Gli appunti verranno condivisi sulla pagina del corso

https://elearning.uniroma1.it/user/index.php?id=4304

First part: Lebesgue integration, Lebesgue measure.

<u>References:</u>

Royden : Real Analysis.

Fusco-Marcellini-Sbordone: Analisi Matematica II - Zanichelli

Giusti - Analisi Matematica II - Bollati Boringhieri.

Lebesgue measure in $\mathbb{R}$ $(\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^N)$

We would like to associate to <u>every</u> subset of $\mathbb{R}$ ($E \subseteq \mathbb{R}$) a measure $m(E) \in [0, +\infty]$ satisfying the following "intutive" properties:

1) if E is an interval, then $m(E)$ is the ordinary lenght of the intervalle (that is, $m((a,b)) = b-a$)

2) if $\{E_n\}$ is a sequence of pairwise disjoint sets ($E_n \cap E_m = \emptyset$ if $n \neq m$)

then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$. In particular, $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \emptyset$

3) m is translation-invariant.

$$m(E) = m(E+y) \quad \forall E \subseteq \mathbb{R} \quad \forall y \in \mathbb{R}.$$

$$E+y = \{x+y : x \in E\}$$

Unfortunately, it can be proved that a measure satisfying 1), 2), 3) cannot be defined for all sets $E \subseteq \mathbb{R}$

$\Rightarrow$ Our aim is to define a measure for a very large class $\mathcal{M}$ of subsets of $\mathbb{R}$.
This class $\mathcal{M}$ must be closed with respect to some operations, for instance:

- $\mathcal{M}$ must be closed with respect to union, intersection of countably many sets.
- $\mathcal{M}$ must be closed with respect to differences of sets.
  $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$, $A \setminus B = \{a \in A \text{ s.t. } a \notin B\}$

Lenght of an interval.

If $I = (a,b) \Rightarrow \quad \ell(I) = b-a$.

External measure of a set

Let $E \subseteq \mathbb{R}$. We define

$\{I_n\}$ is a covering of E

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \overbrace{E \subset \bigcup_{n=1}^{\infty} I_n}, \ I_n \text{ are open intervals} \right\}$$

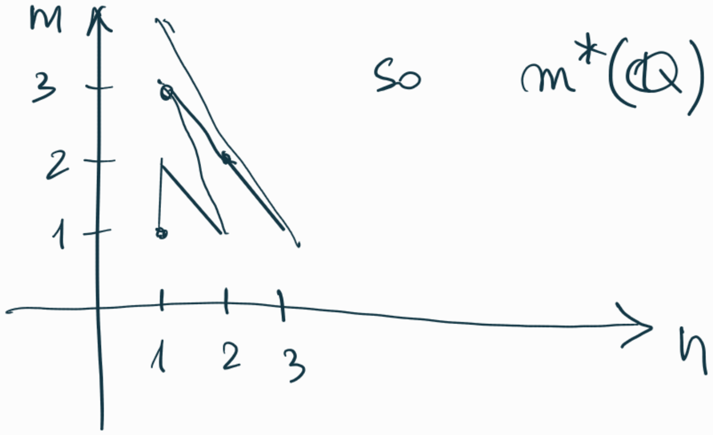
OSS $\quad m^*(\phi) = 0$

$m^*(\{x\}) = 0$

$m^*(E) = 0 \quad$ if E is a finite set or a countable set.

Let E be a countable set. $=$ a set which can be "counted", i.e., put in a bijection with the set $\mathbb{N} = \{0, 1, 2, \dots\}$.

For instance $\quad \mathbb{Q} = \{$rational numbers$\}$ is countable



so $\quad m^*(\mathbb{Q}) = 0$

Proof

If E is countable $\quad E = \{a_n\}$ then we take

$$I_1 = \left(a_1 - \frac{\varepsilon}{2}, a_1 + \frac{\varepsilon}{2}\right), \quad I_2 = \left(a_2 - \frac{\varepsilon}{4}, a_2 + \frac{\varepsilon}{4}\right)$$

$$I_3 = \left(a_3 - \frac{\varepsilon}{8}, a_3 + \frac{\varepsilon}{8}\right) \dots \quad I_n = \left(a_n - \frac{\varepsilon}{2^n}, a_n + \frac{\varepsilon}{2^n}\right)$$

$\ell(I_1) = \varepsilon$, $\ell(I_2) = \frac{\varepsilon}{2}$ $\ell(I_3) = \frac{\varepsilon}{4}$

$\ell(I_n) = \frac{\varepsilon}{2^{n-1}}$

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n-1}} = \varepsilon \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}}_{= 2} = 2\varepsilon$$

But ε is arbitrary, so the inf is 0.

<u>Theorem</u> m^* is countably sub-additive

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

Proof will be on the notes.

<u>Remark</u> This m^* is not satisfactory because one could find two <u>disjoint</u> sets of $\mathbb{R}$ such that

$$m^*(E \cup F) < m^*(E) + m^*(F)$$

We define <u>measurable sets</u> according to Lebesgue.

<u>DEF</u>. We say that $E \subseteq \mathbb{R}$ is measurable if

$$\boxed{\forall F \subseteq \mathbb{R} \qquad m^*(F) = m^*(F \cap E) + m^*(F \cap E^c)}$$

$F \setminus E$

$E^c = \{x \in \mathbb{R} \text{ s.t. } x \notin E\}$

Note that if E is measurable then E^c is also measurable.

Remarks about the external measure m^*

1) if $E \subseteq F$ $\quad m^*(E) \leq m^*(F)$

(monotonicity)

2) $m^*(I) = \ell(I)$ $\quad \forall I$ interval

(it is a Theorem, not obvious)

<u>Remark</u>. if $m^*(E) = 0$, then E is measurable, i.e

$$\forall F \subseteq \mathbb{R} \qquad m^*(F) = m^*(F \cap E) + m^*(F \cap E^c)$$

$\leq$ always true, so we only need to prove $\geq$

by subadditivity

$$m^*(F \cap E) + m^*(F \cap E^c) \leq 0 + m^*(F) = m^*(F)$$

monotonicity

$m^*(F \cap E) \leq m^*(E) = 0 \qquad m^*(F \cap E^c) \leq m^*(F)$

<u>PROP</u>. If E_1, E_2 are measurable, then $E_1 \cup E_2$ is measurable, $E_1 \cap E_2$ is measurable, E_1^c is measurable

$E_2 \setminus E_1$ is measurable

So the set $\mathcal{M}$ of measurable sets is closed with respect to finite unions, finite intersections, differences.

Briefly, $\mathcal{M}$ is <u>an algebra of sets</u>.

Actually, there is more:

Theorem The union of a sequence of measurable sets is still measurable.

$$E_n \subset \mathbb{R},\ \text{measurable}\ (n \in \mathbb{N}) \implies \bigcup_{n=1}^{\infty} E_n \text{ is measurable.}$$

Proof is on the notes.

The same is true for $\bigcap_{n=1}^{\infty} E_n$.

So the class $\mathcal{M}$ of measurable sets is closed with respect to countable unions, countable intersections, difference. Shortly, $\boxed{\mathcal{M} \text{ is a } \sigma\text{-algebra}}$ (σ is a reference to sums)

However, the class $\mathcal{M}$ of Lebesgue-measurable sets does not contain all possible subsets of $\mathbb{R}$.

We define the $\boxed{\text{Lebesgue measure:}}$

for all measurable $E \subseteq \mathbb{R}$, we define

$$m(E) = m^*(E)$$

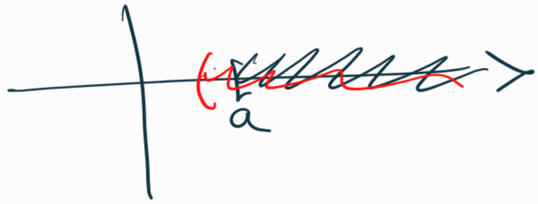
Now, what sets are measurable?

Prop. The half-line $(a, +\infty)$ is measurable

(proof is on the notes).

Consequences: $(a,+\infty)^c = (-\infty, a]$ is measurable $\forall a \in \mathbb{R}$

$$[a,+\infty) = \bigcap_{n=1}^{\infty} \left(a-\frac{1}{n}, +\infty\right)$$ is also measurable



$(-\infty, a)$ measurable

$$(a,b) = (a,+\infty) \cap (-\infty, b)$$ is also measurable

All intervals are measurable.

All open sets are measurable, since they are countable unions of open intervals
All closed sets are measurable.

So, the class of measurable sets contains all open, closed sets ...

Finally, the Lebesgue-measure is countably additive.

Theorem
If E_n are measurable and pairwise disjoint, then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

PROPOSITION. (Measurable sets are "near" to nice sets).

Let $E \subseteq \mathbb{R}$. Then the following statements are equivalent:

i) E is measurable,
ii) $\forall \varepsilon > 0$ $\exists A \overset{\text{open}}{\supseteq} E$ such that $m^*(A \setminus E) < \varepsilon$
iii) $\forall \varepsilon > 0$ $\exists C$ closed set, $C \subseteq E$ such that $m^*(E \setminus C) < \varepsilon$

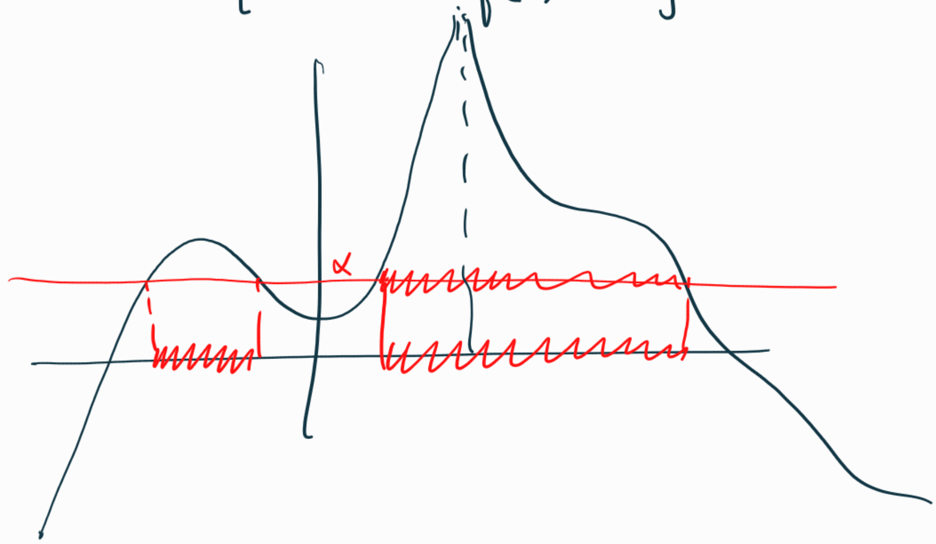
Measurable functions

DEF Let $E \subseteq \mathbb{R}$ be measurable, and let

$$f : E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$ be a function on E.

We will say that f is measurable if one of the following six equivalent conditions is true.

i) $\forall \alpha \in \mathbb{R}$, the set $\{x \in E : f(x) > \alpha\}$ is measurable



ii) $\forall \alpha \in \mathbb{R}$, the set $\{x \in E : f(x) \geq \alpha\}$ is measurable

iii) " " $\{ \qquad < \alpha\}$ "

iv) " " " $\{ \qquad \leq \alpha\}$ "

v) $\forall A$ open set $\subseteq \mathbb{R}$, the set $\{x \in E : f(x) \in A\} = f^{-1}(A)$ is measurable

vi) $\forall C$ closed set $\subseteq \mathbb{R}$, the set $\{ \quad : \quad \in C\} = f^{-1}(C)$ "

Note. any of the conditions i) - vi) implies the condition

$\forall \alpha \in \mathbb{R}$, the set $\underbrace{\{x \in E : f(x) = \alpha\} \text{ is measurable}}$

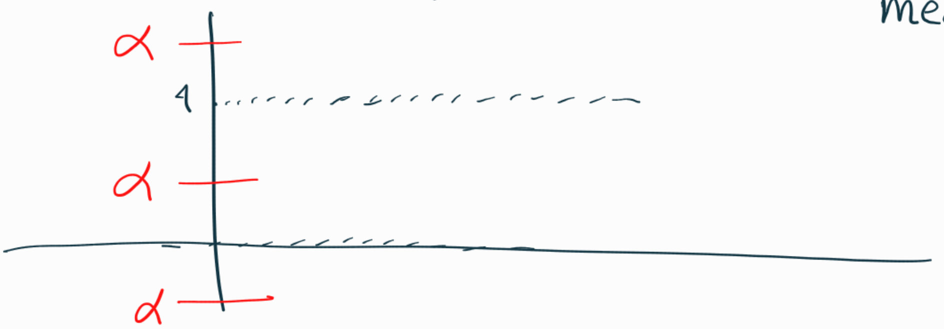
but this is not enough for measurability

Examples of measurable functions:

- Every function $f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous is measurable.
  Indeed
  $\forall \alpha \in \mathbb{R}$, $\{x \in \mathbb{R} : f(x) > \alpha\}$ is open, thus measurable.

- $$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

measurable



- We can generalize this. Take any measurable set $E \subseteq \mathbb{R}$ and take

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in \mathbb{R} \setminus E \end{cases}$$

characteristic function of E.

This is a measurable functions.

Prop. Let $E \subseteq \mathbb{R}$ be measurable. Let $f, g : E \to \mathbb{R}$, $c \in \mathbb{R}$
Then the following functions are measurable

$$f(x) + g(x), \quad c f(x), \quad f(x) g(x), \quad \frac{f(x)}{g(x)}$$

on the measurable set
$\{x : g(x) \neq 0\}$

If $f_n : E \xrightarrow{n \in \mathbb{N}} \mathbb{R}$ are measurable, then

$\sup_n f_n(x)$, $\inf_n f_n(x)$, $\lim_{n \to +\infty} f_n(x)$ (supposing the limit exists)

are also measurable.

<u>DEF</u> We say that some statement is true a.e. (almost everywhere) (q.o. = quasi ovunque) if the points where the statement is false form a set of zero measure

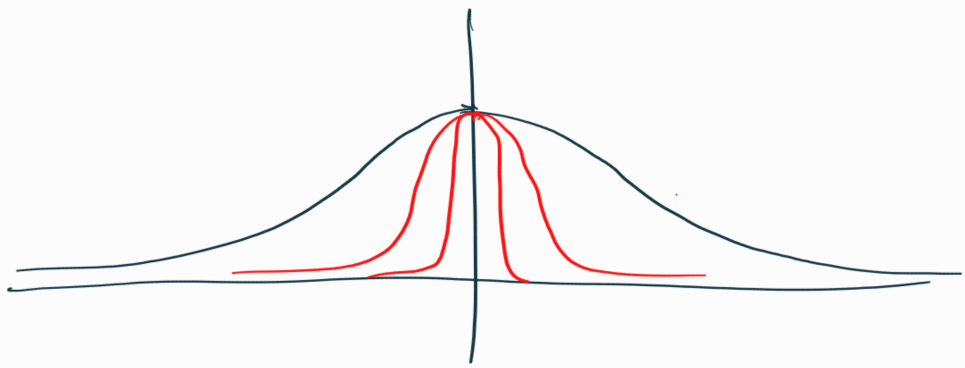
<u>Example</u> 1. The Dirichlet function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

then $\chi_{\mathbb{Q}}(x) = 0$ a.e.

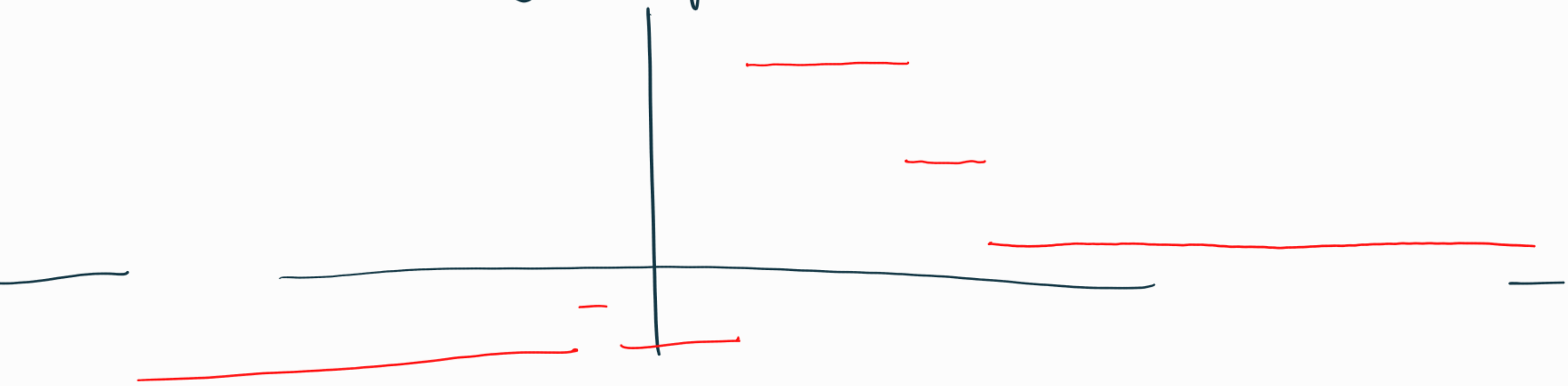
2. The sequence $f_n(x) = \dfrac{1}{1+n^2x^2}$ for $n \to +\infty$

converges to 0 a.e. (i.e. $\forall x \neq 0$)



PROP If f is measurable, and $f(x)=g(x)$ a.e., then g is measurable

Def. A function φ is called simple if it is measurable and assumes only a finite number of values.



Every simple function can be written as linear combination of characteristic functions

$$\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$$

where E_i are measurable sets. pairwise disjoint.

$$E_i = \{x \in \mathbb{R} : \varphi(x) = \alpha_i\}$$

Lebesgue integral.

1) For simple functions.

Let $\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)$ $\Bigg\{$ E_i measurable disjoint. $m(E_i) < \infty$

We define

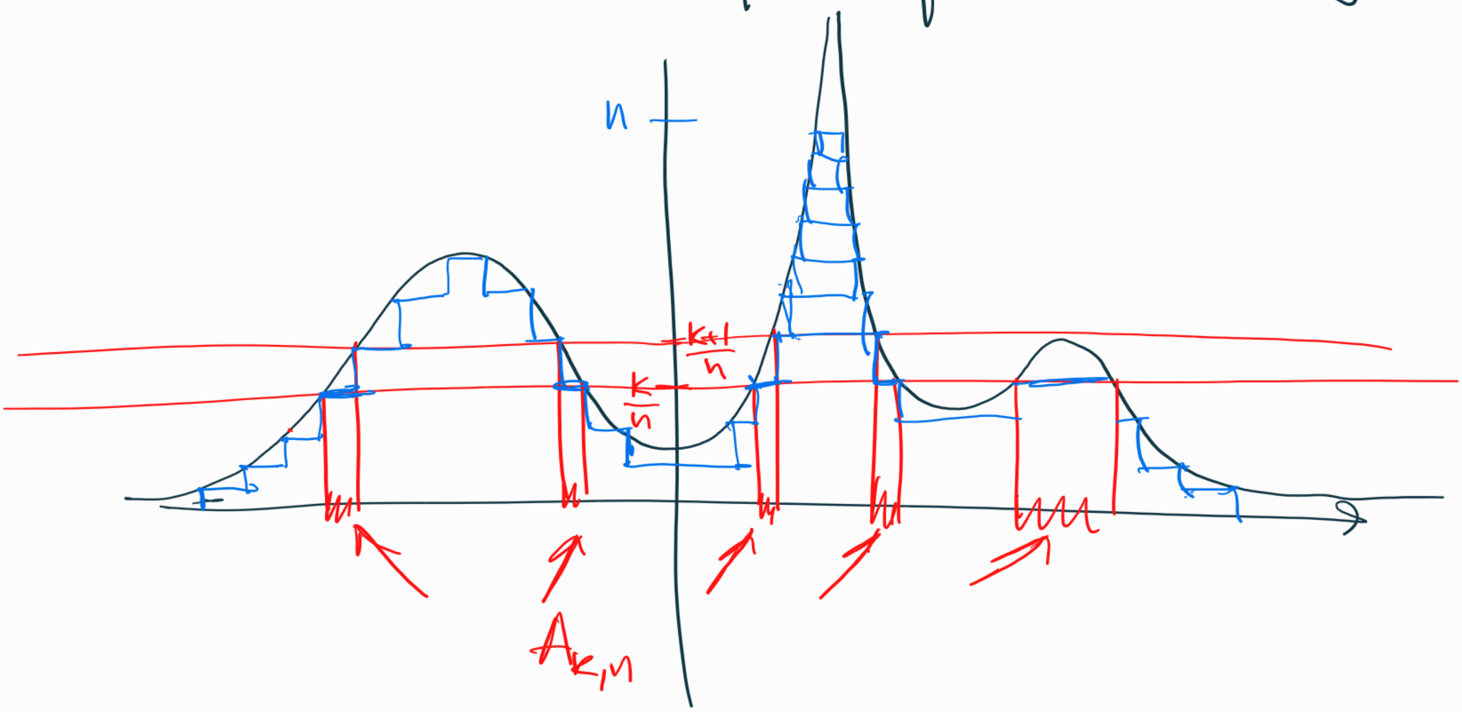
$$\int_{\mathbb{R}} \varphi(x)\,dx = \sum_{i=1}^{n} \alpha_i\, m(E_i)$$

2) For nonnegative functions:

If $f: \mathbb{R} \to [0,+\infty]$ f measurable

We define

$$\int_{\mathbb{R}} f(x)dx = \sup \left\{ \int_{\mathbb{R}} \varphi(x)dx \,,\ \varphi(x) \text{ simple function s.t.: } 0 \le \varphi(x) \le f(x) \quad \forall x \in E \right\}$$



How do you make this approximation practically?

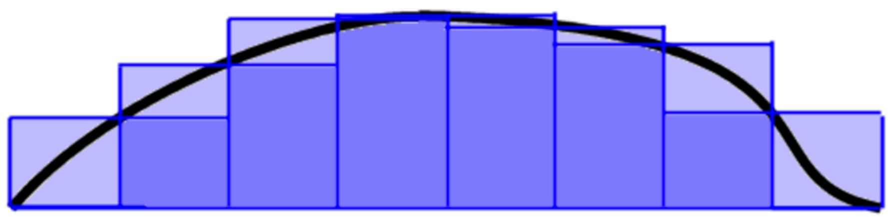
We divide the range of f in smaller and smaller intervals.

We fix $n \in \mathbb{N}$ and we consider the sets

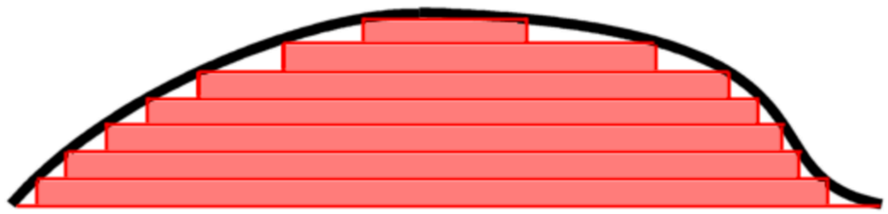
$$A_{n,k} = \left\{ x \in \mathbb{R} : \ \frac{k}{n} \le f(x) < \frac{k+1}{n} \right\}, \quad k = 0, 1, \dots, n^2$$

And we take $\varphi_n(x) = \sum_{k=1}^{n^2} \frac{k}{n} \chi_{A_{n,k}}(x)$

$$\int \varphi_n(x)dx = \sum_{k=1}^{n^2} \frac{k}{n} m(A_{n,k})$$



Riemann



Lebesgue

$$\int_E f(x)\,dx = \sup\Big\{ \int \varphi(x)\,dx \ : \ \varphi(x) \text{ simple function}$$

measurable

$$\varphi(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x) \text{ s.t. } E_i \subseteq E$$

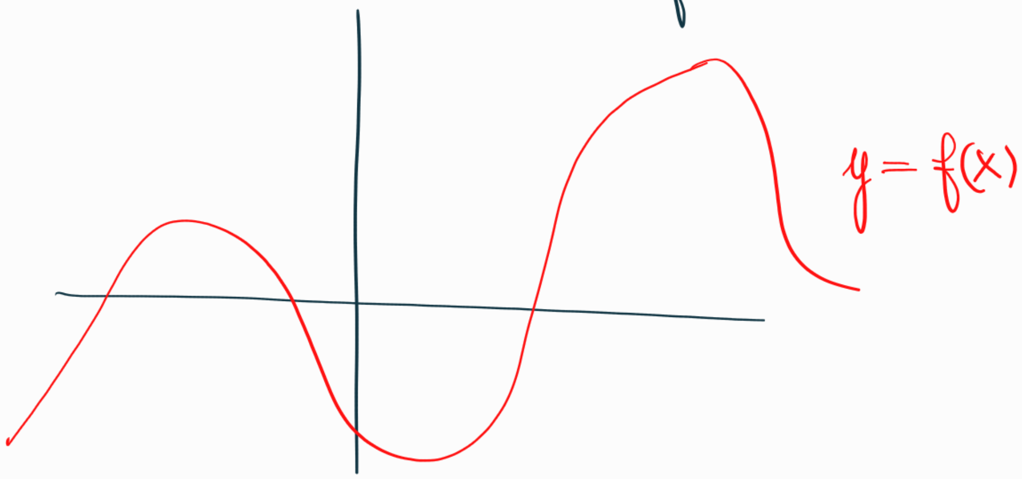
$$0 \le \varphi(x) \le f(x) \ \forall x \in E \Big\}$$

That is

$$\int_E f(x)\,dx = \int_{\mathbb{R}} f(x)\cdot \chi_E(x)\,dx$$

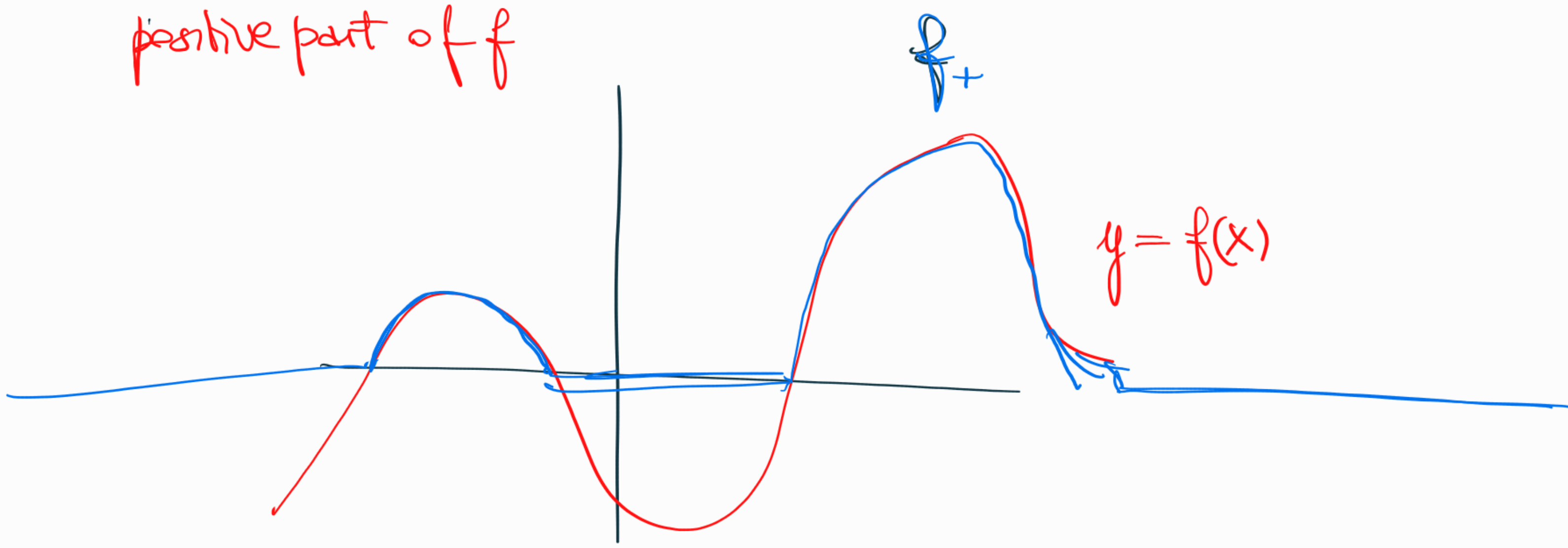
3) Integral of functions of any sign.

$f : E \to \overline{\mathbb{R}}$ — E measurable, f measurable.

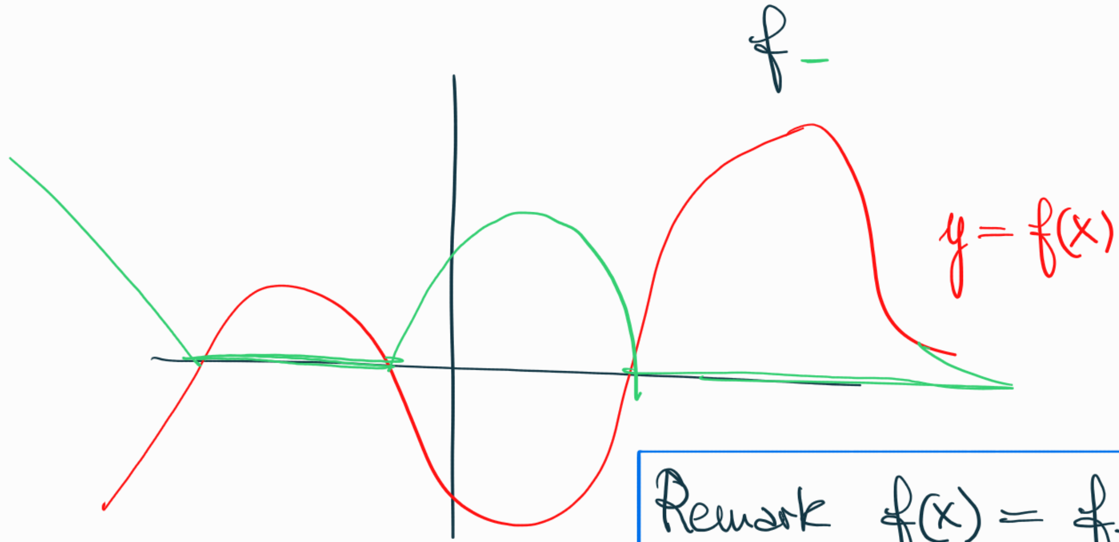


We define $f_+(x) = \max\{0, f(x)\} = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ 0 & \quad\ \ < 0 \end{cases}$

positive part of f



$$f_-(x) = \max\{0, -f(x)\} = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$



Remark $f(x) = f_+(x) - f_-(x)$

$$|f(x)| = f_+(x) + f_-(x)$$

So we define

$$\int_E f(x)\,dx = \int_E f_+(x)\,dx - \int_E f_-(x)\,dx$$

if at least one of these integral is finite.

We say that the function f is <u>summable on</u> E if both the integral of f_+ and the integral of f_- are finite, that is, if $\int_E |f|\,dx$ is finite.