ESPONENZIALE PER MATRICI

(a) Definizione

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$ [la serie converge per qualsiasi A]

(b) Valgono

$$\begin{cases} V e^A V^{-1} = \exp(VAV^{-1}) \\ U e^A U^+ = \exp(UAU^+) \end{cases}$$ U unitaria

Infatti

$$V e^A V^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} V A^n V^{-1}$$

Ora
$$VA^2V^{-1} = VAAV^{-1} = VA\overbrace{V^{-1}V}^{Id.}AV = (VAV^{-1})^2$$
$$VA^3V^{-1} = VAAAV^{-1} = VAV^{-1}VAV^{-1}VAV^{-1} = (VAV^{-1})^3$$
$$\vdots$$
$$VA^nV^{-1} = (VAV^{-1})^n$$

Quindi

$$V e^A V^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} (VAV^{-1})^n = \exp(VAV^{-1})$$

La dimostrazione per UAU^+ è analoga.

c) Derivata

Se $A(t)$ è una matrice i cui coefficienti dipendono da t, definiamo $A'(t)$ [derivata] la matrice che ha come coefficienti le derivate dei coefficienti

$$A(t) = \{a_{nm}(t)\} \qquad \frac{dA}{dt} = \left\{\frac{da_{nm}}{dt}\right\}$$

Vale

$$\frac{d}{dt}(A(t)B(t)) = \frac{dA}{dt}B + A\frac{dB}{dt}$$

Se $[A(t), A'(t)] = 0$ allora

$$\frac{d}{dt} e^{A(t)} = A'(t) e^{A(t)} = e^{A(t)} A'(t)$$

DIM: $\frac{d}{dt}(A^2) = \frac{dA}{dt} A + A \frac{dA}{dt} = 2A'A$ [dato che $[A', A] =$

$$\frac{d}{dt}(A^3) = \frac{dA^2}{dt} A + A^2 \frac{dA}{dt} = 2A'A^2 + A^2 A'$$
$$= 3A'A^2$$

ecc.

$$\frac{d}{dt} e^{A(t)} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dt} A^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} A' A^{n-1} = A' e^A$$

d) inversa ed aggiunta

$$(e^A)^{-1} = e^{-A} \qquad (e^A)^+ = e^{A^+}$$

Per dimostrare la prima definiamo

$$f(t) = e^{tA} e^{-tA}$$

$$f'(t) = e^{tA} A e^{-tA} + e^{tA}(-A) e^{-tA} = 0$$

Quindi $f(t) = f(0) = \mathbb{1}$

Segue $e^A e^{-A} = \mathbb{1} \qquad (e^A)^{-1} = e^{-A}$

Per dimostrare la seconda basta osservare $(A^n)^+ = (A^+)^n$

e) Relazione

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]]$$

$$+ \dots \frac{1}{n!} \underbrace{[A, \dots [A, B] \dots]}_{n \text{ operatori } A}$$

Dimostrazione

Definiamo

$$f(t) = e^{tA} B e^{-tA}$$

e calcoliamo preliminarmente

$$\frac{d}{dt} e^{tA} C e^{-tA} = e^{tA} A C e^{-tA} + e^{tA} C(-A) e^{-tA}$$
$$= e^{tA} [A, C] e^{-tA} \qquad (*)$$

Quindi, usando $(*)$ con $C = B$

$$f'(t) = e^{tA} [A, B] e^{-tA}$$

Usando $(*)$ con $C = [A, B]$

$$f''(t) = \frac{d}{dt} f'(t) = e^{tA} [A, [A, B]] e^{-tA}$$

Usando $(*)$ con $C = [A, [A, B]]$

$$f'''(t) = \frac{d}{dt} f''(t) = e^{tA} [A, [A, [A, B]]] e^{-tA}$$

Quindi, (serie di Taylor)

$$e^{A} B e^{-A} = f(1) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)$$
$$= B + \underbrace{[A, B]}_{f'(0)} + \frac{1}{2!} \underbrace{[A, [A, B]]}_{f''(0)} + \ldots$$

f) Se A, B soddisfano $[A, [A, B]] = [B, [A, B]] = 0$ allora

$$e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A, B]}$$
$$= e^{-\frac{1}{2}[A, B]} e^{A} e^{B}$$

Definiamo

$$f(t) = e^{-tB} e^{-tA} e^{t(A+B)}$$

$$\frac{d}{dt} f(t) = e^{-tB} (-B) e^{-tA} e^{t(A+B)}$$

$$e^{-tB} e^{-tA} (-A) e^{t(A+B)}$$

$$e^{-tB} e^{-tA} (A+B) e^{t(A+B)}$$

$$= e^{-tB} (-B) e^{-tA} e^{t(A+B)}$$

$$e^{-tB} \underbrace{e^{-tA} B} e^{t(A+B)}$$

Ora

$$e^{-tA} B = (e^{-tA} B e^{tA}) e^{-tA}$$

$$= (B - [tA, B]) e^{-tA}$$

(non vi sono altri termini dato che $[A,[A,B]] = 0$)

$$= B e^{-tA} - t[A,B] e^{-tA}$$

Quindi

$$\frac{df}{dt} = e^{-tB} (-B) e^{-tA} e^{t(A+B)}$$

$$e^{-tB} B e^{-tA} e^{t(A+B)}$$

$$-t e^{-tB} [A,B] e^{-tA} e^{t(A+B)}$$

$$= -t e^{-tB} [A,B] e^{-tA} e^{t(A+B)} =$$

(B e quindi e^{-tB} commuta con $[A,B]$)

$$= -t [A,B] f(t)$$

Ne segue $\quad f(t) = e^{-\frac{t^2}{2}[A,B]} K \quad$ K matrice costante

$$f(0) = \mathbb{1} \implies K = \mathbb{1}$$

$$\underline{f(1) = e^{-\frac{1}{2}[A,B]}} \quad \text{QED}$$

Quindi abbiamo (mettiamo $t=1$)

$$e^{-B} e^{-A} e^{A+B} = e^{-\frac{1}{2}[A,B]}$$

$$\overline{e^{B} e^{-B}} e^{-A} e^{A+B} = e^{B} e^{-\frac{1}{2}[A,B]}$$

[Abbiamo moltiplicato ambo i membri per e^{B}]

$$e^{A} e^{-A} e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}$$ [Abbiamo moltiplicato ambo i membri per e^{A}]

$$e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}$$

Dato che $[A,B]$ commuta con A, B, allora $[A,B]$ commuta con e^{A} ed e^{B}

Vale pure

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B}$$

g) Se A, B soddisfano $[A,[A,B]] = [B,[A,B]] = 0$ allora

$$e^{A} e^{B} = e^{B} e^{A} e^{[A,B]} = e^{[A,B]} e^{A} e^{B}$$

Infatti

$$\begin{aligned} e^{A} e^{B} &= e^{A+B} e^{\frac{1}{2}[A,B]} \\ &= e^{B+A} e^{\frac{1}{2}[A,B]} \\ &= e^{B} e^{A} e^{-\frac{1}{2}[B,A]} e^{\frac{1}{2}[A,B]} \\ &= e^{B} e^{A} e^{[A,B]} \end{aligned}$$

Approfondimento [FACOLTATIVO]

Si supponga che tutti i commutatori che coinvolgono 4 matrici A e B siano nulli. Per esempio $[A,[A,[A,B]]] = 0$ ecc.

Allora $e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]} e^{a[A,[A,B]] + b[B,[A,B]]}$

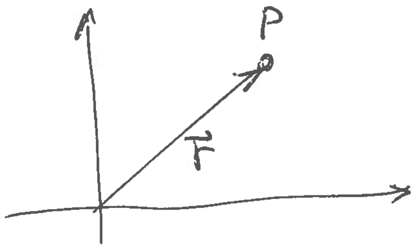
Si calcolino a e b

Sugg: $f(t) = e^{\frac{t^2}{2}[A,B]} e^{-tB} e^{-tA} e^{t(A+B)}$

Si trova

$$f'(t) = 3t^{3}\left(a[A,[A,B]] + b[B,[A,B]]\right) f(t)$$

Consideriamo un sistema di coordinate (ξ_1, ξ_2, ξ_3). Per fissare le idee, possiamo sempre pensare alle coordinate sferiche ed identificare $\xi_1 = r$, $\xi_2 = \theta$, $\xi_3 = \phi$.



Il punto P ha coordinate (ξ_1, ξ_2, ξ_3). Il vettore $\vec{r}$ che unisce l'origine a P è quindi una funzione di (ξ_1, ξ_2, ξ_3).

Calcoliamo i tre vettori

$$\vec{v}_i = \frac{\partial \vec{r}}{\partial \xi_i}$$

i loro moduli $M_i = |\vec{v}_i|$ ed i versori unitari $\hat{u}_i$ tali che

$$\vec{v}_i = M_i \hat{u}_i$$

Le coordinate sono un sistema di coordinate ORTOGONALI se

$$\vec{v}_i \cdot \vec{v}_j = 0 \qquad \text{per } i \neq j$$

Segue che, in ogni punto P, $\{\hat{u}_i\}$ è una base ORTONORMALE

$$\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$$

Esempio: coordinate sferiche

$$\vec{r}(r, \theta, \varphi) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \leftarrow \text{componenti cartesiane}$$

- $$\vec{v}_1 = \frac{\partial \vec{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$|v_1|^2 = \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta = 1 \qquad M_1 = 1 = M_r$$

$$\hat{u}_1 = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \hat{u}_r$$

- $\overline{v}_2 = \frac{\partial \overline{r}}{\partial \theta} = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta)$

$$|v_2|^2 = r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta = r^2 \qquad M_2 = r = M_\theta$$

$$\hat{u}_2 = \frac{1}{M_2}\overline{v}_2 = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = \hat{u}_\theta$$

- $v_3 = \frac{\partial \overline{r}}{\partial \varphi} = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)$

$$|v_3|^2 = r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi = r^2\sin^2\theta \qquad M_3 = r\sin\theta = M_\phi$$

$$\hat{u}_3 = \frac{1}{M_3}\overline{v}_3 = (-\sin\varphi, \cos\varphi, 0)$$

PROPRIETÀ GENERALE

Vale $\frac{\partial}{\partial \xi_i}\frac{\partial \overline{r}}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\frac{\partial \overline{r}}{\partial \xi_i}$ [SCAMBIATO ORDINE DERIVATE]

$$\frac{\partial}{\partial \xi_i}(M_j \hat{u}_j) = \frac{\partial}{\partial \xi_j}(M_i \hat{u}_i)$$

$$\hat{u}_i \cdot \frac{\partial}{\partial \xi_i}(M_j \hat{u}_j) = \hat{u}_i \cdot \left(\frac{\partial M_i}{\partial \xi_j}\hat{u}_i + M_i \frac{\partial \hat{u}_i}{\partial \xi_j}\right)$$

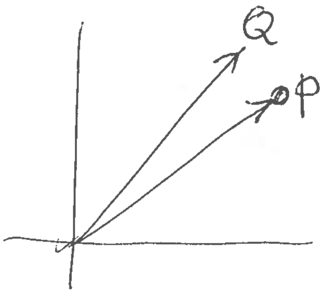
Per qualsiasi versore vale

$$\hat{u}_i \cdot \frac{\partial \hat{u}_i}{\partial \xi_j} = \frac{1}{2}\frac{\partial}{\partial \xi_j}(\hat{u}_i \cdot \hat{u}_i) = 0$$

Quindi

$$\hat{u}_i \cdot \frac{\partial}{\partial \xi_i}(M_j \hat{u}_j) = \frac{\partial M_i}{\partial \xi_j}$$

GRADIENTE



P di coordinate (ξ_1, ξ_2, ξ_3)

Q di coordinate $(\xi_1 + \Delta\xi_1, \xi_2 + \Delta\xi_2 + \xi_3 + \Delta\xi_3)$

Sia $f(P)$ una funzione

$$f(Q) - f(P) = (\nabla f)\big|_P \cdot (\bar{r}(Q) - \bar{r}(P)) \quad \text{per } Q \to P$$

Questa relazione definisce il gradiente nel punto P.

(a) $f(Q) - f(P) = f(\xi_1 + \Delta\xi_1, \xi_2 + \Delta\xi_2, \xi_3 + \Delta\xi_3) - f(\xi_1, \xi_2, \xi_3)$

$$= \sum_i \frac{\partial f}{\partial \xi_i} \Delta\xi_i$$

(b) Decomponiamo il gradiente nella base $\hat{u}_i$

$$\nabla f\big|_P = (\nabla f)_1 \hat{u}_1 + (\nabla f)_2 \hat{u}_2 + (\nabla f)_3 \hat{u}_3$$

(c)

$$(\bar{r}(Q) - \bar{r}(P)) = \sum_i \frac{\partial \bar{r}}{\partial \xi_i} \Delta\xi_i = \sum_i \hat{u}_i M_i \Delta\xi_i$$

(d)

$$\nabla f\big|_P \cdot (\bar{r}(Q) - \bar{r}(P)) = \sum_i (\nabla f)_i M_i \Delta\xi_i$$

Quindi

$$\sum_i \frac{\partial f}{\partial \xi_i} \Delta\xi_i = \sum_i (\nabla f)_i M_i \Delta\xi_i$$

Segue

$$\frac{\partial f}{\partial \xi_i} = (\nabla f)_i M_i \qquad (\nabla f)_i = \frac{1}{M_i} \frac{\partial f}{\partial \xi_i}$$

$$\nabla f = \sum_i \frac{\hat{u}_i}{M_i} \frac{\partial f}{\partial \xi_i}$$

COORDINATE SFERICHE: GRADIENTE

$$\nabla f = \frac{\hat{u}_r}{M_r}\frac{\partial f}{\partial r} + \frac{\hat{u}_\theta}{M_\theta}\frac{\partial f}{\partial \theta} + \frac{\hat{u}_\phi}{M_\phi}\frac{\partial f}{\partial \phi}$$

$$= \hat{u}_r\frac{\partial f}{\partial r} + \frac{\hat{u}_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{u}_\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

OPERATORE ∇

$$\overline{\nabla} = \sum_i \frac{\hat{u}_i}{M_i}\frac{\partial}{\partial \xi_i}$$

DIVERGENZA

Dato un vettore $\overline{V}$, sia

$$\overline{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3 = \sum_i V_i\hat{u}_i$$

la sua decomposizione nella base.

Calcoliamo

$$\overline{\nabla}\cdot\overline{V} = \sum_i \frac{\hat{u}_i}{M_i}\frac{\partial}{\partial \xi_i}\left(\sum_j V_j\hat{u}_j\right)$$

$$= \sum_{ij}\frac{\hat{u}_i}{M_i}\cdot\frac{\partial}{\partial \xi_i}\left(\frac{V_j}{M_j}\cdot M_j\hat{u}_j\right)$$

$$= \sum_{ij}\frac{\hat{u}_i}{M_i}\cdot\left[\frac{\partial}{\partial \xi_i}\left(\frac{V_j}{M_j}\right)M_j\hat{u}_j + \frac{V_j}{M_j}\frac{\partial}{\partial \xi_i}(M_j\hat{u}_j)\right]$$

$$= \sum_{ij}\frac{M_j}{M_i}\frac{\partial}{\partial \xi_i}\left(\frac{V_j}{M_j}\right)\boxed{\hat{u}_i\cdot\hat{u}_j} \longrightarrow \delta_{ij}$$

$$+ \sum_{ij}\frac{V_j}{M_iM_j}\boxed{\hat{u}_i\cdot\frac{\partial}{\partial \xi_i}(M_j\hat{u}_j)} \longrightarrow \frac{\partial M_i}{\partial \xi_j}$$

$$= \sum_j \frac{\partial}{\partial \xi_i}\left(\frac{V_j}{M_j}\right) + \sum_{ij}\frac{V_j}{M_iM_j}\frac{\partial M_i}{\partial \xi_j} = \sum_j\left[\frac{\partial}{\partial \xi_j}\left(\frac{V_j}{M_j}\right) + \frac{V_j}{M_j}\sum_i\frac{1}{M_i}\frac{\partial M_i}{\partial \xi_j}\right.$$

Scriviamo esplicitamente il termine con $j=1$

$$\frac{\partial}{\partial \xi_1}\left(\frac{V_1}{M_1}\right)+\frac{V_1}{M_1}\left(\frac{1}{M_1}\frac{\partial M_1}{\partial \xi_1}+\frac{1}{M_2}\frac{\partial M_2}{\partial \xi_1}+\frac{1}{M_3}\frac{\partial M_3}{\partial \xi_3}\right)$$

$$\frac{\partial}{\partial \xi_1}\left(\frac{V_1}{M_1}\right)+\frac{V_1}{M_1}\frac{1}{M_1M_2M_3}\left[M_2M_3\frac{\partial M_1}{\partial \xi_1}+M_1M_3\frac{\partial M_2}{\partial \xi_1}+M_1M_2\frac{\partial M_3}{\partial \xi_1}\right]$$

$$=\frac{1}{M_1M_2M_3}\left[M_1M_2M_3\frac{\partial}{\partial \xi_1}\left(\frac{V_1}{M_1}\right)+\frac{V_1}{M_1}\frac{\partial}{\partial \xi_1}(M_1M_2M_3)\right]$$

$$=\frac{1}{M_1M_2M_3}\frac{\partial}{\partial \xi_1}\left[\frac{V_1}{M_1}M_1M_2M_3\right]$$

Se definiamo $M_p=M_1M_2M_3$ abbiamo

$$\cancel{\frac{\partial}{\partial \xi_1}\left(\frac{V_1}{M_1}\right)}\quad \frac{1}{M_p}\frac{\partial}{\partial \xi_1}\left(\frac{V_1}{M_1}M_p\right)$$

Analogamente i termini con $j=2$ e 3 diventano

$$\frac{1}{M_p}\frac{\partial}{\partial \xi_2}\left(\frac{V_2}{M_2}M_p\right)\quad e \quad \frac{1}{M_p}\frac{\partial}{\partial \xi_3}\left(\frac{V_3}{M_3}M_p\right)$$

Quindi

$$\bar{\nabla}\cdot\bar{V}=\frac{1}{M_p}\sum_i\frac{\partial}{\partial \xi_i}\left(\frac{V_i}{M_i}M_p\right)$$

DIVERGENZA IN COORDINATE SFERICHE

$$M_p=M_r M_\theta M_\phi=r^2\sin\theta$$

$$\nabla\cdot V=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(V_r r^2\sin\theta\right)+\frac{\partial}{\partial\theta}\left(\frac{V_\theta}{r}r^2\sin\theta\right)+\frac{\partial}{\partial\phi}\left(\frac{V_\phi}{r\sin\theta}r^2\sin\theta\right)\right]$$

$$\bar{\nabla}\cdot\bar{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

$$= \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r + \frac{1}{\cancel{r\sin\theta}}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r}\cot\theta\, V_\theta + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

LAPLACIANO IN COORD. SFERICHE

$$\bar{V} = \nabla f \qquad V_r = \frac{\partial f}{\partial r} \qquad V_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta} \qquad V_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Ora

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) = \frac{1}{r}\frac{\partial}{\partial r}(rf' + f) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$$

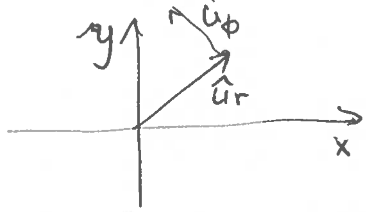
Quindi

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2}\left[\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\frac{\partial f}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right]$$

Versori e prodotti vettore

I tre versori $\hat{u}_r, \hat{u}_\theta, \hat{u}_\varphi$ sono ortogonali.
Vogliamo capire in quale ordine formano una terna destrorsa

Nel piano xy $(\theta = \pi/2)$ vale la figura



per cui

$$\hat{u}_r \times \hat{u}_\phi = \hat{u}_z \text{ (versore asse } z\text{)}$$

D'altra parte

$$\hat{u}_\theta = (\cos\varphi\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \overset{\theta=\pi/2}{=} (0,0,-1) = -\hat{u}_z$$

Quindi

$$\hat{u}_r \times \hat{u}_\phi = -\hat{u}_\theta \implies \hat{u}_\phi \times \hat{u}_r = \hat{u}_\theta$$

Quindi la terna destrorsa è $(\hat{u}_r, \hat{u}_\theta, \hat{u}_\phi)$ per cui

$$\hat{u}_r \times \hat{u}_\theta = \hat{u}_\phi$$
$$\hat{u}_\theta \times \hat{u}_\phi = \hat{u}_r$$
$$\hat{u}_\phi \times \hat{u}_r = \hat{u}_\theta$$