(a) Dato che $[S_z, H]=0$ possiamo considerare una base $\varphi(x)|S_z\rangle$. Per $S_z=\hbar/2$ la particella è soggetta al potenziale

$$V_+(x) = \frac{\hbar\omega}{2}\rho^2 = \frac{1}{2}m\omega^2x^2$$

Per $S_z=-\hbar/2$ è soggetta al potenziale

$$V_- = \frac{\hbar\omega}{2}(\rho^6-2\rho^4-4\rho^2)$$

In entrambi i casi $\lim_{x\to\pm\infty} V_\pm(x) = +\infty$.
Quindi tutti i moti classici sono limitati.
Lo spettro è discreto, non degenere (salvo degenerazioni accidentali tra stati con autovalore S_z diverso) e le autofunzioni sono L_2 (stati legati)

(b) Se $|\varphi_-\rangle$ è autofunzioni di H deve valere

$$\left(\frac{p^2}{2m}+V_-\right)\varphi = E\varphi \quad \text{per un qualche } E.$$

$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} = -\frac{\hbar\omega}{2}\frac{d^2}{d\rho^2} \quad \text{per cui}$$

$$-\frac{1}{2}\frac{d^2\varphi}{d\rho^2} + \frac{V_-}{\hbar\omega}\varphi = \frac{E}{\hbar\omega}\varphi$$

$$\varphi = N\rho\exp\left(-\frac{\rho^4}{4}-\frac{b}{2}\rho^2\right)$$

$$\frac{d\varphi}{d\rho} = N\left[1+\rho(-\rho^3-b\rho)\right]\exp\left(-\frac{\rho^4}{4}-\frac{b}{2}\rho^2\right)$$

$$= N(-\rho^4-b\rho^2+1)\exp\left(-\frac{\rho^4}{4}-\frac{b}{2}\rho^2\right)$$

$$\varphi'' = N\left(-4\rho^3 - 2b\rho\right)\exp\left(-\frac{\rho^4}{4} - \frac{b}{2}\rho^2\right)$$

$$+N\left(-\rho^4 - b\rho^2 + 1\right)\left(-\rho^3 - b\rho\right)\exp\left(-\frac{\rho^4}{2} - \frac{b}{2}\rho^2\right)$$

$$= \cancel{N}\left(-4\rho^2 - 2b\right)\varphi + \left(\rho^4 + b\rho^2 - 1\right)\left(\rho^2 + b\right)\varphi$$

$$= \left(-4\rho^2 - 2b\right) + \left(\rho^6 + b\rho^4 + b\rho^4 + b^2\rho^2 - \rho^2 - b\right)\varphi$$

$$= \left[\rho^6 + 2b\rho^4 + (b^2 - 5)\rho^2 - 3b\right]\varphi$$

Quindi

$$-\frac{1}{2}\varphi'' + V\varphi =$$

$$\left(-\frac{1}{2}\rho^6 - b\rho^4 - \frac{1}{2}(b^2 - 5)\rho^2 + \frac{3b}{2}\right)\varphi$$

$$+\left(\frac{1}{2}\rho^6 - \rho^4 - 2\rho^2\right)\varphi$$

$$= \left(-(b+1)\rho^4 - \frac{1}{2}(b^2 - 1)\rho^2 + \frac{3b}{2}\right)\varphi$$

Perché φ sia autofunzione deve valere

$$\begin{cases} b + 1 = 0 \\ b^2 - 1 = 0 \end{cases} \Longrightarrow b = -1$$

Per questo valore $-\frac{1}{2}\varphi'' + V\varphi = -\frac{3}{2}\varphi \Longrightarrow E = -\frac{3}{2}\hbar\omega$

~~Quindi, in unità ordinarie~~ $\cancel{E = -\frac{3}{2}\hbar\omega}$

c) Riscriviamo $\rho^2 e^{-\rho^2/2}$ in termini di autofunzioni dell'oscillatore armonico

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\rho^2/2}$$

$$\psi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{2\sqrt{2}} (4\rho^2 - 2)\, e^{-\rho^2/2}$$

Quindi

$$\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \psi_0 = e^{-\rho^2/2}$$

$$\left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \psi_2 = \sqrt{2}\rho^2 e^{-\rho^2/2} - \frac{1}{\sqrt{2}} e^{-\rho^2/2}$$

$$\rho^2 e^{-\rho^2/2} = \frac{1}{\sqrt{2}} \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \psi_2 + \frac{1}{2} \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \psi_0$$

$$= \frac{1}{\sqrt{2}} \left(\frac{\pi\hbar}{m\omega}\right)^{1/4} \left(\psi_2 + \frac{\sqrt{2}}{2}\psi_0\right)$$

Se definiamo $B = N_+ \frac{1}{\sqrt{2}} \left(\frac{\pi\hbar}{m\omega}\right)^{1/4}$ abbiamo

$$\langle \rho | \psi \rangle = B\left(\psi_2 + \frac{\sqrt{2}}{2}\psi_0\right)|+\rangle + \frac{1}{\sqrt{2}} \varphi(\rho)|-\rangle$$

Per normalizzazione $|B|^2\left(1 + \frac{1}{2}\right) + \frac{1}{2} = 1$

$$\frac{3}{2}|B|^2 = \frac{1}{2} \qquad |B| = \frac{1}{3}$$

$$\langle \rho | \psi \rangle = \frac{1}{\sqrt{3}}\left(\psi_2 + \frac{\sqrt{2}}{2}\psi_0\right)|+\rangle + \frac{1}{\sqrt{2}} \varphi(\rho)|-\rangle$$

Quindi

$\mathrm{Prob}\left(E = -\frac{3}{2}\hbar\omega\right) = \frac{1}{2}$ (stato φ)

$\mathrm{Prob}\left(E = \frac{1}{2}\hbar\omega\right) = \frac{1}{6}$ (stato ψ_0)

$\mathrm{Pro}\left(E - \frac{5}{2}\hbar\omega\right) = \frac{1}{3}$ (stato ψ_2)

d)

$$S_x = \frac{1}{2}(S_+ + S_-) \qquad |\psi\rangle = f(\rho)|+\rangle + \frac{1}{\sqrt{2}}\varphi(\rho)|-\rangle$$

$$\frac{1}{\hbar}S_x|\psi\rangle = \frac{1}{2}f(\rho)|-\rangle + \frac{1}{2\sqrt{2}}\varphi(\rho)|+\rangle \qquad \begin{bmatrix} S_+|-\rangle = |+\rangle \\ S_-|+\rangle = |-\rangle \end{bmatrix}$$

Quindi

$$\langle\psi|S_x|\psi\rangle = \hbar\int dx\, f(\rho)^* \left(\frac{1}{2\sqrt{2}}\varphi(\rho)\right)$$
$$+\hbar\int dx \left(\frac{1}{\sqrt{2}}\varphi(\rho)\right)\frac{1}{2}f(\rho).$$

Ora

$$\int dx\,\varphi(\rho) f(\rho) = \int dx\, \underbrace{\varphi(\rho)}_{\text{dispari per } \rho\to-\rho} N_+ \rho^2 \underbrace{e^{-\rho^2/2}}_{\text{pari per } \rho\to-\rho} = 0$$

Quindi $\langle\psi|S_x|\psi\rangle = 0$

Esercizio 2

(a) Riscriviamo la funzione $f(r,\theta,\varphi)$ in termini di funzioni radiali ed angolari normalizzate

Definiamo $R_1(r) = a e^{-r/r_0}$ $\quad R_2(r) = b\, r\, e^{-r/r_0}$

con a, b costanti da determinare in modo che le funzioni siano normalizzate

$$1 = \int_0^\infty dr\, r^2 R_1^2 = a^2 \int_0^\infty dr\, r^2 e^{-2r/r_0} = \qquad r = \frac{r_0}{2} x$$

$$= a^2 \left(\frac{r_0}{2}\right)^3 \int_0^\infty dx\, x^2 e^{-x} = a^2 \frac{r_0^3}{8} \cdot 2! = a^2 \frac{r_0^3}{4} \Rightarrow a = \frac{2}{r_0^{3/2}}$$

$$1 = \int_0^\infty dr\, r^2 R_2^2 = b^2 \int dr\, r^4 e^{-2r/r_0} =$$

$$= b^2 \left(\frac{r_0}{2}\right)^5 \int_0^\infty dx\, x^4 e^{-x} = b_0^2 \frac{r_0^5}{32} 4! = \frac{3}{4} b^2 r_0^5 \Rightarrow b = \frac{2}{\sqrt{3}} \frac{1}{r_0^{5/2}}$$

Quindi

$$f(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \frac{1}{r_0^{3/2}} e^{-r/r_0} + \frac{A}{r_0^{3/2}} \frac{r}{r_0} e^{-r/r_0} \frac{1}{\sqrt{4\pi}} \sin\theta\, e^{i\varphi}$$

$$= \frac{1}{2} R_1(r) Y_0^0 + A \frac{\sqrt{3}}{2} R_2(r) \frac{1}{\sqrt{4\pi}} \left(-\sqrt{\frac{8\pi}{3}} Y_1^1\right)$$

$$= \frac{1}{2} R_1(r) Y_0^0 - A \frac{\sqrt{2}}{2} R_2(r) Y_1^1$$

Sotto scambio/parità $\quad (\theta \to \pi - \theta, \ \varphi \to \varphi + \pi)$

$$f(r,\theta,\varphi) \to f(r, \pi-\theta, \varphi+\pi)$$

$$= \frac{1}{2} R_1(r) Y_0^0 + A \frac{\sqrt{2}}{2} R_2(r) Y_1^1$$

Per il principio di Pauli, la funzione d'onda [totale, ossia spin incluso] deve essere dispari sotto scambio

Quindi

$$f(r,\pi-\theta,\varphi+\pi)|-\rangle|+\rangle + g(r,\pi-\theta,\varphi+\pi)|+\rangle|-\rangle =$$

$$-\left[f(r,\theta,\varphi)|+\rangle|-\rangle + g(r,\theta,\varphi)|-\rangle|+\rangle\right]$$

Questo implica

$$\begin{cases} f(r,\pi-\theta,\varphi+\pi) = -g(r,\theta,\varphi) \\ g(r,\pi-\theta,\varphi+\pi) = -f(r,\theta,\varphi) \end{cases}$$

condizioni equivalenti.

Quindi

$$g(r,\theta,\varphi) = -f(r,\pi-\theta,\varphi+\pi) = \left[\frac{1}{2}R_1(r)\,Y_0^0 + A\frac{\sqrt{2}}{2}R_2(r)\,Y_1^1\right](-1)$$

Segue

$$\psi = \left(\frac{1}{2}R_1 Y_0^0 - A\frac{\sqrt{2}}{2}R_2 Y_1^1\right)|+\rangle|-\rangle$$

$$- \left(\frac{1}{2}R_1 Y_0^0 + A\frac{\sqrt{2}}{2}R_2 Y_1^1\right)|-\rangle|+\rangle$$

$$= -A\frac{\sqrt{2}}{2}R_2 Y_1^1\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$$

$$+\frac{1}{2}R_1 Y_0^0\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

Passiamo ora alla base $|S\,S_z\rangle$

$$\overset{S\,S_z}{|0\,0\rangle} = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$$

Segue

$$\psi = -A\,R_2(r)\,Y_1^1\,|1\,0\rangle + \frac{1}{\sqrt{2}}R_1(r)\,Y_0^0\,|0\,0\rangle$$

Imponendo $\langle\psi|\psi\rangle = 1$ abbiamo

$$\langle\psi|\psi\rangle = |A|^2 + \frac{1}{2} = 1 \Rightarrow |A| = \frac{1}{\sqrt{2}} \qquad A = \frac{1}{\sqrt{2}}$$

In notazione di Dirac

$$|\psi\rangle = -\frac{1}{\sqrt{2}} R_2(r) |\overset{L}{1}\ \overset{L_z}{1}\rangle |\overset{S}{1}\ \overset{S_z}{0}\rangle + \frac{1}{\sqrt{2}} R_1(r) |\overset{L}{0}\ \overset{L_z}{0}\rangle |\overset{S}{0}\ \overset{S_z}{0}\rangle$$

(b) Per calcolare la probabilità cambiamo base

$$|L\ L_z\rangle |S\ S_z\rangle \longrightarrow |L\ S\ J\ J_z\rangle_J$$

Abbiamo

$$|\psi\rangle = -\frac{1}{\sqrt{2}} R_2 \left[\frac{1}{\sqrt{2}} |\overset{L}{1}\ \overset{S}{1}\ \overset{J}{2}\ \overset{J_z}{1}\rangle_J + \frac{1}{\sqrt{2}} |1\ 1\ 1\ 1\rangle_J \right]$$

$$+ \frac{1}{\sqrt{2}} R_1 |0\ 0\ 0\ 0\rangle_J$$

Quindi

$$\text{Prob}(J^2 = 6\hbar^2) = \frac{1}{4}$$

$$\text{Prob}(J^2 = 2\hbar^2) = \frac{1}{4}$$

$$\text{Prob}(J^2 = 0) = \frac{1}{2}$$

c) Hamiltoniana coulombiana con $\alpha = \frac{\hbar^2}{\mu r_0}$

$$r_B = \frac{\hbar^2}{\alpha\mu} = r_0$$

$$E_0 = \frac{\alpha}{2 r_B} = \frac{\hbar^2}{2\mu r_0^2} \qquad E_n = -\frac{E_0}{n^2}$$

Per $\beta=0$, gli autostati relativi ai livelli più bassi (8)
sono:

$$\begin{array}{lll} n=1 & |1\ \overset{l}{0}\ \overset{l_z}{0}\rangle & \text{pari sotto parità} \\ n=2 & |2\ 0\ 0\rangle & \text{pari sotto parità} \\ & |2\ 1\ m\rangle & \text{dispari sotto parità} \end{array}$$

$\left[\begin{array}{l} |n\,l\,m\rangle \\ \text{corrisponde a} \\ R_{nl}\, Y_l^m \end{array}\right]$

Quindi per $\beta=0$, tenendo conto del principio di Pauli, abbiamo

$$\begin{array}{lll} n=1 & |1\ 0\ 0\rangle|\overset{S}{0}\ \overset{S_z}{0}\rangle & E=-E_0 \\ n=2 & |2\ 0\ 0\rangle|0\ 0\rangle & E=-\dfrac{E_0}{4} \\ & |2\ 1\ m\ \ |1\ S_z\rangle & \end{array}$$

Il termine proporzionale a β è dato da

$$H_1 = 2\beta(\bar{L}\cdot\bar{S}+L^2) =$$
$$=\beta\left[(\bar{L}+\bar{S})^2-L^2-S^2\right]+2\beta L^2$$
$$=\beta\left[J^2+L^2-S^2\right]$$

Calcoliamo il nuovo spettro

$n=1$ $\quad L=0,\ S=0 \Rightarrow J=0$ nessun contributo prop. a β

$E=-E_0$ livello non degenere

$n=2$ $\quad |2\,0\,0\rangle|0\,0\rangle$ $\quad L=0,\ S=0 \Rightarrow J=0$ nessun contrib $\propto \beta$

$E=-\dfrac{E_0}{4}$ livello non degenere

$n=2 \qquad |2\ 1\ m\rangle|1\ S_z\rangle$

Quindi $L=1,\ S=1 \longrightarrow J=\begin{cases}0\\1\\2\end{cases}$

NUOVA BASE: $\overset{n\ \ell\ s}{|2\ 1\ 1}\ J\ J_z\rangle_J$

Stati $|2\ 1\ 1\ 0\ 0\rangle \qquad E=-\dfrac{E_0}{4} \qquad$ non deg.

$|2\ 1\ 1\ 1\ J_z\rangle \qquad E=-\dfrac{E_0}{4}+2\beta\hbar^2 \quad$ deg. 3

$|2\ 1\ 1\ 2\ J_z\rangle \qquad E=-\dfrac{E_0}{4}+6\beta\hbar^2 \quad$ deg. 5

Quindi i 4 livelli più bassi sono:

① $\overset{n\ \ell\ m}{|1\ 0\ 0\rangle}\overset{s\ s_z}{|0\ 0\rangle} \qquad E=-E_0$

② $\begin{cases}\overset{n\ \ell\ m}{|2\ 0\ 0\rangle}\overset{s\ s_z}{|0\ 0\rangle}\\ \underset{n\ \ell\ s\ J\ J_z}{|2\ 1\ 1\ 0\ 0\rangle_J}\end{cases} \qquad E=-\dfrac{E_0}{4} \qquad$ deg. 2

③ $|2\ 1\ 1\ 1\ J_z\rangle_J,\ J_z=-1,0,1 \qquad E=-\dfrac{E_0}{4}+2\beta\hbar^2 \quad$ deg. 3

④ $|2\ 1\ 1\ 2\ J_z\rangle_J,\ J_z=\pm2,\pm1,0 \qquad E=-\dfrac{E_0}{4}+6\beta\hbar^2 \quad$ deg. 5

d) La probabilità richiesta è

$$\text{Prob}=\sum_{J_z}\left|{}_J\langle 2\ 1\ 1\ 1\ J_z|\psi\rangle\right|^2$$

Lo stato $|\psi\rangle$ ha un solo contributo con $J=1$.
Se definiamo

$$|\hat{\psi}\rangle=-\frac{1}{2}R_2(r)\,\overset{\ell\ s\ J\ J_z}{|1\ 1\ 1\ 1\rangle_J}$$

$$\text{Prob}=\left|{}_J\langle 2\ 1\ 1\ 1\ 1|\hat{\psi}\rangle\right|^2$$

Separiamo la parte angolare dalla parte radiale in $|2\ 1\ 1\ 1\ 1\rangle = R_{21}(r)\,|1\ 1\ 1\ 1\rangle$

Segue

$$\langle 2\ 1\ 1\ 1\ 1|\hat{\psi}\rangle = -\frac{1}{2}\int dr\, r^2\, R_2(r)\, R_{21}(r)$$

$$= -\frac{1}{2}\,\frac{2}{\sqrt{3}}\,\frac{1}{r_0^{3/2}}\,\frac{1}{\sqrt{3}}\,\frac{1}{2\sqrt{2}\,r_0^{3/2}}\int dr\, r^2\,\frac{r}{r_0}e^{-r/r_0}\,\frac{r}{r_0}e^{-r/2r_0}$$

$$= -\frac{1}{6\sqrt{2}}\,\frac{1}{r_0^3}\int dr\,\frac{r^4}{r_0^2}\,e^{-3r/2r_0} \qquad x = \frac{3r}{2r_0} \quad r = \frac{2r_0}{3}x$$

$$= -\frac{1}{6\sqrt{2}}\left(\frac{2}{3}\right)^5\int dx\, x^4 e^{-x}$$

$$= -\frac{32}{2\sqrt{2}}\,\frac{1}{3^6}\,4! = -\frac{128}{3^5\sqrt{2}} = -\frac{128}{243\sqrt{2}}$$

$$\text{Prob} = \frac{2^{13}}{3^{10}} \approx 0.139 < 1 \quad \underline{\text{ok}}$$