definition *A sequence of real numbers $\{a_n\}$ is said to be* **Cauchy** *provided for each $\epsilon > 0$, there is an index N for which*

$$\text{if } n, m \geq N, \text{ then } |a_m - a_n| < \epsilon. \tag{4}$$

Theorem 17 (**the Cauchy Convergence Criterion for Real Sequences**) *A sequence of real numbers converges if and only if it is Cauchy.*

Proof First suppose that $\{a_n\} \to a$. Observe that for all natural numbers n and m,

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \le |a_n - a| + |a_m - a|. \tag{5}$$

Let $\epsilon > 0$. Since $\{a_n\} \to a$, we may choose a natural number N such that if $n \ge N$, then $|a_n - a| < \epsilon/2$. We infer from (5) that if $n, m \ge N$, then $|a_n - a_m| < \epsilon$. Therefore the sequence $\{a_n\}$ is Cauchy. To prove the converse, let $\{a_n\}$ be a Cauchy sequence. We claim that it is bounded. Indeed, for $\epsilon = 1$, choose N such that if $n, m \ge N$, then $|a_n - a_m| < 1$. Thus

$$|a_n| = |(a_n - a_N) + a_N| \le |a_n - a_N| + |a_N| \le 1 + |a_N| \text{ for all } n \ge N.$$

Define $M = 1 + \max\{|a_1|, \ldots, |a_N|\}$. Then $|a_n| \le M$ for all n. Thus $\{a_n\}$ is bounded. The Bolzano-Weierstrass Theorem tells us there is a subsequence $\{a_{n_k}\}$ which converges to a real number a. We claim that the whole sequence converges to a. Indeed, let $\epsilon > 0$. Since $\{a_n\}$ is Cauchy we may choose a natural number N such that

$$\text{if } n, m \ge N, \text{ then } |a_n - a_m| < \epsilon/2.$$

On the other hand, since $\{a_{n_k}\} \to a$ we may choose a natural number n_k such that $|a - a_{n_k}| < \epsilon/2$ and $n_k \ge N$. Therefore

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \le |a_n - a_{n_k}| + |a - a_{n_k}| < \epsilon \text{ for all } n \ge N.$$ □

Theorem 18 (**Linearity and Monotonicity of Convergence of Real Sequences**) *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences of real numbers. Then for each pair of real numbers α and β, the sequence $\{\alpha \cdot a_n + \beta \cdot b_n\}$ is convergent and*

$$\lim_{n \to \infty} [\alpha \cdot a_n + \beta \cdot b_n] = \alpha \cdot \lim_{n \to \infty} a_n + \beta \cdot \lim_{n \to \infty} b_n. \tag{6}$$

Moreover,

$$\text{if } a_n \le b_n \text{ for all } n, \text{ then } \lim_{n \to \infty} a_n \le \lim_{n \to \infty} b_n. \tag{7}$$

Proof Define

$$\lim_{n \to \infty} a_n = a \text{ and } \lim_{n \to \infty} b_n = b.$$

Observe that

$$\big|[\alpha \cdot a_n + \beta \cdot b_n] - [\alpha \cdot a + \beta \cdot b]\big| \le |\alpha| \cdot |a_n - a| + |\beta| \cdot |b_n - b| \text{ for all } n. \tag{8}$$

Let $\epsilon > 0$. Choose a natural number N such that

$$|a_n - a| < \epsilon/[2 + 2|\alpha|] \text{ and } |b_n - b| < \epsilon/[2 + 2|\beta|] \text{ for all } n \ge N.$$

We infer from (8) that

$$|[\alpha \cdot a_n + \beta \cdot b_n] - [\alpha \cdot a + \beta \cdot b]| < \epsilon \text{ for all } n \ge N.$$

Therefore (6) holds. To verify (7), set $c_n = b_n - a_n$ for all n and $c = b - a$, Then $c_n \ge 0$ for all n and, by linearity of convergence, $\{c_n\} \to c$. We must show $c \ge 0$. Let $\epsilon > 0$. There is an N such that

$$-\epsilon < c - c_n < \epsilon \text{ for all } n \ge N.$$

In particular, $0 \le c_N < c + \epsilon$. Since $c > -\epsilon$ for every positive number $\epsilon, c \ge 0$. □

If a sequence $\{a_n\}$ has the property that for each real number c, there is an index N such that if $n \geq N$, then $a_n \geq c$, we say that $\{a_n\}$ **converges to infinity**, call ∞ the limit of $\{a_n\}$, and write $\lim_{n\to\infty} a_n = \infty$. Similar definitions are made at $-\infty$. With this extended concept of convergence we may assert that any monotone sequence $\{a_n\}$ of real numbers, bounded or unbounded, converges to an extended real number and therefore $\lim_{n\to\infty} a_n$ is properly defined.

The extended concept of supremum and infimum of a set and of convergence for any monotone sequence of real numbers allows us to make the following definition.

Definition *Let $\{a_n\}$ be a sequence of real numbers. The limit superior of $\{a_n\}$, denoted by $\limsup\{a_n\}$, is defined by*

$$\limsup\{a_n\} = \lim_{n\to\infty} [\sup\{a_k \mid k \geq n\}].$$

The limit inferior of $\{a_n\}$, denoted by $\liminf\{a_n\}$, is defined by

$$\liminf\{a_n\} = \lim_{n\to\infty} [\inf\{a_k \mid k \geq n\}].$$

We leave the proof of the following proposition as an exercise.

Proposition 19 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.*

(i) $\limsup\{a_n\} = \ell \in \mathbf{R}$ *if and only if for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$ and only finitely many indices n for which $a_n > \ell + \epsilon$.*

(ii) $\limsup\{a_n\} = \infty$ *if and only if $\{a_n\}$ is not bounded above.*

(iii) $\limsup\{a_n\} = -\liminf\{-a_n\}$.

(iv) A sequence of real numbers $\{a_n\}$ converges to an extended real number a if and only if

$$\liminf\{a_n\} = \limsup\{a_n\} = a.$$

(v) If $a_n \leq b_n$ for all n, then

$$\limsup\{a_n\} \leq \limsup\{b_n\}.$$

For each sequence $\{a_k\}$ of real numbers, there corresponds a sequence of **partial sums** $\{s_n\}$ defined by $s_n = \sum_{k=1}^{n} a_k$ for each index n. We say that the series $\sum_{k=1}^{\infty} a_k$ is **summable** to the real number s provided $\{s_n\} \to s$ and write $s = \sum_{k=1}^{\infty} a_k$.

We leave the proof of the following proposition as an exercise.

Proposition 20 *Let $\{a_n\}$ be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ is summable if and only if for each $\epsilon > 0$, there is an index N for which

$$\left| \sum_{k=n}^{n+m} a_k \right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ also is summable.

(iii) If each term a_k is non-negative, then the series $\sum_{k=1}^{\infty} a_k$ is summable if and only if the sequence of partial sums is bounded.

Consider the series $\sum_{k=1}^{\infty} a_k$. It is said to be **absolutely convergent** provided that the series $\sum_{k=1}^{\infty} |a_k|$ converges. Given a permutation $\pi\colon \mathbf{N} \to \mathbf{N}$, the series $\sum_{k=1}^{\infty} a_{\pi(k)}$ is called a **rearrangement** of $\sum_{k=1}^{\infty} a_k$.

Theorem 21 (**the Riemann Rearrangement Theorem**) *If a series converges absolutely, then every rearrangement converges to the same sum. If a series converges, but does not converge absolutely, then for every real number s, there is a rearrangement that converges to s.*

A proof of this remarkable theorem may be found in Terence Tao's Analysis 1.

PROBLEMS

38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.
39. Prove Proposition 19.
40. Show that a sequence $\{a_n\}$ is convergent to an extended real number if and only if there is exactly one extended real number that is a cluster point of the sequence.
41. Show that $\liminf a_n \leq \limsup a_n$.
42. Prove that if, for all n, $a_n \geq 0$ and $b_n \geq 0$, then
$$\limsup\, [a_n \cdot b_n] \leq (\limsup\ a_n) \cdot (\limsup\ b_n),$$
provided the product on the right is not of the form $0 \cdot \infty$.
43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.
44. Let p be a natural number greater than 1, and x a real number, $0 \leq x \leq 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that
$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$
and that this sequence is unique except when x is of the form $q/p^n, 0 < q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series
$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$
converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal* expansion of x. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

45. Prove Proposition 20.
46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

1.6 CONTINUOUS REAL-VALUED FUNCTIONS OF A REAL VARIABLE

Let f be a real-valued function defined on a set E of real numbers. We say that f is **continuous at the point** x in E provided that for each $\epsilon > 0$, there is a $\delta > 0$ for which

$$\text{if } x' \in E \text{ and } |x' - x| < \delta, \text{ then } |f(x') - f(x)| < \epsilon.$$

The function f is said to be **continuous** (on E) provided it is continuous at each point in its domain E. The function f is said to be **Lipschitz** provided there is a $c \geq 0$ for which

$$|f(x') - f(x)| \leq c \cdot |x' - x| \text{ for all } x', x \in E.$$

It is clear that a Lipschitz function is continuous. Indeed, for a number $x \in E$ and any $\epsilon > 0$, $\delta = \epsilon/c$ responds to the ϵ challenge regarding the criterion for the continuity of f at x. Not all continuous functions are Lipschitz. For example, if $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$, then f is continuous on $[0, 1]$ but is not Lipschitz.

We leave as an exercise the proof of the following characterization of continuity at a point in terms of sequential convergence.

Proposition 22 *A real-valued function f defined on a set E of real numbers is continuous at the point $x_* \in E$ if and only if whenever a sequence $\{x_n\}$ in E converges to x_*, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.*

We have the following characterization of continuity of a function on all of its domain.

Proposition 23 *Let f be a real-valued function defined on a set E of real numbers. Then f is continuous on E if and only if for each open set $\mathcal{O}$,*

$$f^{-1}(\mathcal{O}) = E \cap \mathcal{U} \text{ where } \mathcal{U} \text{ is an open set.} \tag{9}$$

Proof First assume the inverse image under f of any open set is the intersection of the domain with an open set. Let x belong to E. To show that f is continuous at x, let $\epsilon > 0$. The interval $I = (f(x) - \epsilon,\ f(x) + \epsilon)$ is an open set. Therefore, there is an open set $\mathcal{U}$ such that

$$f^{-1}(I) = \{x' \in E \mid f(x) - \epsilon < f(x') < f(x) + \epsilon\} = E \cap \mathcal{U}.$$

In particular, $f(E \cap \mathcal{U}) \subseteq I$ and x belongs to $E \cap \mathcal{U}$. Since $\mathcal{U}$ is open, there is a $\delta > 0$ such that $(x - \delta,\ x + \delta) \subseteq \mathcal{U}$. Thus if $x' \in E$ and $|x' - x| < \delta$, then $|f(x') - f(x)| < \epsilon$. Hence f is continuous at x.

Suppose now that f is continuous. Let $\mathcal{O}$ be an open set and x belong to $f^{-1}(\mathcal{O})$. Then $f(x)$ belongs to the open set $\mathcal{O}$ so that there is an $\epsilon > 0$, such that $(f(x) - \epsilon,\ f(x) + \epsilon) \subseteq \mathcal{O}$.

Since f is continuous at x, there is a $\delta > 0$ such that if x' belongs to E and $|x' - x| < \delta$, then $|f(x') - f(x)| < \epsilon$. Define $I_x = (x - \delta, x + \delta)$. Then $f(E \cap I_x) \subseteq \mathcal{O}$. Define

$$\mathcal{U} = \bigcup_{x \in f^{-1}(\mathcal{O})} I_x.$$

Since $\mathcal{U}$ is the union of open sets it is open. It has been constructed so that (9) holds. □

The Extreme Value Theorem *A continuous real-valued function on a non-empty, closed, bounded set of real numbers takes a minimum and maximum value.*

Proof Let f be a continuous real-valued function on the non-empty closed bounded set E of real numbers. We first show that f is bounded on E, that is, there is a real number M such that

$$|f(x)| \leq M \text{ for all } x \in E. \tag{10}$$

Let x belong to E. Let $\delta > 0$ respond to the $\epsilon = 1$ challenge regarding the criterion for continuity of f at x. Define $I_x = (x - \delta,\, x + \delta)$. Therefore, if x' belongs to $E \cap I_x$, then $|f(x') - f(x)| < 1$ and so $|f(x')| \leq |f(x)| + 1$. The collection $\{I_x\}_{x \in E}$ is an open cover of E. The Heine-Borel Theorem tells us that there are a finite number of points $\{x_1, \ldots, x_n\}$ in E such that $\{I_{x_k}\}_{k=1}^n$ also covers E. Define $M = 1 + \max\{|f(x_1)|, \ldots, |f(x_n)|\}$. We claim that (10) holds for this choice of E. Indeed, let x belong to E. There is an index k such that x belongs to I_{x_k} and therefore $|f(x)| \leq 1 + |f(x_k)| \leq M$. To see that f takes a maximum value on E, define $m = \sup f(E)$. If f failed to take the value m on E, then the function $x \mapsto 1/(f(x) - m), x \in E$ is a continuous function on E which is unbounded. This contradicts what we have just proved. Therefore, f takes a maximum value on E. Since $-f$ is continuous, $-f$ takes a maximum value, that is, f takes a minimum value on E. □

The Intermediate Value Theorem *Let f be a continuous real-valued function on the closed, bounded interval $[a, b]$ for which $f(a) < c < f(b)$. Then there is a point x_0 in (a, b) at which $f(x_0) = c$.*

Proof We will define by induction a descending countable collection $\{[a_n, b_n]\}_{n=1}^\infty$ of closed intervals whose intersection consists of a single point $x_0 \in (a, b)$ at which $f(x_0) = c$. Define $a_1 = a$ and $b_1 = b$. Consider the midpoint m_1 of $[a_1, b_1]$. If $c < f(m_1)$, define $a_2 = a_1$ and $b_2 = m_1$. If $f(m_1) \geq c$, define $a_2 = m_1$ and $b_2 = b_1$. Therefore, $f(a_2) \leq c \leq f(b_2)$ and $b_2 - a_2 = [b_1 - a_1]/2$. We inductively continue this bisection process to obtain a descending collection $\{[a_n, b_n]\}_{n=1}^\infty$ of closed intervals such that

$$f(a_n) \leq c \leq f(b_n) \text{ and } b_n - a_n = [b - a]/2^{n-1} \text{ for all } n. \tag{11}$$

According to the Nested Set Theorem, $\bigcap_{n=1}^\infty [a_n, b_n]$ is non-empty. Choose x_0 in $\bigcap_{n=1}^\infty [a_n, b_n]$. Observe that

$$|a_n - x_0| \leq b_n - a_n = [b - a]/2^{n-1} \text{ for all } n.$$

Therefore $\{a_n\} \to x_0$. By the continuity of f at x_0, $\{f(a_n)\} \to f(x_0)$. Since $f(a_n) \leq c$ for all n, and the set $(-\infty, c]$ is closed, $f(x_0) \leq c$. By a similar argument, $f(x_0) \geq c$. Hence $f(x_0) = c$. □

Definition *A real-valued function f defined on a set E of real numbers is said to be* ***uniformly continuous*** *provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for all x, x' in E,*

$$\text{if } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.$$

Theorem 24 *A continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous.*

Proof Let f be a continuous real-valued function on a closed bounded set E of real numbers. Let $\epsilon > 0$. For each $x \in E$, there is a $\delta_x > 0$ such that if $x' \in E$ and $|x' - x| < \delta_x$, then $|f(x') - f(x)| < \epsilon/2$. Define I_x to be the open interval $(x - \delta_x/2, x + \delta_x/2)$. Then $\{I_x\}_{x \in E}$ is an open cover of E. According to the Heine-Borel Theorem, there is a finite subcollection $\{I_{x_1}, \ldots, I_{x_n}\}$ which covers E. Define

$$\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}.$$

We claim that this $\delta > 0$ responds to the $\epsilon > 0$ challenge regarding the criterion for f to be uniformly continuous on E. Indeed, let x and x' belong to E with $|x - x'| < \delta$. Since $\{I_{x_1}, \ldots, I_{x_n}\}$ covers E, there is an index k for which $|x - x_k| < \delta_{x_k}/2$. Since $|x - x'| < \delta \leq \delta_{x_k}/2$,

$$|x' - x_k| \leq |x' - x| + |x - x_k| < \delta_{x_k}/2 + \delta_{x_k}/2 = \delta_{x_k}.$$

By the definition of δ_{x_k}, since $|x - x_k| < \delta_{x_k}$ and $|x' - x_k| < \delta_{x_k}$ we have $|f(x) - f(x_k)| < \epsilon/2$ and $|f(x') - f(x_k)| < \epsilon/2$. Therefore,

$$|f(x) - f(x')| \leq |f(x) - f(x_k)| + |f(x') - f(x_k)| < \epsilon/2 + \epsilon/2 = \epsilon. \qquad \square$$

Definition *A real-valued function f defined on a set E of real numbers is said to be* **increasing** *provided $f(x) \leq f(x')$ whenever x, x' belong to E and $x \leq x'$, and* **decreasing** *provided $-f$ is increasing. It is called* **monotone** *if it is either increasing or decreasing.*

Let f be a monotone real-valued function defined on an open interval I that contains the point x_0. We infer from Theorem 15 and its proof that if $\{x_n\}$ is a decreasing sequence in $I \cap (x_0, \infty)$ which converges to x_0, then the sequence $\{f(x_n)\}$ converges to a real number and the limit is independent of the choice of sequence $\{x_n\}$. We denote the limit by $f(x_0^+)$. Similarly, we define $f(x_0^-)$. Then clearly f is continuous at x_0 if and only if $f(x_0^-) = f(x_0) = f(x_0^+)$. If f fails to be continuous at x_0, then the only point of the image of f that lies strictly between $f(x_0^+)$ and $f(x_0^-)$ is $f(x_0)$ and f is said to have a **jump discontinuity** at x_0. Thus, by the Intermediate Value Theorem, a monotone function on an open interval is continuous if and only if its image is an interval (see Problem 55).

PROBLEMS

47. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E. Show that there is a function g defined and continuous on all of $\mathbf{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take g to be linear on each of the intervals of which $\mathbf{R} \sim E$ is composed.)

48. Define the real-valued function f on $\mathbf{R}$ by setting

$$f(x) = \begin{cases} x & \text{if } x \text{ irrational} \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

49. Let f and g be continuous real-valued functions with a common domain E.
   (i) Show that the sum, $f+g$, and product, fg, are also continuous functions.
   (ii) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.
   (iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.
   (iv) Show that $|f|$ is continuous.

50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

51. A continuous function φ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which φ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function φ on $[a, b]$ with $|f(x) - \varphi(x)| < \epsilon$ for all $x \in [a, b]$.

52. Show that a non-empty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

53. Show that a set E of real numbers is closed and bounded if and only if every open cover of E has a finite subcover.

54. Show that a non-empty set E of real numbers is an interval if and only if every continuous real-valued function on E has an interval as its image.

55. Show that a monotone function on an open interval is continuous if and only if its image is an interval.

56. Let f be a real-valued function defined on $\mathbf{R}$. Show that the set of points at which f is continuous is a G_δ set.

57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbf{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.

58. Let f be a continuous real-valued function on $\mathbf{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.

59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.

C H A P T E R 2

Lebesgue Measure

Contents

2.1 INTRODUCTION

The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function by step-functions, which are constructed by partitioning its domain into subintervals. The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function by simple functions, which are constructed by partitioning into intervals the range of the function and considering preimages of these intervals. In this chapter, properties of individual measurable sets and of the collection of measurable sets are established.

Each interval is Lebesgue measurable. The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing step-functions. This leads to a larger class of functions that are Lebesgue integrable over very general domains and an integral that has better properties. For instance, under quite general circumstances, we prove that if a sequence of functions converges pointwise to a limiting function, then the integral of the limit function is the limit of the integrals of the approximating functions.

The length $\ell(I)$ of an interval I is defined to be the difference of the end-points of I if I is bounded, and ∞ if I is unbounded. Length is an example of a *set-function*, that is, a function that associates an extended real number to each set in a collection of sets. In the case of length, the domain is the collection of all intervals. In this chapter, the set-function length is extended to a large collection of sets of real numbers. For instance, the "length" of an open set is the sum of the lengths of the countable number of open intervals of which it is composed. However, the collection of sets consisting of intervals and open sets is still too limited for our purposes. We construct a collection of sets called **Lebesgue measurable sets**, and a set-function of this collection called **Lebesgue measure**, denoted by m. The collection of Lebesgue measurable sets is a σ-algebra[1] which contains all open sets and all closed sets. The set-function m possesses the following three properties.

[1]Recall that a collection of subsets of $\mathbf{R}$ is called a σ-algebra provided it contains $\mathbf{R}$ and is closed with respect to complements and countable unions; by De Morgan's Identities, such a collection is also closed with respect to countable intersections.

The measure of an interval is its length Each interval I is Lebesgue measurable and

$$m(I) = \ell(I).$$

Measure is translation invariant If E is Lebesgue measurable and y is any number, then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and

$$m(E + y) = m(E).$$

Measure is countably additive over countable, disjoint unions of sets[2] If $\{E_k\}_{k=1}^{\infty}$ is a countable, disjoint collection of Lebesgue measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set-function that possesses the above three properties and is defined for all sets of real numbers. The response to this limitation is to construct a set-function on a very rich class of sets that does possess the above three properties. The construction has two stages.

We first construct a set-function called **outer-measure**, which we denote by m^*. It is defined for any set, and thus, in particular, for any interval. The outer-measure of an interval is its length. Outer-measure is translation invariant. However, outer-measure is not finitely additive, much less countably additive (see Corollary 24). But it is countably monotone in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, that covers a set E, then

$$m^*(E) \le m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k).$$

The second stage in the construction is to determine what it means for a set to be **Lebesgue measurable** and show that the collection of Lebesgue measurable sets is a σ-algebra that contains all open sets, and all sets of outer-measure zero. We then restrict the set-function m^* to the collection of Lebesgue measurable sets, denote it by m, and prove m is countably additive. We call m **Lebesgue measure**.

PROBLEMS

In the first three problems, let μ be a set-function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$, and μ be countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $\mu(A) \le \mu(B)$. This property is called *monotonicity*.
2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $\mu(A) < \infty$, then $\mu(\emptyset) = 0$.
3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $\mu(\bigcup_{k=1}^{\infty} E_k) \le \sum_{k=1}^{\infty} \mu(E_k)$.
4. A set-function c is defined on all subsets of $\mathbf{R}$ as follows: define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be the number of members in E if E is finite, and define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set-function. This set-function is called the **counting measure**.

[2] For a collection of sets to be disjoint, we mean what is sometimes called pairwise disjoint, that is, that each pair of sets in the collection has empty intersection.

2.2 OUTER-MEASURE

Let I be an interval of real numbers. Define its length, $\ell(I)$, to be ∞ if I is unbounded and otherwise define its length to be the difference of its end-points. For a set A of real numbers, consider all countable collections $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals that cover A, in the sense that $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. The **outer-measure** of $A, m^*(A)$, is defined to be the infimum of all such sums, that is,

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}.$$

It follows immediately from the definition of outer-measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer-measure is **monotone** in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example A countable set C has outer-measure zero. Indeed, let C be enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each k, define $I_k = (c_k - \epsilon/2^{k+1}, c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. Therefore,

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$, and so $m^*(C) = 0$. According to Proposition 25, there is a non-countable set, the Cantor set, which has measure zero.

Lemma 1 (**von Neumann**) *For a bounded set E, define its* **integral count**, $\mu^{int}(E)$, *to be the number of integers in E. For each $\epsilon > 0$, define the ϵ-dilation T_ϵ: $\mathbf{R} \to \mathbf{R}$ by $T_\epsilon(x) = \epsilon \cdot x$. Then for each bounded interval I,*

$$\lim_{\epsilon \to \infty} \frac{\mu^{int}(T_\epsilon(I))}{\epsilon} = \ell(I). \tag{1}$$

Proof If I has end-points a and b, there is the following estimate for $\mu^{int}(I)$:

$$(b-a) - 1 \leq \mu^{int}(I) \leq (b-a) + 1.$$

For $\epsilon > 0$, replace the interval I by the dilated interval $T_\epsilon(I)$ to obtain the estimate

$$\epsilon \cdot (b-a) - 1 \leq \mu^{int}(T_\epsilon(I)) \leq \epsilon \cdot (b-a) + 1].$$

Divide this inequality by ϵ and take the limit as $\epsilon \to \infty$ to establish (1). □

Proposition 2 *If the bounded interval I is covered by a finite collection $\{I_k\}_{k=1}^{n}$ of bounded intervals, then*

$$\ell(I) \leq \sum_{k=1}^{n} \ell(I_k).$$

Proof For each $\epsilon > 0$, the bounded interval $T_\epsilon(I)$ is covered by the collection of bounded intervals $\{T_\epsilon(I^k)\}_{k=1}^m$. It is clear that the integer count is finitely monotone, and so

$$\mu^{int}(T_\epsilon(I)) \leq \sum_{k=1}^{m} \mu^{int}(T_\epsilon(I^k)) \text{ for all } \epsilon.$$

Divide each side by ϵ, take the limit as $\epsilon \to \infty$ and, by the preceding lemma, conclude the proof. □

Proposition 3 *The outer-measure of an interval is its length.*

Proof We begin with the case of a closed, bounded interval $[a, b]$. Let $\epsilon > 0$. Since the open interval $(a - \epsilon, b + \epsilon)$ contains $[a, b]$, we have $m^*([a, b]) \leq \ell((a - \epsilon, b + \epsilon)) = b - a + 2\epsilon$. This holds for any $\epsilon > 0$, and so $m^*([a, b]) \leq b - a$. It remains to verify this inequality in the opposite direction, which this is equivalent to showing that if $\{I_k\}_{k=1}^\infty$ is any countable collection of open, bounded intervals covering $[a,\, b]$, then

$$b - a \leq \sum_{k=1}^{\infty} \ell(I_k).$$

By the Heine-Borel Theorem[3], any collection of open intervals covering $[a, b]$ has a finite subcollection that also covers $[a, b]$. Choose an index n for which $\{I_k\}_{k=1}^n$ covers $[a,\, b]$. To verify the above inequality, it suffices to show that

$$b - a \leq \sum_{k=1}^{n} \ell(I_k).$$

However, this follows from the preceding proposition. Now consider the case of a general bounded interval I. Given $\epsilon > 0$, there are two closed, bounded intervals J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2 \text{ while } \ell(I) - \epsilon < \ell(J_1) \text{ and } \ell(J_2) < \ell(I) + \epsilon.$$

By the equality of outer-measure and length for closed bounded intervals and the monotonicity of outer-measure,

$$\ell(I) - \epsilon < \ell(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(J_2) < \ell(I) + \epsilon.$$

This holds for each $\epsilon > 0$. Therefore, $\ell(I) = m^*(I)$.

If I is an unbounded interval, then for each natural number n, there is an interval $J \subseteq I$ with $\ell(J) = n$. We have $m^*(I) \geq m^*(J) = \ell(J) = n$. This holds for each n, and therefore $m^*(I) = \infty$. □

Proposition 4 *Outer-measure is* **translation invariant***, in the sense that for any set E and any c, if $E + c = \{x + c | x \in E\}$,*

$$m^*(E + c) = m^*(E).$$

[3]See page 18.

Proof Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$ covers E if and only if $\{I_k + c\}_{k=1}^{\infty}$ covers $E + c$. Moreover, if each I_k is an open interval, then each $I_k + c$ is an open interval of the same length and so

$$\sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k + c).$$

It follows that $m^*(E + c) = m^*(E)$. □

Proposition 5 *Outer-measure is* **countably monotone**, *in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, that covers a set E, then*

$$m^*(E) \le m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k).$$

Proof The left-hand inequality follows from the monotonicity of outer-measure. If one of the E_k's has infinite outer-measure, the right-hand inequality holds trivially. We therefore assume that each of the E_k's has finite outer-measure. Let $\epsilon > 0$. For each k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open, bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i} \text{ and } \sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \epsilon/2^k.$$

Now, $\{I_{k,i}\}_{1 \le k, i \le \infty}$ is a countable collection of open, bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$, and consequently,

$$\begin{aligned} m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{1 \le k, i < \infty} \ell(I_{k,i}) &= \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \ell(I_{k,i})\right] \\ &< \sum_{k=1}^{\infty}\left[m^*(E_k) + \epsilon/2^k\right] \\ &= \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \epsilon. \end{aligned}$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. □

PROBLEMS

5. By using properties of outer-measure, prove that the interval $[0, 1]$ is not countable.
6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.
7. A set of real numbers is said to be a G_δ set provided that it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

8. (Jordan content) The Jordan content of a set is defined as is outer-measure m^*, except that only finite coverings of the set by open, bounded intervals are considered. Prove that if the set $Q \cap [0, 1]$ is covered by the finite collection $\{\ell(I_k)\}_{k=1}^n$, then $\sum_{k=1}^n \ell(I_k) \geq 1$.
9. Suppose that outer-measure is defined by covering sets by countable collections of closed, bounded intervals rather than coverings by open, bounded intervals. Show that the outer-measure remains unchanged.
10. Prove that if $m^*(A) = 0$, then, for any set B, $m^*(A \cup B) = m^*(B)$.
11. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.3 THE σ-ALGEBRA OF LEBESGUE MEASURABLE SETS

Outer-measure has four virtues: (i) it is defined for all sets of real numbers; (ii) the outer-measure of an interval is its length, (iii) it is countably monotone, and (iv) it is translation invariant. But outer-measure fails even to be finitely additive. According to Corollary 24, there are disjoint sets A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{2}$$

To ameliorate this fundamental defect, we select a σ-algebra of sets, called the Lebesgue measurable sets, which contains all open sets and all sets of outer-measure zero, and the restriction of the set-function outer-measure to the collection of Lebesgue measurable sets is countably additive. There are a number of ways to define what it means for a set to be measurable. We follow an approach due to Constantin Carathéodory.

Definition *A set E is said to be* ***Lebesgue measurable****, or simply* ***measurable****, provided that for any set A*[4]*,*

$$m^*(A) = m^*(A \cap E) + m^*(A \sim E).$$

The collection of measurable sets is denoted by $\mathcal{M}$.

We immediately see one advantage possessed by measurable sets, namely, that the strict inequality (2) cannot occur if one of the sets is measurable. Indeed, if, say, A is measurable and B is any set disjoint from A, then

$$m^*(A \cup B) = m^*([A \cup B] \cap A)) + m^*([A \cup B)] \sim A) = m^*(A) + m^*(B).$$

So, if a set E is measurable, then outer-measure m^* is additive over particular partitions of any set A, namely, as $A = [A \cap E] \cup [A \sim E]$. Since outer-measure is finitely monotone and $A = [A \cap E] \cup [A \sim E]$, we always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \sim E).$$

[4]Recall that we denote by $A \sim B$ the set $\{x \in A \mid x \notin B\}$, the **relative complement** of B in A.

Therefore, E is measurable if and only if for each set A,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \sim E). \tag{3}$$

This inequality trivially holds if $m^*(A) = \infty$. Consequently, it suffices to establish (3) for sets A for which $m^*(A) < \infty$.

Observe that the definition of measurability is symmetric in E and $\mathbf{R} \sim E$, and therefore a set is measurable if and only if its complement is measurable. Clearly, the set $\mathbf{R}$ is measurable.

Lebesgue outer-measure has the following **excision property**.

Proposition 6 *If $E_0 \subseteq E$, E_0 is measurable and $m^*(E_0) < \infty$, then*

$$m^*(E \sim E_0) = m^*(E) - m^*(E_0). \tag{4}$$

Proof Since E_0 is measurable,

$$m^*(E) = m^*(E_0) + m^*(E \sim E_0),$$

and therefore, since $m^*(E_0) < \infty$, (4) holds. □

Proposition 7 *Any set of outer-measure zero is measurable.*

Proof Assume that $m^*(E) = 0$. Let A be any set. Since

$$A \cap E \subseteq E \text{ and } A \sim E \subseteq A,$$

by the monotonicity of outer-measure,

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \sim E) \leq m^*(A).$$

Therefore,

$$m^*(A) \geq m^*(A \sim E) = 0 + m^*(A \sim E) = m^*(A \cap E) + m^*(A \sim E),$$

and so E is measurable. □

Proposition 8 *The translate $E + c$ of a measurable set E is measurable.*

Proof By the translation invariance of outer-measure and the measurability of E, for any set A,

$$\begin{aligned} m^*(A) = m^*(A - c) &= m^*([A - c] \cap E) + m^*\left([A - c] \sim E\right) \\ &= m^*(A \cap [E + c]) + m^*\left(A \sim [E + c]\right). \end{aligned}$$

Therefore, $E + c$ is measurable. □

Proposition 9 *The union of a finite collection of measurable sets is measurable.*

Proof We show that the union of two measurable sets E_1 and E_2 is measurable. The general case follows by induction. Let A be any set. First using the measurability of E_1, and then the measurability of E_2, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \sim E_1)$$
$$= m^*(A \cap E_1) + m^*([A \sim E_1] \cap E_2) + m^*([A \sim E_1] \sim E_2).$$

There are the following set identities:

$$[A \sim E_1] \sim E_2 = A \sim [E_1 \cup E_2]$$

and

$$[A \cap E_1] \cup [[A \sim E_1] \cap E_2] = A \cap [E_1 \cup E_2].$$

It follows from these identities and the finite monotonicity of outer-measure that

$$m^*(A) = m^*(A \cap E_1) + m^*([A \sim E_1] \cap E_2) + m^*([A \sim E_1] \sim E_2)$$
$$= m^*(A \cap E_1) + m^*([A \sim E_1] \cap E_2) + m^*(A \sim [E_1 \cup E_2])$$
$$\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \sim [E_1 \cup E_2]).$$

Therefore, $E_1 \cup E_2$ is measurable. □

A collection of subsets of **R** is called an **algebra** provided that it contains **R** and is closed with respect to relative complements and finite unions; by De Morgan's Identities, such a collection is also closed with respect to finite intersections. It follows from this proposition, together with the measurability of the complement of a measurable set, that the collection $\mathcal{M}$ of measurable sets is an algebra.

Proposition 10 *If A is any set and $\{E_k\}_{k=1}^n$ is a finite, disjoint collection of measurable sets, then*

$$m^*\left(A \cap \left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n m^*(A \cap E_k).$$

In particular,

$$m^*\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n m^*(E_k).$$

Proof The proof proceeds by induction on n. It is clearly true for $n = 1$. Assume that it is true for $n - 1$. Since the collection $\{E_k\}_{k=1}^n$ is disjoint,

$$A \cap \left[\bigcup_{k=1}^n E_k\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{k=1}^n E_k\right] \sim E_n = A \cap \left[\bigcup_{k=1}^{n-1} E_k\right].$$

Consequently, by the measurability of E_n and the induction assumption,

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= m^*(A \cap E_n) + \sum_{k=1}^{n-1} m^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} m^*(A \cap E_k).$$

□

Definition *A countable, disjoint collection* $\{E_k\}_{k=1}^{\infty}$ *of measurable subsets of* E *is called a* **measurable partition** *of* E *provided that* $E = \bigcup_{k=1}^{\infty} E_k$.

Lemma 11 *Let* $E = \bigcup_{k=1}^{\infty} E_k$, *a countable union of measurable sets. Then there is a measurable partition* $\{E'_k\}_{k=1}^{\infty}$ *of* E *for which each* $E'_k \subseteq E_k$.

Proof Define $E'_1 = E_1$ and for each $k \geq 2$, define

$$E'_k = E_k \sim \bigcup_{i=1}^{k-1} E_i.$$

Since the collection of measurable sets is an algebra, each E'_k is measurable, The collection $\{E'_k\}_{k=1}^{\infty}$ was constructed to be disjoint, and it is a measurable partition of E, since for each $x \in E$, there is a first index k for which $x \in E_k$. and so $x \in E'_k$. □

Proposition 12 *The union of a countable collection of measurable sets is measurable.*

Proof Let E be the union of a countable collection of measurable sets. By the preceding lemma, there is a measurable partition $\{E_k\}_{k=1}^{\infty}$ of E. Let $A \subseteq \mathbf{R}$ be any set. For each n, define $F_n = \bigcup_{k=1}^{n} E_k$. The measurable sets are an algebra, and therefore F_n is measurable, so that, by the inclusion $F_n \subseteq E$,

$$m^*(A) = m^*(A \cap F_n) + m^*(A \sim F_n) \geq m^*(A \cap F_n) + m^*(A \sim E).$$

By the preceding proposition,

$$m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Therefore,

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \sim E).$$

The left-hand side of this inequality is independent of n, so that

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \sim E).$$

Consequently, by the countable monotonicity of outer-measure,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \sim E),$$

and therefore E is measurable. □

Proposition 13 *Every interval is measurable.*

Proof Since the collection of measurable sets is a σ-algebra, to show that every interval is measurable, it suffices to show that every interval of the form (a, ∞) is measurable (see Problem 12). Consider such an interval. Let A be any set. Assume that a does not belong to A. Otherwise, replace A by $A \sim \{a\}$, and, of course, if $A \sim \{a\}$ is measurable, so is A. We must show that

$$m^*(A_1) + m^*(A_2) \leq m^*(A), \tag{5}$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty).$$

By the definition of $m^*(A)$ as an infimum, to verify (5), it is necessary and sufficient to show that for any countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals that covers A,

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k). \tag{6}$$

For such a covering and each index k, define

$$I'_k = I_k \cap (-\infty, a) \text{ and } I''_k = I_k \cap (a, \infty).$$

Then I'_k and I''_k are intervals and

$$\ell(I_k) = \ell(I'_k) + \ell(I''_k).$$

Since $\{I'_k\}_{k=1}^{\infty}$ and $\{I''_k\}_{k=1}^{\infty}$ are countable collections of open, bounded intervals that cover A_1 and A_2, respectively,

$$m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I'_k) \text{ and } m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I''_k).$$

Consequently,

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{k=1}^{\infty} \ell(I'_k) + \sum_{k=1}^{\infty} \ell(I''_k). \\ &= \sum_{k=1}^{\infty} [\ell(I'_k) + \ell(I''_k)] \\ &= \sum_{k=1}^{\infty} \ell(I_k). \end{aligned}$$

Therefore, (6) holds and the proof is complete. □

A collection of subsets of $\mathbf{R}$ is called an **σ-algebra** provided that it is an algebra and is closed with respect to countable unions; by De Morgan's Identities, such a collection is also closed with respect to countable intersections. We have shown that the collection of measurable sets, $\mathcal{M}$, is an algebra, and so it follows from Proposition 12 that it is a

σ-algebra. Since every open set is the union of a countable collection of open intervals[5], by the preceding two propositions, every open set is measurable. Every closed set, being the complement of an open set, is measurable. A set of real numbers is said to be a G_δ **set** provided that it is the intersection of a countable collection of open sets and said to be an F_σ **set** provided that it is the union of a countable collection of closed sets. Since $\mathcal{M}$ is a σ-algebra, every G_δ set and every F_σ set is measurable.

Let $\mathcal{S}$ be a collection of subsets of $\mathbf{R}$. Then $\mathcal{S}$ is contained in the σ-algebra of all subsets of $\mathbf{R}$. Define $\mathcal{A}$ to be the intersection of all σ-algebras that contain $\mathcal{S}$. Then $\mathcal{A}$ is the smallest σ-algebra that contains $\mathcal{S}$, smallest in the sense that $\mathcal{A}$ is a σ-algebra that contains $\mathcal{S}$ and is contained in any other σ-algebra that contains $\mathcal{S}$. The **Borel σ-algebra** $\mathcal{B}$ is defined to be the smallest σ-algebra that contains all open sets. Since $\mathcal{M}$ is such a σ-algebra, we have, by minimality, the inclusion $\mathcal{B} \subseteq \mathcal{M}$. According to Proposition 28, there are measurable sets that are not Borel sets. The following theorem has been proven.

Theorem 14 *The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the Borel σ-algebra and all sets of outer-measure zero.*

PROBLEMS

12. Prove that if a σ-algebra of subsets of $\mathbf{R}$ contains intervals of the form (a, ∞), then it contains all intervals.
13. Show that (i) the translate of an F_σ set is also F_σ, (ii) the translate of a G_δ set is also G_δ, and (iii) the translate of a set of measure zero also has measure zero.
14. Show that if a set E has positive outer-measure, then there is a bounded subset of E that also has positive outer-measure.
15. Show that if $m(E) < \infty$ and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

2.4 FINER PROPERTIES OF MEASURABLE SETS

Definition *The restriction of the set-function outer-measure to the σ-algebra of measurable sets $\mathcal{M}$ is called* **Lebesgue measure** *or simply* **measure**. *It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by*

$$m(E) = m^*(E).$$

Theorem 15 (**the Regularity of Lebesgue Measure**) *If E is Lebesgue measurable and $\epsilon > 0$, then there is a closed set F and an open set $\mathcal{O}$ for which*

$$F \subseteq E \subseteq \mathcal{O},\ \ m(\mathcal{O} \sim E) < \epsilon \text{ and } m(E \sim F) < \epsilon. \tag{7}$$

Proof We establish the open outer-approximation in two steps. First, consider the case $m(E) < \infty$. By the definition of outer-measure, there is a countable collection of open,

[5] See page 17.

bounded intervals $\{I_k\}$ that covers E and for which $\sum_{k=1}^{\infty} \ell(I_k) < m(E) + \epsilon$. Define $\mathcal{O} = \cup_{k=1}^{\infty} I_k$. Then E is contained in the open set $\mathcal{O}$. Moreover, by the countable monotonicity of measure, $m(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k)$. Consequently, by the excision property of measure, since $m(E) < \infty$,

$$m(\mathcal{O} \sim E) = m(\mathcal{O}) - m(E) < \epsilon.$$

So the open outer-approximation holds if $m(E) < \infty$. Now consider the case $m(E) = \infty$. Fix k and define $E_k \equiv E \cap [-k, k]$. Then $m(E_k) < \infty$ and so, by what has just been established, there if an open set $\mathcal{O}_k$ for which

$$E_k \subseteq \mathcal{O}_k \text{ and } m(\mathcal{O}_k \sim E_k) < \epsilon/2^k.$$

Define $\mathcal{O} \equiv \bigcup_{k=1}^{\infty} \mathcal{O}_k$, so that $\mathcal{O}$ is open and $\mathcal{O} \sim E \subseteq \bigcup_{k=1}^{\infty} (\mathcal{O}_k \sim E_k)$. By the countable monotonicity of measure,

$$m(\mathcal{O} \sim E) \leq \sum_{k=1}^{\infty} m(\mathcal{O}_k \sim E_k) < \epsilon.$$

So the open outer-approximation property holds.

To establish the closed inner-approximation property of E, we use the open outer-approximation property of the measurable set $\mathbf{R} \sim E$. There is an open set $\mathcal{U}$ for which

$$\mathbf{R} \sim E \subseteq \mathcal{U} \text{ and } m(\mathcal{U} \sim (\mathbf{R} \sim E)) < \epsilon.$$

Define $F = \mathbf{R} \sim \mathcal{U}$. Then F is a closed subset of E and

$$E \sim F = E \sim (\mathbf{R} \sim \mathcal{U}) = E \cap U = \mathcal{U} \sim (\mathbf{R} \sim E).$$

Therefore,

$$m(E \sim F) = m(\mathcal{U} \sim (\mathbf{R} \sim E)) < \epsilon. \qquad \square$$

Corollary 16 *If E is measurable, then there is a G_δ set G and an F_σ set F for which*

$$F \subseteq E \subseteq G, \; m(G \sim E) = 0 \text{ and } m(E \sim F) = 0. \tag{8}$$

Proof For each n, by the preceding theorem, there is a closed set F_n and an open set $\mathcal{O}_n$ for which

$$F_n \subseteq E \subseteq \mathcal{O}_n, \; m(\mathcal{O}_n \sim E) < 1/n \text{ and } m(E \sim F_n) < 1/n. \tag{9}$$

Define

$$G \equiv \bigcap_{n=1}^{\infty} \mathcal{O}_n \text{ and } F \equiv \bigcup_{n=1}^{\infty} F_n$$

Then G is a G_δ set, F is an F_σ set, and, by (9), $m(G \sim E) = m(E \sim F) = 0$. $\square$

We will frequently use the following elementary characterization of measurability.

Corollary 17 *A set of real numbers is measurable if and only if it is a G_δ set from which a set of measure zero has been excised.*

Proof Let E be measurable. According to the preceding corollary, there is a G_δ set G for which $E \subseteq G$ and $m(G \sim E) = 0$. Therefore, $E = G \sim [G \sim E]$, a G_δ set from which a set of measure zero has been excised. Since $\mathcal{M}$ is a σ-algebra, any such set is measurable. □

Theorem 18 *The collection $\mathcal{M}$ of Lebesgue measurable, sets is the smallest σ-algebra that contains the Borel σ-algebra and all sets of outer-measure zero.*

Proof Let $\mathcal{S}$ be the union of the Borel σ-algebra and all sets of outer-measure zero, and let $\mathcal{A}$ be the smallest σ-algebra that contains $\mathcal{S}$. Since every open set and every set of outer-measure zero is measurable, by the minimality of $\mathcal{A}$, $\mathcal{A} \subseteq \mathcal{M}$. According to the preceding corollary, if E is a measurable set, then E is a G_δ set from which a set of measure zero has been excised and so E belongs to $\mathcal{A}$. Consequently, $\mathcal{M} \subseteq \mathcal{A}$, and so $\mathcal{M} = \mathcal{A}$. □

The following property of measurable sets of finite measure asserts that such sets are, roughly speaking, "nearly" equal to the disjoint union of a finite collection of open, bounded intervals.

Theorem 19 *If $m(E) < \infty$ and $\epsilon > 0$, then there is a finite, disjoint collection of open, bounded intervals $\{I_k\}_{k=1}^n$ for which, if $\mathcal{U} \equiv \bigcup_{k=1}^n I_k$, then*[6]

$$m(\mathcal{U} \cup E) - m(\mathcal{U} \cap E) = m(\mathcal{U} \sim E) + m(E \sim \mathcal{U}) < \epsilon.$$

Proof According to Theorem 15, there is an open set $\mathcal{O}$ that contains E and

$$m(\mathcal{O} \sim E) < \epsilon/2. \tag{10}$$

Since $m(E) < \infty$, by the excision property of measure, $m(\mathcal{O}) < \infty$. Now $\mathcal{O}$, being open, is the disjoint union of a countable collection $\{I_k\}_{k=1}^\infty$ of open intervals and, by the countable additivity of measure,

$$\sum_{k=1}^{\infty} \ell(I_k) = m(\mathcal{O}) < \infty.$$

Choose an n for which

$$\sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Define $\mathcal{U} = \bigcup_{k=1}^n I_k$. Since $\mathcal{U} \sim E \subseteq \mathcal{O} \sim E$, it follows from (10) that

$$m(\mathcal{U} \sim E) < \epsilon/2.$$

On the other hand, since $E \subseteq \mathcal{O}$,

$$E \sim \mathcal{U} \subseteq \mathcal{O} \sim \mathcal{U} = \bigcup_{k=n+1}^{\infty} I_k,$$

[6] For two sets A and B, the **symmetric difference**, which is denoted by $A \Delta B$, is defined to be the set $[A \sim B] \cup [B \sim A]$. Using this notation, the conclusion is that $m(E \,\Delta\, \mathcal{U}) < \epsilon$.

so that

$$m(E \sim \mathcal{O}) \leq \sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Therefore,

$$m(\mathcal{U} \sim E) + m(E \sim \mathcal{U}) < \epsilon.$$ □

PROBLEMS

16. Show that a set E is measurable if for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m(\mathcal{O} \sim F) < \epsilon$.
17. Assume that $m^*(E) < \infty$. Show that there is a G_δ set G that contains E and $m(G) = m^*(E)$. Show that E is measurable if and only if there is a G_δ set that contains G and $m^*(G \sim E) = 0$.
18. Assume that $m^*(E) < \infty$. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer-measure and for which

$$m^*(\mathcal{O} \sim E) > m^*(\mathcal{O}) - m^*(E).$$

19. (Lebesgue's definition of measurability) Let E have finite outer-measure. Show that E is measurable if and only if for each open, bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \sim E).$$

20. For any set A, define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf \left\{ m^*(\mathcal{O}) \;\middle|\; \mathcal{O} \supseteq A, \mathcal{O} \text{ open} \right\}.$$

How is this set-function m^{**} related to outer-measure m^*?

21. For any set A, define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup \left\{ m^*(F) \;\middle|\; F \subseteq A, F \text{ closed} \right\}.$$

How is this set-function m^{***} related to outer-measure m^*?

2.5 COUNTABLE ADDITIVITY AND CONTINUITY OF MEASURE, AND THE BOREL-CANTELLI LEMMA

Theorem 20 **(the Countable Additivity of Measure)** *Lebesgue measure is* **countably additive**, *in the sense that if $\{E_k\}_{k=1}^{\infty}$ is a measurable partition of E, then*

$$m(E) = \sum_{k=1}^{\infty} m(E_k).$$

Proof According to Proposition 12, $\bigcup_{k=1}^{\infty} E_k$ is measurable, and according to Proposition 5, outer-measure is countably monotone. Therefore,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k). \tag{11}$$

It remains to prove this inequality in the opposite direction. By the finite additivity of measure, for each n,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k).$$

Since $\bigcup_{k=1}^{\infty} E_k$ contains $\bigcup_{k=1}^{n} E_k$, by the monotonicity of outer-measure and the preceding equality,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{n} m(E_k) \text{ for each } n.$$

The left-hand side of this inequality is independent of n. Consequently,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k). \tag{12}$$

From the inequalities (11) and (12), it follows that these are equalities. □

According to Proposition 3, the outer-measure of an interval is its length while according to Proposition 4, outer-measure is translation invariant. Therefore, the preceding proposition completes the proof of the following theorem.

Theorem 21 *The set-function Lebesgue measure, defined on the σ-algebra of Lebesgue measurable sets $\mathcal{M}$, assigns length to any interval, is translation invariant, and is countably additive.*

A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be **ascending** provided that for each k, $E_k \subseteq E_{k+1}$, and said to be **descending** provided that for each k, $E_{k+1} \subseteq E_k$.

Theorem 22 **(the Continuity of Measure)**

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k). \tag{13}$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k). \tag{14}$$

Proof We first prove (i). If there is an index k_0 for which $m(A_{k_0}) = \infty$, then, by the monotonicity of measure, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ and $m(A_k) = \infty$ for all $k \geq k_0$. Therefore, (13) holds since each side equals ∞. It remains to consider the case that $m(A_k) < \infty$ for all k. Define $A_0 = \emptyset$ and then define $C_k = A_k \sim A_{k-1}$ for each $k \geq 1$. By construction, since the collection $\{A_k\}_{k=1}^{\infty}$ is ascending,

$$\{C_k\}_{k=1}^{\infty} \text{ is disjoint and } \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k.$$

By the countable additivity of m,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}). \tag{15}$$

The collection $\{A_k\}_{k=1}^{\infty}$ is ascending, and since each $m(A_k) < \infty$, it follows from the excision property of measure that

$$\begin{aligned} \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) &= \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \\ &= \lim_{n\to\infty} [m(A_n) - m(A_0)]. \end{aligned} \tag{16}$$

Since $m(A_0) = m(\emptyset) = 0$, (13) follows from (15) and (16). To prove (ii), define $D_k = B_1 \sim B_k$ for each k. Since the collection $\{B_k\}_{k=1}^{\infty}$ is descending, the collection $\{D_k\}_{k=1}^{\infty}$ is ascending. By part (i),

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k).$$

According to De Morgan's Identities,

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^{\infty} B_k.$$

On the other hand, by the excision property of measure, for each k, since $m(B_k) < \infty$, $m(D_k) = m(B_1) - m(B_k)$. Therefore,

$$m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty} [m(B_1) - m(B_n)].$$

Once more using the excision property of measure, we obtain (14). □

The Borel-Cantelli Lemma *If $\{E_k\}_{k=1}^{\infty}$ is a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$, then almost all $x \in \mathbf{R}$ belong to at most finitely many of the E_k's.*

Proof For each n, by the countable monotonicity of m,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \le \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Consequently, by the continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore, almost all $x \in \mathbf{R}$ fail to belong to $\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]$, which means that they belong to at most finitely many E_k's. □

PROBLEMS

22. Show that if E_1 and E_2 are measurable sets of finite measure, then

$$m(E_1 \cup E_2) - m(E_1 \cap E_2) = m(E_1 \sim E_2) + m(E_2 \sim E_1).$$

23. Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

24. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

25. Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbf{R}$ and m' a set-function on $\mathcal{M}'$ that takes values in $[0, \infty)$, is countably additive, and such that $m'(\emptyset) = 0$.
    (i) Show that m' is finitely additive, monotone, countably monotone, and possesses the excision property.
    (ii) Show that m' possesses the same continuity properties as Lebesgue measure.

26. Show that the continuity of measure together with the finite additivity of measure are equivalent to the countable additivity of measure.

27. (Arzelà) Let $\{E_k\}_{k=1}^{\infty}$ be a collection of measurable subsets of $[a, b]$, and assume there is a $\delta > 0$ such that $m(E_k) \geq \delta$ for each k. Show that there is an $x \in [a, b]$ that belongs to infinitely many such E_k's.

2.6 VITALI'S EXAMPLE OF A NON-MEASURABLE SET

We have considered properties of individual measurable sets and of the collection of measurable sets. It is only natural to ask if, in fact, there are any sets that fail to be measurable. The answer is not at all obvious. If a set E has outer-measure zero, then it is measurable, and since any subset also has outer-measure zero, every subset of E is measurable. This is the best that can be said regarding the inheritance of measurability through the relation of set inclusion. We now show that if E has positive outer-measure, then there are subsets of E that fail to be measurable.

For any non-empty set E of real numbers, define two points in E to be **rationally equivalent** provided that their difference is a rational number. Clearly, this is an equivalence relation, that is, it is reflexive, symmetric, and transitive. We call it the rational equivalence relation on E. For this relation, there is the disjoint decomposition of E into the collection of equivalence classes. A **choice set** for the rational equivalence relation on E is a set $\mathcal{C}_E$ comprising exactly one member of each equivalence class. It follows from the Axiom of Choice[7] that there is such a set.

Theorem 23 **(Vitali)** *If E is a set of real numbers for which $m^*(E) > 0$, then there is a subset of E that is not measurable.*

Proof In view of the countable monotonicity of m^*, by possibly replacing E by a bounded subset of positive outer-measure, we assume that E is bounded. Choose $\mathcal{C}_E \subseteq E$ to be a choice set for the rational equivalence relation on E, that is,

$$\text{the countable collection } \{\mathcal{C}_E + q\}_{q \in \mathbf{Q}} \text{ is disjoint and } E \subseteq \bigcup_{q \in \mathbf{Q}} [\mathcal{C}_E + q].$$

[7]See page 5.

We claim that C_E is not measurable, and to verify this, we assume otherwise and obtain a contradiction. Choose $\mathbf{Q}_0$ to be a countably infinite, bounded set of rational numbers. The set $\bigcup_{q\in\mathbf{Q}_0}[C_E+q]$ is bounded and measurable. Consequently, by the countable additivity of measure, since $\{C_E+q\}_{q\in\mathbf{Q}_0}$ is a countable, disjoint collection of measurable sets,

$$m\Big(\bigcup_{q\in\mathbf{Q}_0}[C_E+q]\Big)=\sum_{q\in\mathbf{Q}_0}m(C_E+q)<\infty.$$

However, by the translation invariance of measure, $m(C_E+q)=m(C_E)$, for every $q\in\mathbf{Q}_0$, and therefore, since $\mathbf{Q}_0$ is countably infinite, $m(C_E)=0$. By the countable monotonicity of m^*, the countability of the rationals and the inclusion $E\subseteq\bigcup_{q\in\mathbf{Q}}[C_E+q]$, we obtain the following contradiction:

$$0<m^*(E)\le\sum_{q\in\mathbf{Q}}m^*(C_E+q)=\sum_{q\in\mathbf{Q}}m(C_E+q)=0.$$

□

Corollary 24 *There are disjoint sets of real numbers A and B for which*

$$m^*(A\cup B)<m^*(A)+m^*(B).$$

Proof If there were not two such sets, then, by the very definition of measurable set, every set of real numbers would be measurable, which contradicts the preceding theorem. □

PROBLEMS

28. (i) Show that rational equivalence defines an equivalence relation on any set.
    (ii) Explicitly find a choice set for the rational equivalence relation on $\mathbf{Q}$, and also on $\mathbf{R}\sim\mathbf{Q}$.
    (iii) Define two numbers to be irrationally equivalent provided that their difference is irrational or zero. Is this an equivalence relation on $\mathbf{R}$? Is this an equivalence relation on $\mathbf{Q}$?
29. Show that any choice set for the rational equivalence relation on a set of positive outer-measure must be uncountably infinite.

2.7 THE CANTOR SET AND THE CANTOR-LEBESGUE FUNCTION

In this section, we construct a set called the Cantor set and a function called the Cantor-Lebesgue function. These provide two interesting examples: a continuous, increasing function, which has a derivative that vanishes almost everywhere, and yet the function is not constant, and a measurable set that is not a Borel set.

Consider the closed, bounded interval $I=[0,1]$. The first step in the construction of the Cantor set is to subdivide I into three intervals of equal length $1/3$ and remove the interior of the middle interval, that is, remove the interval $(1/3,2/3)$ from the interval $[0,1]$ to obtain the closed set C_1, which is the disjoint union of two disjoint, closed intervals, each of length $1/3$:

$$C_1=[0,1/3]\cup[2/3,1].$$

Repeat this "open middle one-third removal" on each of the two intervals in C_1 to obtain a closed set C_2, which is the disjoint union of 2^2 closed intervals, each of length $1/3^2$:

$$C_2=[0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1].$$

Now repeat this "open middle one-third removal" on each of the four intervals in C_2 to obtain a closed set C_3, which is the disjoint union of 2^3 closed intervals, each of length $1/3^3$. Continue this removal operation countably many times to obtain a countable collection of sets $\{C_k\}_{k=1}^{\infty}$. Finally, define the Cantor set $\mathbf{C}$ by

$$\mathbf{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^{\infty}$ possesses the following two properties:

(i) $\{C_k\}_{k=1}^{\infty}$ is a countable, descending collection of closed subsets of $[0, 1]$, and
(ii) for each k, C_k is the disjoint union of 2^k closed intervals, each of length $1/3^k$.

Proposition 25 *The Cantor set* $\mathbf{C}$ *is a closed, uncountable set of measure zero.*

Proof The intersection of any collection of closed sets is closed. Therefore, $\mathbf{C}$ is closed. Each closed set is measurable, so that each C_k and $\mathbf{C}$ itself is measurable. Now each C_k is the disjoint union of 2^k intervals, each of length $1/3^k$, so that by the finite additivity of measure,

$$m(C_k) = (2/3)^k.$$

By the monotonicity of measure, since $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$, for all k, $m(\mathbf{C}) = 0$. It remains to show that $\mathbf{C}$ is uncountable. To do so, suppose otherwise, and let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. One of the two disjoint Cantor intervals whose union is C_1 fails to contain the point c_1; denote it by F_1. One of the two disjoint Cantor intervals in C_2 whose union is F_1 fails to contain the point c_2; denote it by F_2. Continuing in this way, construct a countable collection of sets $\{F_k\}_{k=1}^{\infty}$, which, for each index k, possesses the following three properties: (i) F_k is closed and $F_{k+1} \subseteq F_k$; (ii) $F_k \subseteq C_k$; and (iii) $c_k \notin F_k$. From (i) and the Nested Set Theorem[8], it follows that the intersection $\bigcap_{k=1}^{\infty} F_k$ is non-empty. Let $x \in \bigcap_{k=1}^{\infty} F_k$. By property (ii),

$$\bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = \mathbf{C},$$

and therefore $x \in \mathbf{C}$. However, $\{c_k\}_{k=1}^{\infty}$ is an enumeration of $\mathbf{C}$, so that $x = c_n$ for some index n. Consequently, $c_n = x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_n$. This contradicts the choice of c_n. Therefore, $\mathbf{C}$ is uncountable. □

We now define the Cantor-Lebesgue function, which is a continuous, increasing function $\varphi\colon [0, 1] \to \mathbf{R}$ with the remarkable property that, despite the fact that $\varphi(1) > \varphi(0)$, its derivative exists and is zero on a set of measure 1. For each k, let $\mathcal{O}_k$ be the union of the $2^k - 1$ intervals which have been removed during the first k stages of the Cantor deletion process. Therefore, $C_k = [0, 1] \sim \mathcal{O}_k$. Define $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$. Then, by De Morgan's Identities, $\mathbf{C} = [0, 1] \sim \mathcal{O}$. We begin by defining φ on $\mathcal{O}$, and then define it on $\mathbf{C}$.

Fix k. Define φ on $\mathcal{O}_k$ to be the increasing function on $\mathcal{O}_k$ that is constant on each of its $2^k - 1$ open intervals and takes the $2^k - 1$ values

$$\{1/2^k, 2/2^k, 3/2^k, \ldots, [2^k - 1]/2^k\}.$$

[8] See page 18

On the three intervals that are removed in the first two stages, the prescription for φ is

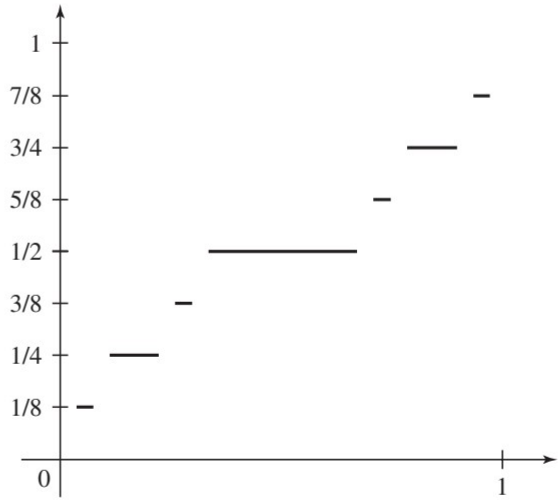
$$\varphi(x) = \begin{cases} 1/4 & \text{if } x \in (1/9,\, 2/9) \\ 2/4 & \text{if } x \in (3/9,\, 6/9) = (1/3, 2/3) \\ 3/4 & \text{if } x \in (7/9,\, 8/9) \end{cases}$$

Extend φ to all of $[0, 1]$ by defining it on $\mathbf{C}$ as follows:

$$\varphi(0) = 0 \text{ and } \varphi(x) = \sup\{\varphi(t) \mid t \in \mathcal{O} \cap [0, x)\} \text{ if } x \in \mathbf{C} \sim \{0\}.$$

Proposition 26 *The Cantor-Lebesgue function φ: $[0, 1] \to \mathbf{R}$ is an increasing, continuous function that maps $[0, 1]$ onto $[0, 1]$. Its derivative exists on the open set $\mathcal{O}$, the complement in $[0, 1]$ of the Cantor set,*

$$\varphi' = 0 \text{ on } \mathcal{O}, \text{ yet } m(\mathcal{O}) = 1 \text{ and } \varphi(1) > \varphi(0).$$



The graph of the Cantor-Lebesgue function on $\mathcal{O}_3 = [0, 1] \sim C_3$

Proof Since φ is increasing on $\mathcal{O}$, its extension to $[0, 1]$ also is increasing. As for continuity, φ certainly is continuous at each point in $\mathcal{O}$, since each such point belongs to an open interval on which it is constant. Now consider a point $x_0 \in \mathbf{C}$ with $x_0 \neq 0, 1$. Since $x_0 \in \mathbf{C}$, it is not a member of the $2^k - 1$ intervals removed in the first k stages of the removal process, the union of which is denoted by $\mathcal{O}_k$. Therefore, x_0 lies between two consecutive intervals in $\mathcal{O}_k$: choose a_k in the lower of these and b_k in the upper one. The function φ was defined to increase by $1/2^k$ across two consecutive intervals in $\mathcal{O}_k$. Therefore, for each k,

$$a_k < x_0 < b_k \text{ and } \varphi(b_k) - \varphi(a_k) = 1/2^k.$$

So the function φ fails to have a jump discontinuity at x_0. For an increasing function, a jump discontinuity is the only possible type of discontinuity. Therefore, φ is continuous at x_0. If x_0 is an end-point of $[0, 1]$, a similar argument establishes continuity at x_0.

Since φ is constant on each of the intervals removed at any stage of the removal process, its derivative exists and equals 0 at each point in $\mathcal{O}$. Since $\mathbf{C}$ has measure zero, its complement in $[0, 1]$, $\mathcal{O}$, has measure 1. Finally, since $\varphi(0) = 0, \varphi(1) = 1$ and φ is increasing and continuous, it follows from the Intermediate Value Theorem that φ maps $[0, 1]$ onto $[0, 1]$. □

Proposition 27 *Let $\varphi\colon [0, 1] \to \mathbf{R}$ be the Cantor-Lebesgue function and define the function $\psi\colon [0, 1] \to \mathbf{R}$ by*

$$\psi(x) = \varphi(x) + x \text{ for all } x \in [0, 1].$$

Then ψ is a strictly increasing, continuous function that maps $[0, 1]$ onto $[0, 2]$,

(i) maps the Cantor set onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a non-measurable set.

Proof The function ψ is continuous, since it is the sum of two continuous functions and is strictly increasing since it is the sum of an increasing and a strictly increasing function. Moreover, since $\psi(0) = 0$ and $\psi(1) = 2$, by the Intermediate Value Theorem, $\psi([0, 1]) = [0, 2]$. For $\mathcal{O} = [0, 1] \sim \mathbf{C}$, there is the disjoint decomposition of the domain

$$[0, 1] = \mathbf{C} \cup \mathcal{O},$$

which, since ψ is one-to-one, lifts to the disjoint decomposition of its image,

$$[0, 2] = \psi(\mathcal{O}) \cup \psi(\mathbf{C}). \tag{17}$$

A strictly increasing, continuous function defined on an interval has a continuous inverse. Therefore, $\psi(\mathbf{C})$ is closed and $\psi(\mathcal{O})$ is open, so both are measurable. We will show that $m(\psi(\mathcal{O})) = 1$ from which it will follow from (17) that $m(\psi(\mathbf{C})) = 1$ and thereby prove (i). Let $\{I_k\}_{k=1}^{\infty}$ be an enumeration (in any manner) of the collection of intervals that are removed in the Cantor removal process. We have $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Since φ is constant on each I_k, ψ maps I_k onto a translated copy of itself of the same length. Since ψ is one-to-one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By the countable additivity of measure,

$$m(\psi(\mathcal{O})) = \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = m(\mathcal{O}).$$

Therefore, $m(\psi(\mathcal{O})) = 1$ and so, by (17), $m(\psi(\mathbf{C})) = 1$. We have established (i). To verify (ii), observe that, according to Theorem 23, $\psi(\mathbf{C})$ contains a set W that is not measurable. The set $\psi^{-1}(W)$ is measurable and has measure zero, since it is a subset of the Cantor set. The set $\psi^{-1}(W)$, a subset C, is mapped by ψ to a non-measurable set. □

Proposition 28 *There is a measurable set, a subset of the Cantor set, that is not a Borel set.*

Proof The function $\psi\colon [0, 1] \to \mathbf{R}$ defined in the preceding proposition maps a measurable set A onto a non-measurable set. A strictly increasing, continuous function defined on an interval maps Borel sets onto Borel sets, since ψ^{-1} is continuous. Therefore, the set A is not a Borel set, since otherwise its image under ψ would be a Borel set and therefore would be measurable. □

PROBLEMS

30. Let f: $[a, b] \to \mathbf{R}$ be continuous and strictly increasing. Show that f^{-1}: $[f(a), f(b)] \to \mathbf{R}$ is continuous.

31. Let f: $[a, b] \to \mathbf{R}$ be continuous. Show that f maps closed sets to closed sets and maps F_σ sets to F_σ sets. Does f map measurable sets to measurable sets?

32. Let the function f: $[a, b] \to \mathbf{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f is continuous and maps sets of measure zero to sets of measure zero. Conclude that f maps measurable set to measurable sets. (Suggestion: consider the preceding problem and the regularity of measure.)

33. Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0, 1] \sim F$ is dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

34. Show that there is an open set of real numbers which, contrary to intuition, has a boundary of positive measure. (Suggestion: Consider the complement of the generalized Cantor set in the preceding problem.)

35. A subset A of $\mathbf{R}$ is said to be **nowhere dense** in $\mathbf{R}$ provided that the closure of every open set $\mathcal{O}$ has a non-empty open subset that is disjoint from A. Show that the Cantor set is nowhere dense in $[0, 1]$.

36. Let f: $[a, b] \to \mathbf{R}$ be continuous and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set.

CHAPTER 3

Lebesgue Measurable Functions

Contents

3.1 SUMS, PRODUCTS, AND COMPOSITIONS

We continue to denote by $\mathcal{M}$ the σ-algebra of Lebesgue measurable sets and by m the set-function Lebesgue measure. In Part 11, we consider a general concept of measure. However, in the first eight chapters, we only consider Lebesgue measure for subsets of $\mathbf{R}$, and so, in these, we often drop the adjective Lebesgue. Recall that a property is said to hold **almost everywhere** on a measurable set E provided that it holds on $E_0 \subseteq E$, where $m(E \sim E_0) = 0$. The extended real numbers is the set $\mathbf{R} \cup \{\pm\infty\}$ which we denote by $\overline{\mathbf{R}}$. Given two functions $h\colon E \to \overline{\mathbf{R}}$ and $g\colon E \to \overline{\mathbf{R}}$, we write "$h \leq g$ on E" to mean that $h(x) \leq g(x)$ for all $x \in E$. We say that a sequence of functions $\{f_n\colon E \to \overline{\mathbf{R}}\}$ is increasing provided that $f_n \leq f_{n+1}$ on E for each index n.

Many arguments will depend on inverse images of functions. For any mapping $f\colon X \to Y$, if $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a collection of subsets of Y, parametrized by a space Λ, then

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} Y_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(Y_\lambda) \text{ and } f^{-1}\left(\bigcap_{\lambda \in \Lambda} Y_\lambda\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(Y_\lambda).$$

Definition *A function $f\colon E \to \overline{\mathbf{R}}$ is said to be* **Lebesgue measurable**, *or simply* **measurable**, *provided that its domain E is a measurable subset of $\mathbf{R}$ and for each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.*

Proposition 1 *If $f\colon E \to \overline{\mathbf{R}}$ is measurable, then for every interval I of real numbers, $f^{-1}(I)$ is measurable.*

Proof Every bounded interval is the intersection of two unbounded intervals, and so it suffices to show that $f^{-1}(I)$ is measurable if I is unbounded. Fix a number c. We have

$$f^{-1}[c, \infty) = X \sim f^{-1}(-\infty, c) \text{ and } f^{-1}(c, \infty) = \bigcup_{1 \leq k < \infty} f^{-1}[c + 1/k, \infty).$$

Now, $\mathcal{M}$ is a σ-algebra, Therefore, $f^{-1}[c, \infty)$, being the complement of a measurable set, is measurable, and $f^{-1}(c, \infty)$, being the countable union of measurable sets, is measurable, as is the following countable intersection of measurable sets:

$$f^{-1}(-\infty, c] = \bigcap_{1 \leq k < \infty} f^{-1}(-\infty, c + 1/k). \qquad \square$$

Proposition 2 *A function $f\colon E \to \mathbf{R}$ is measurable if and only if for each open set $\mathcal{O}$, the inverse image of $\mathcal{O}$ under f, $f^{-1}(\mathcal{O})$, is a measurable set.*

Proof If the inverse image of each open set is measurable, then $E = f^{-1}(\mathbf{R})$ is measurable, and since each interval $(-\infty, c)$ is open, the function f is measurable. Conversely, suppose that f is measurable. Let $\mathcal{O}$ be open. According to Proposition 9 of Chapter 1, $\mathcal{O}$ is the union of a countable collection of intervals. Therefore, the measurable sets being a σ-algebra, it follows from the preceding proposition that $f^{-1}(\mathcal{O})$ is measurable. □

The following proposition follows from its predecessor by recalling that, for a continuous function $f\colon E \to \mathbf{R}$ on a measurable set E, if $\mathcal{O}$ is open, then $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$, where U is an open subset of $\mathbf{R}$, and so $f^{-1}(\mathcal{O})$ is measurable.

Proposition 3 *If E is measurable, then every continuous function $f\colon E \to \mathbf{R}$ is measurable.*

Proposition 4 *If $E = A \cup B$, where A and B are measurable, then $f\colon E \to \overline{\mathbf{R}}$ is measurable if and only if its restrictions to A and B are measurable. In particular, if $E_0 \subseteq E$ and $m(E \sim E_0) = 0$, then*

$$f\colon E \to \overline{\mathbf{R}} \text{ is measurable if and only } f\colon E_0 \to \overline{\mathbf{R}} \text{ is measurable.} \tag{1}$$

Proof Let $f\colon E \to \overline{\mathbf{R}}$ be measurable. For each $c \in \mathbf{R}$,

$$\{x \in A \mid f(x) < c\} = \{x \in E \mid f(x) < c\} \cap A,$$

so that the restriction of f to A is measurable. Similarly, the restriction of f to B is measurable. On the other hand, if both restrictions are measurable, then for each $c \in \mathbf{R}$,

$$\{x \in E \mid f(x) < c\} = \{x \in A \mid f(x) < c\} \cup \{x \in B \mid f(x) < c\},$$

so that $f\colon E \to \overline{\mathbf{R}}$ is measurable. If $m(E_0) = 0$, then every subset of E_0 is measurable, and therefore every function on E_0 is measurable, and so (1) holds. □

It follows from these two propositions that, for a real-valued function defined on a measurable set, if the set of points at which it fails to be continuous has measure zero, then the function is measurable. According to the first theorem of the following chapter, a monotone function on an interval is continuous except for at most a countable collection of points, and so is measurable.

Theorem 5 *If $f\colon E \to \mathbf{R}$ and $g\colon E \to \mathbf{R}$ are measurable functions, then for any α and β,*

(Linearity)

$$\alpha f + \beta g\colon E \to \mathbf{R} \text{ is measurable.}$$

(Products)

$$f \cdot g\colon E \to \mathbf{R} \text{ is measurable.}$$

Proof We consider the case $\alpha = \beta = 1$, and leave the case of general coefficients as an exercise. For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so, by the density of the set of rational numbers $\mathbf{Q}$ in $\mathbf{R}$, there is a rational number q for which $f(x) < q < c - g(x)$. Consequently,

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbf{Q}} \{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}.$$

However, the rational numbers are countable, and so $\{x \in E \mid f(x) + g(x) < c\}$ is measurable, since $\mathcal{M}$ is a σ-algebra. Therefore, $f + g$ is measurable. To prove that the product of measurable functions is measurable, observe that

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

Since we have established linearity, to show that the product of two measurable functions is measurable, it suffices to show that the square of a measurable function is measurable. For $c > 0$,

$$\{x \in E \mid f^2(x) < c\} = \{x \in E \mid -\sqrt{c} < f(x) < \sqrt{c}\},$$

while for $c \leq 0$, $\{x \in E \mid f^2(x) < c\} = \emptyset$. Therefore, f^2 is measurable. □

We identify two measurable functions with a common domain that agree almost everywhere. In particular, in view of Proposition 4, we identify a measurable function that is finite almost everywhere with a real-valued measurable function. This identification permits the extension of measurability results for real-valued functions to hold for extended real-valued functions that are finite almost almost everywhere, regardless of the possibility that functional values of sums and products may not be properly defined at points at which one of the functions takes an infinite value. In particular, the above theorem extends to measurable functions that are finite almost everywhere.

If A is any set, the **characteristic function** of A, χ_A, is the function on $\mathbf{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

It is clear that the function χ_A is measurable if and only if the set A is measurable. Consequently, since there are non-measurable sets, there are non-measurable functions.

Many of the properties of functions considered in calculus, such as continuity and differentiability, are preserved under the operation of composition of functions. As the next example shows, the composition of measurable functions may not be measurable.

Example There are two measurable real-valued functions, each defined on all of $\mathbf{R}$, for which the composition fails to be measurable. Indeed, by Proposition 27 of the preceding chapter, there is a continuous, strictly increasing function $\psi\colon [0, 1] \to \mathbf{R}$ and a measurable subset A of $[0, 1]$ for which $\psi(A)$ is non-measurable. Extend ψ to a continuous, strictly increasing function that maps $\mathbf{R}$ onto $\mathbf{R}$. The function ψ^{-1} is continuous and therefore is measurable. On the other hand, A is a measurable set and so its characteristic function χ_A is a measurable function. However, the composition

$$f \equiv \chi_A \circ \psi^{-1} = \chi_{\psi(A)}$$

is not measurable, since the set $\psi(A)$ is not measurable.

There is the following useful result regarding the preservation of measurability under composition (also see Problem 12)[1].

Proposition 6 *If $f: E \to \mathbf{R}$ is a measurable function and $g: \mathbf{R} \to \mathbf{R}$ is continuous, then the composition $g \circ f: \mathbf{R} \to \mathbf{R}$ is measurable.*

Proof According to Proposition 2, a real-valued function is measurable if and only if the inverse image of each open set is measurable. Let $\mathcal{O}$ be open. Then

$$(g \circ f)^{-1}(\mathcal{O}) = f^{-1}(g^{-1}(\mathcal{O})).$$

Since g is continuous, the set $\mathcal{U} = g^{-1}(\mathcal{O})$ is an open subset of $\mathbf{R}$. By the measurability of f, $f^{-1}(\mathcal{U})$ is measurable. Therefore, $(g \circ f)^{-1}(\mathcal{O})$ is measurable, and so $g \circ f$ is measurable. □

An immediate consequence of the above composition result is that if $f: E \to \mathbf{R}$ is measurable, then $|f|$ also is measurable, and indeed

$$|f|^p: E \to \mathbf{R} \text{ is measurable for each } p > 0.$$

For a finite collection of functions $\{f_k: E \to \mathbf{R}\}_{k=1}^n$, define

$$\max\{f_1, \ldots, f_n\}(x) \equiv \max\{f_1(x), \ldots, f_n(x)\} \text{ for } x \in E.$$

The function $\min\{f_1, \ldots, f_n\}$ is defined the same way. We leave the proof of the following result as an exercise.

Proposition 7 *For a finite collection $\{f_k: E \to \mathbf{R}\}_{k=1}^n$ of measurable functions, the functions $\max\{f_1, \ldots, f_n\}$ and $\min\{f_1, \ldots, f_n\}$ also are measurable.*

For a measurable function $f: E \to \mathbf{R}$, there are associated non-negative measurable functions f^+ and f^-, called the positive part and the negative part of f, defined on E by

$$f^+(x) = \max\{f(x), 0\},\ f^-(x) = \max\{-f(x), 0\}.$$

These provide the following frequently used expressions of f and $|f|$ as the difference and sum, respectively, of non-negative measurable functions:

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^- \text{ on } E.$$

[1]In general, it is important, but not always easy, to determine if the composition of measurable functions is measurable. In Chapter 6, we define what it means for a function $f: [a, b] \to R$ to be absolutely continuous; Lipschitz functions, for example, are absolutely continuous. We prove a theorem of von Neumann which asserts that if $f: [a, b] \to R$ is a strictly increasing, continuous function, then $g \circ f: [a, b] \to R$ is measurable whenever $g: R \to R$ is measurable if and only if its inverse $f^{-1}: [f(a), f(b)] \to R$ is absolutely continuous.

PROBLEMS

1. Let A and B be measurable sets. Is $f\colon A \cup B \to \mathbf{R}$ continuous if and only if its restrictions to A and B are continuous?
2. Show that if $h\colon E \to \mathbf{R}$ is measurable, then for any α, so is $\alpha \cdot h\colon E \to \mathbf{R}$.
3. Provide an example of a function $f\colon [a, b] \to \mathbf{R}$ that is not measurable, while both $|f|$ and f^2 are measurable.
4. Let $\{f_n\colon E \to \mathbf{R}\}$ be a sequence of measurable functions. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges to a real number. Is the set E_0 measurable?
5. Let E be measurable and $f\colon E \to \mathbf{R}$ be continuous except at a countable number of points. Show that f is measurable.
6. If E is measurable and the function $f\colon E \to \mathbf{R}$ has the property that $f^{-1}(c)$ is measurable for each number c, is f necessarily measurable?
7. If the function $f\colon E \to \mathbf{R}$ has the property that $\{x \in E \mid f(x) > c\}$ is a measurable set for each rational number c, is f necessarily measurable?
8. Show that $f\colon E \to \mathbf{R}$ is measurable if and only if the function $\hat{f}\colon \mathbf{R} \to \mathbf{R}$ is measurable, where $\hat{f}(x) = f(x)$ for $x \in E$ and $\hat{f}(x) = 0$ for $x \notin E$.
9. Show that $f\colon E \to \mathbf{R}$ is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable.
10. (Borel measurability) A function $f\colon E \to \mathbf{R}$ is said to be **Borel measurable** provided that its domain E is a Borel set and for each c, the set $\{x \in E \mid f(x) < c\}$ is a Borel set. Verify that Proposition 1 and Theorem 5 remain valid if we replace "(Lebesgue) measurable set" by "'Borel set." Verify the following: (i) every Borel measurable function is Lebesgue measurable; (ii) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set; (iii) if f and g are Borel measurable, so is $f \circ g$; and (iv) if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.
11. Assume that $f\colon \mathbf{R} \to \mathbf{R}$ is measurable and $g\colon \mathbf{R} \to \mathbf{R}$ is continuous. Is the composition $f \circ g$ necessarily measurable?
12. Suppose that $f\colon [a, b] \to R$ is strictly increasing, continuous function that has a Lipschitz inverse. Show that $g \circ f\colon [a, b] \to R$ is measurable whenever $g\colon R \to R$ is measurable. (Suggestion: Show that a Lipschitz function maps measurable sets to measurable sets, by examining images of F_σ sets and sets of measure zero.)

3.2 SEQUENTIAL POINTWISE LIMITS AND SIMPLE APPROXIMATION

For a sequence of functions $\{f_n\colon E \to \overline{\mathbf{R}}\}$ and function $f\colon E \to \overline{\mathbf{R}}$, there are several ways in which it is necessary to consider what it means to state that

"the sequence $\{f_n\}$ converges to f."

In this chapter, we consider the concept of pointwise convergence. In later chapters, we consider many other modes of convergence.

Definition *A sequence of functions $\{f_n\colon E \to \overline{\mathbf{R}}\}$ is said to converge pointwise to the function $f\colon E \to \overline{\mathbf{R}}$ provided that*

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in E.$$

The pointwise limit of continuous functions may not be continuous. The pointwise limit of Riemann integrable functions may not be Riemann integrable. However, for measurable functions, we have the following indispensable theorem.

Theorem 8 *If $\{f_n\colon E \to \overline{\mathbf{R}}\}$ is a sequence of measurable functions that converges pointwise almost everywhere to the function $f\colon E \to \overline{\mathbf{R}}$, then f is measurable.*

Proof In view of Proposition 4, we assume the convergence is on all of E. Fix a number c. It must be shown that $\{x \in E \mid f(x) < c\}$ is measurable. Observe that for a point $x \in E$, since $\lim_{n\to\infty} f_n(x) = f(x)$, $f(x) < c$ if and only if there are indices n and k for which

$$f_j(x) < c - 1/n \text{ for all } j \geq k.$$

But for any n and j, $\{x \in E \mid f_j(x) < c - 1/n\}$ is a measurable set, since f_j is a measurable function. Therefore, for any k, since $\mathcal{M}$ is a σ-algebra, the set

$$\bigcap_{j=k}^{\infty} \{x \in E \mid \; f_j(x) < c - 1/n\}$$

is measurable, as is the set

$$\{x \in E \mid \; f(x) < c\} = \bigcup_{1 \leq k, n < \infty} \left[\bigcap_{j=k}^{\infty} \{x \in E \mid \; f_j(x) < c - 1/n\} \right]. \qquad \square$$

Linear combinations of characteristic functions of measurable sets play a role in Lebesgue integration that is foreshadowed by the role of step-functions in Riemann integration, and so we name these functions.

Definition *A real-valued function $\varphi\colon E \to \mathbf{R}$ is said to be* ***simple*** *provided that it is measurable and takes only a finite number of values.*

Clearly, linear combinations and products of characteristic functions are simple. If $\varphi\colon E \to \mathbf{R}$ is simple and takes the distinct values $c_1, \ldots, c_n$, then

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \; \varphi(x) = c_k\}.$$

This particular expression of φ as a linear combination of characteristic functions of measurable sets is called the **canonical representation of the simple function** φ. It is characterized by the E_k's being disjoint and the c_k's being distinct.

The Simple Approximation Lemma *If $f\colon E \to \mathbf{R}$ is a measurable, bounded function, then for each $\epsilon > 0$, there are simple functions $\varphi_\epsilon\colon E \to \mathbf{R}$ and $\psi_\epsilon\colon E \to \mathbf{R}$ for which*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } E.$$

Proof Let $[c, d)$ be a bounded interval that contains the image of E, $f(E)$, and

$$c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$$

be a partition of the closed interval $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$. Define

$$I_k = [y_{k-1}, y_k) \text{ and } E_k = f^{-1}(I_k) \text{ for } 1 \leq k \leq n.$$

Since each I_k is an interval and the function f is measurable, each set E_k is measurable. Define the simple functions φ_ϵ and ψ_ϵ on E by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

Let $x \in E$. Since $f(E) \subseteq [c, d)$, there is a unique $k, 1 \leq k \leq n$, for which $y_{k-1} \leq f(x) < y_k$ and therefore,

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x).$$

But $y_k - y_{k-1} < \epsilon$, and therefore φ_ϵ and ψ_ϵ have the required approximation properties. □

The proof of the above lemma depends on a seminal idea of Henri Lebesgue. In the development of the Riemann integral, approximation of functions by step-functions is done by partitioning into subintervals the *domain* of a function and defining approximations with respect to such a partition. In contrast, the approximation of a bounded measurable function by simple functions is done by partitioning into subintervals an interval containing *image* of a function, and then considering preimages of these subintervals in order to obtain simple approximations.

Definition *A measurable function $f: E \to \mathbf{R}$ is said to be* **finitely supported** *provided that it vanishes on the complement of a set of finite measure.*

The Simple Approximation Theorem *If the function $f: E \to \overline{\mathbf{R}}$ is measurable, then there is a sequence $\{\varphi_n: E \to \mathbf{R}\}$ of finitely supported, simple functions that converges pointwise on E to f and has the property that*

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n.$$

If $f \geq 0$, then, in addition, $\{\varphi_n\}$ is increasing and each $\varphi_n \geq 0$.

Proof Assume that $f \geq 0$ on E. The general case follows by expressing f as the difference of non-negative measurable functions. For each n, define $E_n = \{x \in E \mid f(x) \leq n\}$. Then E_n is a measurable set and the restriction $f: E_n \to \mathbf{R}$ is a non-negative, bounded, measurable function. By the Simple Approximation Lemma, there are simple functions $\varphi_n: E_n \to \mathbf{R}$ and $\psi_n: E_n \to \mathbf{R}$ for which

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Observe that

$$0 \leq \varphi_n \leq f \text{ and } 0 \leq f - \varphi_n \leq \psi_n - \varphi_n < 1/n \text{ on } E_n. \tag{2}$$

Extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) > n$. The function φ_n is a simple function defined on E and $0 \leq \varphi_n \leq f$ on E. We claim that the sequence $\{\varphi_n\}$ converges to f pointwise on E. Let $x \in E$.

Case 1: Assume that $f(x)$ is finite. Choose an N for which $f(x) < N$. Then

$$0 \leq f(x) - \varphi_n(x) < 1/n \text{ for } n \geq N,$$

and therefore $\lim_{n \to \infty} \varphi_n(x) = f(x)$.

Case 2: Assume that $f(x) = \infty$. Then $\varphi_n(x) = n$ for all n, so that $\lim_{n \to \infty} \varphi_n(x) = f(x)$.

Since each $\varphi_n \geq 0$, by replacing each φ_n with the product of $\max\{\varphi_1, \ldots, \varphi_n\}$ and $\chi_{[-n, n]}$, we obtain an increasing sequence of finitely supported, simple functions that converges pointwise to f. □

PROBLEMS

13. (Dini's Theorem) Let $\{f_n\colon [a, b] \to \mathbf{R}\}$ be an increasing sequence of continuous functions that converges pointwise to the continuous function $f\colon [a, b] \to \mathbf{R}$. Show that the convergence is uniform on $[a, b]$. (Suggestion: Let $\epsilon > 0$. For each n, define $E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}$. Show that $\{E_n\}$ is an open cover of $[a, b]$ and use the Heine-Borel Theorem.)
14. A real-valued measurable function is said to be *semisimple* provided that it takes only a countable number of values. Let $f\colon E \to \mathbf{R}$ be measurable. Show that there is a sequence of semisimple functions that converges to f uniformly on E.
15. Assume that $m(E) < \infty$ and let $f\colon E \to \mathbf{R}$ be measurable. Show that for each $\epsilon > 0$, there is a measurable subset E_0 of E such that f is bounded on E_0 and $m(E \sim E_0) < \epsilon$.
16. Assume that $m(E) < \infty$ and let $f\colon E \to \mathbf{R}$ be measurable. Show that for each $\epsilon > 0$, there is a measurable subset E_0 of E and a sequence $\{\varphi\colon E \to \mathbf{R}\}$ of simple functions for which $\{\varphi_n\} \to f$ uniformly on E_0 and $m(E \sim E_0) < \epsilon$. (Suggestion: See the preceding problem.)
17. Let E be a measurable subset of $[a, b]$, and define $f\colon [a, b] \to \mathbf{R}$ by $f = \chi_E$. For each $\epsilon > 0$, show that there is a measurable subset $E_0 \subseteq [a, b]$ and step-function $h\colon [a, b] \to \mathbf{R}$ for which

$$h = f \text{ on } E_0 \text{ and } m([a, b] \sim E_0) < \epsilon.$$

(Suggestion: Use Theorem 19 of the preceding chapter.)

18. Let A and B be any sets. Show that

$$\begin{aligned} \chi_{A \cap B} &= \chi_A \cdot \chi_B \\ \chi_{A \cup B} &= \chi_A + \chi_B - \chi_A \cdot \chi_B \\ \chi_{A \sim B} &= [\chi_A - \chi_B] \cdot \chi_A. \end{aligned}$$

19. For a sequence $\{f_n\colon E \to \mathbf{R}\}$ of measurable functions, show that the functions $\inf\{f_n\}$ and $\sup\{f_n\}$ are measurable.
20. For $f\colon [a, b] \to \mathbf{R}$ measurable, let $E_+ = \{x \in E \mid f(x) \geq 0\}$ and $E_- = \{x \in E \mid f(x) < 0\}$. By considering the restriction of f to E_+ and E_-, prove the general Simple Approximation Theorem based on the special case of a non-negative measurable function.
21. Let $f\colon [a, b] \to \mathbf{R}$ be increasing. Show that f is measurable by first showing that, for each n, the strictly increasing function $x \mapsto f(x) + x/n$ is measurable, and then taking pointwise limits.

3.3 LITTLEWOOD'S THREE PRINCIPLES, EGOROFF'S THEOREM, AND LUSIN'S THEOREM

Speaking of the theory of functions of a real variable, J. E. Littlewood says[2], "The extent of knowledge required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: Every [measurable] set is nearly a finite union of intervals; every [measurable] function is nearly continuous; every pointwise convergent sequence of [measurable] functions is nearly uniformly convergent. Most of the results of [the theory] are fairly intuitive applications of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle the problem if it were 'quite' true, it is natural to ask if the 'nearly' is near enough, and for a problem that is actually solvable it generally is."

Theorem 19 of the preceding chapter is a precise formulation of Littlewood's first principle: it states that given a measurable set E of finite measure, for each $\epsilon > 0$, there is a finite, disjoint collection of open intervals for which the union $\mathcal{U}$ is "nearly equal to" E in the sense that $m(E \sim \mathcal{U}) + m(\mathcal{U} \sim E) < \epsilon$. A precise realization of the third of Littlewood's principle is the following theorem.

Egoroff's Theorem *Assume that $m(E) < \infty$. If $\{f_n\colon E \to \mathbf{R}\}$ is a sequence of measurable functions that converges pointwise on E to the function $f\colon E \to \mathbf{R}$, then for each $\epsilon > 0$, there is a closed set F for which*

$$F \subseteq E, \;\; m(E \sim F) < \epsilon \text{ and } \{f_n\} \to f \text{ uniformly on } F.$$

To prove Egoroff's Theorem, it is convenient to first establish the following lemma.

Lemma 9 *Under the assumptions of Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$, there is a measurable set A and an index N for which*

$$A \subseteq E, \;\; m(E \sim A) < \delta \text{ and } |f_n - f| < \eta \text{ on } A \text{ for all } n \geq N.$$

Proof For each k, since the function $|f - f_k|$ is measurable, the set $\{x \in E \mid |f(x) - f_k(x)| < \eta\}$ is measurable. Since $\mathcal{M}$ is a σ-algebra, for each n,

$$E_n = \{x \in E \mid \; |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\}$$

is a measurable set. Then $\{E_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets, and $E = \cup_{n=1}^{\infty} E_n$, since $\{f_n\}$ converges pointwise to f on E. It follows from the continuity of measure that

$$m(E) = \lim_{n \to \infty} m(E_n).$$

Since $m(E) < \infty$, we may choose an index N for which $m(E_N) > m(E) - \delta$. Define $A = E_N$ and observe that, by the excision property of measure, $m(E \sim A) = m(E) - m(E_N) < \delta$. □

Proof of Egoroff's Theorem By the preceding lemma, for each n, there is a measurable subset A_n of E and an index $N(n)$ for which

$$m(E \sim A_n) < \epsilon / 2^{n+1} \tag{3}$$

[2] Littlewood [Lit41], page 23.

and

$$|f_k - f| < 1/n \text{ on } A_n \text{ for all } k \geq N(n). \tag{4}$$

Define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

By De Morgan's Identities, the countably monotonicity of measure and (3),

$$m(E \sim A) = m\left(\bigcup_{n=1}^{\infty} [E \sim A_n]\right) \leq \sum_{n=1}^{\infty} m(E \sim A_n) < \sum_{n=1}^{\infty} \epsilon/2^{n+1} = \epsilon/2.$$

We claim that $\{f_n\}$ converges to f uniformly on A. Indeed, let $\epsilon > 0$. Choose an index n_0 for which $1/n_0 < \epsilon$. Then, by (4),

$$|f_k - f| < 1/n_0 \text{ on } A_{n_0} \text{ for } k \geq N(n_0).$$

However, $A \subseteq A_{n_0}$ and $1/n_0 < \epsilon$ and therefore,

$$|f_k - f| < \epsilon \text{ on } A \text{ for } k \geq N(n_0).$$

Therefore, $\{f_n\}$ converges to f uniformly on A and $m(E \sim A) < \epsilon/2$. Finally, by the regularity of measure, there is a closed set F contained in A for which $m(A \sim F) < \epsilon/2$. Consequently, $m(E \sim F) < \epsilon$ and $\{f_n\} \to f$ uniformly on F. □

Lemma 10[3] *If F is a closed subset of* $\mathbf{R}$ *and f: $F \to \mathbf{R}$ is a continuous function, then it has a continuous extension to f: $\mathbf{R} \to \mathbf{R}$.*

Proof The complement of F, $\mathbf{R} \sim F$, is open, so $\mathbf{R} \sim F = \bigcup_{n=1}^{\infty} I_n$, a countable, disjoint union of open intervals. If $I_n = (a_n, b_n)$ is bounded, define f on $[a_n, b_n]$ to be any continuous function that agrees f at the end-points. If I_n is unbounded, define f on I_n to be the constant function that agrees with f at the finite end-point of I_n. We leave it as an exercise to verify the continuity of f: $\mathbf{R} \to \mathbf{R}$. □

The following theorem is a confirmation of Littlewood's second principle.

Lusin's Theorem *If f: $E \to \mathbf{R}$ is a measurable function, then for each $\epsilon > 0$, there is a continuous function g: $\mathbf{R} \to \mathbf{R}$ and a closed subset F of $\mathbf{R}$ for which*

$$F \subseteq E,\ m(E \sim F) < \epsilon \text{ and } f = g \text{ on } F.$$

Proof[4] Choose an enumeration $\{I_n\}_{n=1}^{\infty}$ of the countable collection of all open intervals that have rational end-points. Let $\epsilon > 0$. By the regularity of measure, for each n, there are closed subsets of $\mathbf{R}$, F_n, and H_n, for which $F_n \subseteq f^{-1}(I_n)$, $H_n \subseteq E \sim f^{-1}(I_n)$,

$$m(f^{-1}(I_n)) \sim F_n) < 1/\epsilon^{n+1} \text{ and } m((E \sim f^{-1}(I_n)) \sim H_n) < 1/\epsilon^{n+1}.$$

[3]This lemma is a very special case of the Tietze Extension Theorem, which is proven in Chapter 16.

[4]This elegant proof is due to Peter Loeb and Erik Talvila. We remark in passing that this theorem was first stated and proved by Vitali.

Define $F \equiv \cap_{n=1}^{\infty}(F_n \cup H_n)$. Then F, being the intersection of closed sets, is closed. By De Morgan's Identity and the countable monotonicity of measure,

$$m(E \sim F) = m\left(\bigcup_{n=1}^{\infty} E \sim (F_n \cup H_n)\right) < 2 \cdot \sum_{n=1}^{\infty} \epsilon/2^{n+1} = \epsilon.$$

We may assume that $F \neq \emptyset$, for otherwise, $m(E) < \epsilon$ and we simply take F to be a point in E and take g to be the constant function on $\mathbf{R}$ that agrees with f at that point.

We claim that $f\colon F \to R$ is continuous. To do so, for each $x \in F$ and open interval I containing $f(x)$, we need to show that there is an open subset $\mathcal{O}$ of $\mathbf{R}$ for which

$$f(\mathcal{O} \cap F) \subseteq I.$$

For such an x and I, since the rational numbers are a dense subset of $\mathbf{R}$, there is some k for which $f(x) \in I_k \subseteq I$. We claim that the above inclusion holds for the open set $\mathcal{O} = \mathbf{R} \sim H_k$. Indeed, observe that $f(F_k) \subseteq I_k \subseteq I$, and H_k and F_k are disjoint, so that $\mathcal{O} \cap F \subseteq (\mathbf{R} \sim H_k) \cap (F_k \cup H_k) = F_k$. According to the preceding lemma, f has a continuous extension to all of $\mathbf{R}$. □

Corollary 11 *If $f\colon E \to \mathbf{R}$ is a measurable function, then there is a sequence of continuous functions $\{f_n\colon \mathbf{R} \to \mathbf{R}\}$ that converges pointwise almost everywhere on E to f.*

Proof According to Lusin's Theorem, for each n, there is a continuous function $f_n\colon \mathbf{R} \to \mathbf{R}$ for which

$$m\{x \in E \mid f_n(x) \neq f(x)\} < 1/2^n.$$

We deduce from the Borel-Cantelli Lemma that for almost all $x \in E$ there is an n such that $f_k(x) = f(x)$ for all $k \geq n$. □

PROBLEMS

22. Verify the continuity of the function $f\colon \mathbf{R} \to \mathbf{R}$ defined in the proof of Lemma 10.
23. For the function $f\colon E \to \mathbf{R}$ and the set F in the statement of Lusin's Theorem, show that the restriction of f to F is continuous. Must there be any points in E at which $f\colon E \to \mathbf{R}$ is continuous?
24. Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.
25. Show that Egoroff's Theorem continues to hold if f and each f_n is an extended real-valued function that is finite almost everywhere, and the convergence is pointwise almost everywhere on E.
26. Let $\{f_n\colon E \to \mathbf{R}\}$ be a sequence of measurable functions that converges pointwise on E to $f\colon E \to \mathbf{R}$. Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each E_k is measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.
27. Show that in Corollary 11, if f is bounded, then the sequence $\{f_n\}$ may be chosen to be uniformly bounded.