Esercitazione 2026-01-10

Saturday, January 10, 2026 9:58

Dire se la funzione $f(x) = \dfrac{|x| - e}{\log|x| - 1}$ è estendibile con continuità o derivabilità nei punti in cui non è definita.

Dominio di $f = \mathbb{R} \setminus \{0, \pm e\}$

OSS f dipende dal modulo di $|x| \to$ è pari.

$$\lim_{x\to e} f(x) = \lim_{x\to e} \frac{x-e}{\log x - 1} = \lim_{x\to e} \frac{x-e}{\frac{x-e}{e}} = e.$$

$$\log\left(\frac{x}{e}\right) = \log\left(1 + \left(\frac{x}{e} - 1\right)\right) \sim \frac{x}{e} - 1 = \frac{x-e}{e}$$

($\frac{x}{e} \to 1$, $\frac{x}{e} - 1 \to 0$)

Ponendo $f(e) = e$, f risulta continua in $x = e$.

Per la parità lo stesso succede in $x = -e$.

Vediamo in $x = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-e}{\log x - 1} = \left(\frac{-e}{-\infty}\right) = 0$$

($x - e \to -e$, $\log x \to -\infty$)

Quindi (parità) anche $\lim_{x\to 0^-} f(x) = 0$.

In definitiva, ponendo

$$\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq (0, \pm e) \\ 0 & \text{se } x = 0 \\ \dots \end{cases}$$

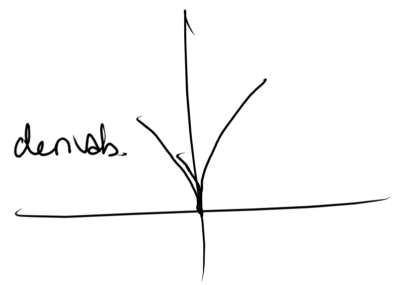
$$\tilde f(x) = \begin{cases} 0 & \text{se } x = 0 \\ e & \text{se } x = \pm e \end{cases}$$

$\tilde f$ è continua in tutto $\mathbb{R}$. Vediamo se questa funzione è anche derivabile

$$\tilde f'_+(0) = \lim_{x\to 0^+} \frac{\overset{f(x)}{\tilde f(x)} - \overset{=0}{\cancel{\tilde f(0)}}}{x} = \lim_{x\to 0^+} \frac{\overset{\to -e}{x-e}}{\underbrace{x(\log x - 1)}_{\sim\, x \log x \,\to\, 0^-}} = \left(\frac{-e}{0^-}\right) = +\infty$$

$$\tilde f'_-(0) = -\infty$$

Si tratta di una cuspide. $\tilde f$ non è derivab. in $x=0$



Si poteva fare anche $\lim_{x\to 0^+} f'(x) =$

$$= \lim_{x\to 0^+} \frac{\log x - 1 - (x-e)\frac{1}{x}}{\underbrace{(\log x - 1)^2}_{\sim (\log x)^2}} = \lim_{x\to 0^+} \frac{\overbrace{\log x - 1 - 1 + \frac{e}{x}}^{\sim e/x}}{(\log x)^2} =$$

$$= \lim_{x\to 0^+} \frac{e}{\underbrace{x(\log x)^2}_{\to 0^+}} = +\infty.$$

derivabilità in $x = e$.

$$\tilde f'_+(e) = \lim_{x\to e^+} \frac{\overset{= f(x)}{\tilde f(x)} - e}{x-e} = \lim_{x\to e} \frac{\frac{x-e}{\log x - 1} - e}{x-e} =$$

$$\tilde f'_+(e) = \lim_{x\to e^+} \frac{\tilde f(x) - e}{x - e} = \lim_{x\to e} \frac{\ldots \log x - 1}{x - e} =$$

$$= \lim_{x\to e^+} \frac{x - e - e(\log x - 1)}{(x-e)(\log x - 1)} = \qquad x - e = t \to 0^+$$

$$= \lim_{t\to 0^+} \frac{t - e(\log(e+t) - 1)}{t\,(\log(e+t) - 1)} = \lim_{t\to 0^+} \frac{\frac{t^2}{2e}}{\frac{t^2}{e}} = \frac{1}{2}$$

$1 = \log e$

$$\log\left(\frac{e+t}{e}\right) = \log\left(1 + \frac{t}{e}\right) \sim \frac{t}{e} \qquad t \to 0$$

tutto il denom. è $\sim \dfrac{t^2}{e}$

Num.: $t - e\,(\log(e+t) - 1) =$

$$= t - e \log\left(1 + \frac{t}{e}\right) = \qquad \log(1+s) = s - \frac{s^2}{2} + o(s^2) \quad s \to 0$$

$$= t - e\left[\frac{t}{e} - \frac{1}{2}\frac{t^2}{e^2} + o(t^2)\right] =$$

$$= \cancel{t} - \cancel{t} + \frac{1}{2}\frac{t^2}{e} + o(t^2) \sim \frac{t^2}{2e}$$

In realtà non abbiamo mai usato che $x \to e^+$, funziona anche se $x \to e^-$. $\Rightarrow \tilde f'(e) = \frac{1}{2}$.

Per parità di f ($\Rightarrow$ disparità di f') $\boxed{\tilde f'(-e) = -\frac{1}{2}.}$

In definitiva: f è prolungabile con continuità in tutto $\mathbb{R}$, In definitiva: f è prolungabile con continuità in tutto $\mathbb{R}$,

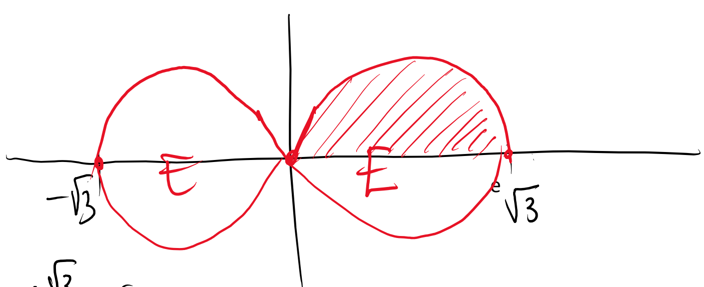
ma prolungabile in modo derivabile in $\mathbb{R}\setminus\{0\}$.

Area di $E=\{(x,y)\in\mathbb{R}^2 : \quad y^2 \le 9x^2 - x^6\}$.

la quantità $9x^2-x^6$ deve essere ≥ 0.

$$9x^2-x^6 \geq 0 \iff 9-x^4 \geq 0 \iff -\sqrt{3}\le x\le \sqrt{3}$$

$$E=\{(x,y) : -\sqrt{3}\le x\le\sqrt{3},\ -\sqrt{9x^2-x^6}\le y\le \sqrt{9x^2-x^6}\}$$



$$\text{area } E = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\sqrt{9x^2-x^6}+\sqrt{9x^2-x^6}\right)dx = 4\int_0^{\sqrt{3}} \sqrt{9x^2-x^6}\,dx =$$

$$= 4\int_0^{\sqrt{3}} x\sqrt{9-x^4}\,dx =$$

$x^2=t$

$2x\,dx = dt$

$x=0 \Rightarrow t=0$

$x=\sqrt{3} \Rightarrow t=3$

$$= 2\int_0^3 \sqrt{9-t^2}\,dt$$

Vari modi:

1) per parti

$$2\int_0^3 1\sqrt{9-t^2}\,dt =$$

$$2\int_0^{1} 1\sqrt{9-t^2}\,dt =$$

$f'(t)=1 \qquad f(t)=t$

$g(t)=\sqrt{9-t^2} \Rightarrow g'(t)=\frac{-2t}{2\sqrt{9-t^2}}=-\frac{t}{\sqrt{9-t^2}}$

$$= 2\left[\, t\sqrt{9-t^2}\,\right]_0^3 + \int_0^3 \frac{(t^2-9)+9}{\sqrt{9-t^2}}\,dt =$$

$$= 2\left[-\int_0^3 \sqrt{9-t^2}\,dt + \int_0^3 \frac{9}{\sqrt{9-t^2}}\,dt \right. =$$

$$\Rightarrow \text{Area } E = 2\int_0^3 \sqrt{9-t^2}\,dt = \int_0^3 \frac{9}{\sqrt{9-t^2}}\,dt =$$

$$= \frac{9}{3}\int_0^3 \frac{dt}{\sqrt{1-\frac{t^2}{9}}} = 3\cdot 3 \arcsin\left(\frac{t}{3}\right)\Big|_0^3 = 9\arcsin\frac{t}{3}\Big|_0^3 =$$

$\frac{t^2}{9} = \left(\frac{t}{3}\right)^2$

$$= \frac{9\pi}{2}$$

Altro modo per calcolare l'integrale

$$\text{Area } E = 2\int_0^3 \sqrt{9-t^2}\,dt =$$

$t = 3\sin\theta$

$t=0 \Rightarrow \theta=0$

$t=3 \Rightarrow \theta=\frac{\pi}{2}$

$dt = 3\cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} 3\cos\theta \cdot 3\cos\theta\, d\theta =$$

$\int^{\pi/2}$

$$= 18 \int_0^{\pi/2} \cos^2\theta \, d\theta =$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= 9 \int_0^{\pi/2} \left(1 + \cos(2\theta)\right) d\theta = \frac{9\pi}{2}.$$

$\cos(2\theta)$: non dà contributo.

$$dt = 3\cos\theta \, d\theta$$

$$\sqrt{9-t^2} = \sqrt{9-9\sin^2\theta} =$$

$$= 3\sqrt{1-\sin^2\theta} = 3|\cos\theta| =$$

$$= 3\cos\theta$$

In alternativa

$$2\int_0^3 \sqrt{9-t^2} \, dt =$$

$$= 2\int_{\pi/2}^{0} 3\sin\theta \cdot (-3\sin\theta) \, d\theta =$$

$$= 18 \int_{\pi/2}^{0} (-\sin^2\theta) \, d\theta = 18 \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$\theta \in [0, \pi]$

$$t = 3\cos\theta$$

$$dt = -3\sin\theta \, d\theta$$

$$\sqrt{9-t^2} = \sqrt{9-9\cos^2\theta} =$$

$$= 3\sqrt{1-\cos^2\theta} = 3|\sin\theta| =$$

$$= 3\sin\theta$$

$$t = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$$t = 3 \Rightarrow \theta = 0$$

$$z^2 + 3\bar{z}\,\mathrm{Re}(z) + 4i = 0 \qquad z = x+iy \qquad x,y \in \mathbb{R}$$

$$z^2 = x^2 - y^2 + 2ixy$$

$$x^2 - y^2 + 2ixy + 3(x-iy)x + 4i = 0$$

$$x^2 - y^2 + 2ixy + 3x^2 - 3ixy + 4i = 0$$

$$x^2 - y^2 + 2ixy + 3x^2 - 3ixy + 4i = 0$$

Questo corrisponde a un sistema nei reali

$$\begin{cases} x^2 - y^2 + 3x^2 = 0 \\ 2xy - 3xy + 4 = 0 \end{cases} \qquad \begin{cases} 4x^2 = y^2 \implies y = \pm 2x \\ 4 - xy = 0 \end{cases}$$

$$\begin{cases} y = 2x \\ 4 - 2x^2 = 0 \end{cases} \qquad \begin{cases} y = -2x \\ 4 + 2x^2 = 0 \end{cases}$$

nessuna sol^{ne} reale

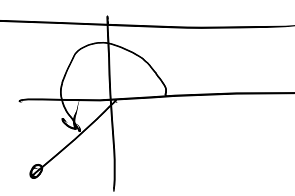
$$\Downarrow$$

$$x^2 = 2 \qquad x = \pm\sqrt{2}$$

$$y = \pm 2\sqrt{2}$$

due sol^{ni}

$$z = \pm\left(\sqrt{2} + 2\sqrt{2}\,i\right) = \pm\sqrt{2}\,(1 + 2i)$$



$$i z^5 - 1 + i = 0$$

$$\iff z^5 = \frac{1 - i}{i} \cdot \frac{-i}{-i} = \frac{-i - 1}{1} = -1 - i = \sqrt{2}\, e^{i\frac{5}{4}\pi}$$

$$z_k = \sqrt[10]{2}\; e^{i\theta_k} \qquad \theta_k = \frac{\frac{5}{4}\pi + 2k\pi}{5} = \frac{\pi}{4} + \frac{2}{5}k\pi$$

$$k = 0, 1, 2, 3, 4$$

$$z_0 = \sqrt[10]{2}\; e^{i\frac{\pi}{4}} = \sqrt[10]{2}\left[\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right] = 2^{-2/5}(1 + i)$$

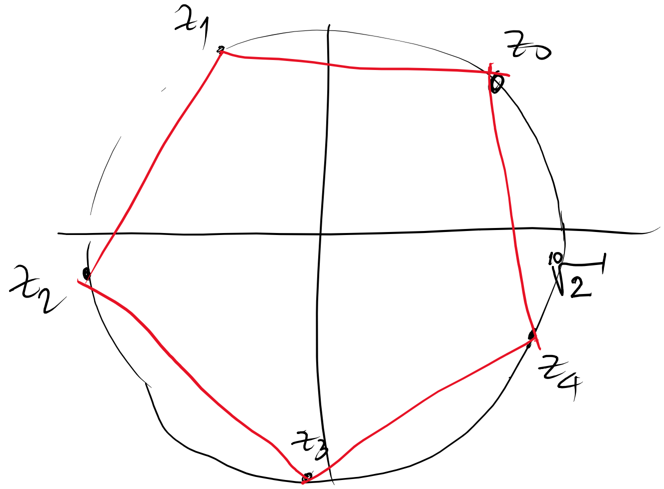
$\sqrt[10]{\ }\; e^{i\,13\pi}$ ...

$$z_1 = \sqrt[10]{2}\ e^{i\frac{13}{20}\pi} = \sqrt[10]{2}\left[\cos\left(\frac{13}{20}\pi\right) + i\sin\left(\frac{13}{20}\pi\right)\right]$$

$$z_2 = \sqrt[10]{2}\ e^{i\frac{21}{20}\pi}$$

$$z_3 = \sqrt[10]{2}\ e^{i\frac{29}{20}\pi}$$

$$z_4 = \sqrt[10]{2}\ e^{i\frac{37}{20}\pi}$$



Al variare di $\alpha \geq 0$, $\beta, \gamma \in \mathbb{R}$, calcolare

$$\lim_{x\to 0^+} \frac{\sin(x^\alpha)}{x+2x^2} = \begin{cases} +\infty & \text{se } \alpha = 0 \\ \lim_{x\to 0^+} \frac{x^\alpha}{x+2x^2} & \alpha > 0 \end{cases}$$

$x+2x^2 \sim x \qquad x^{\alpha-1}$

$$= \begin{cases} +\infty & \text{se } \alpha = 0 \\ +\infty & 0 < \alpha < 1 \\ 1 & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$$

$$\lim_{x\to 0^+} \frac{\beta(e^{2x}-1) + \gamma\log(1+x) - (x^3+5x^2-x)}{x^3} =$$

$$= \lim_{x\to 0^+} \frac{\beta\left(2x + 2x^2 + \frac{4}{3}x^3\right) + \gamma\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - x^3 - 5x^2 + x + o(x^3)}{x^3} =$$

$$= \lim_{x \to 0^+} \frac{[\text{illegible}]}{x^3} =$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + o(x^3) =$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + o(x^3)$$

$$= \lim_{x \to 0^+} \frac{x(2\beta + \gamma + 1) + x^2\left(2\beta - \frac{\gamma}{2} - 5\right) + x^3\left(\frac{4}{3}\beta + \frac{\gamma}{3} - 1\right) + o(x^3)}{x^3} = 1 \quad (*)$$

OSS se $\boxed{2\beta + \gamma + 1 \neq 0}$

$$(*) = \lim_{x \to 0^+} \frac{2\beta + \gamma + 1}{x^2} = \begin{cases} +\infty & \text{se } 2\beta + \gamma + 1 > 0 \\ -\infty & \text{se } 2\beta + \gamma + 1 < 0 \end{cases}$$

Se invece $\boxed{2\beta + \gamma + 1 = 0}$, cioè $\gamma = -1 - 2\beta$

$$(*) = \lim_{x \to 0^+} \frac{x^2\left(2\beta - \frac{\gamma}{2} - 5\right) + x^3\left(\frac{4}{3}\beta + \frac{\gamma}{3} - 1\right) + o(x^3)}{x^3}$$

se $2\beta - \frac{\gamma}{2} - 5 \neq 0 \Rightarrow -\gamma - 1 - \frac{\gamma}{2} - 5 \neq 0$ $\quad \not\beta$

$$\frac{3}{2}\gamma \neq -6 \qquad \boxed{\gamma \neq -4}$$

$$\boxed{2\beta + \gamma + 1 = 0, \; \gamma \neq -4}$$

$$(*) = \lim_{x \to 0^+} \frac{\left(2\beta - \frac{\gamma}{2} - 5\right) x^2}{x^3} = \begin{cases} -\infty & \text{se } 2\beta - \frac{\gamma}{2} - 5 < 0 \\ +\infty & \text{se } 2\beta - \frac{\gamma}{2} - 5 > 0 \end{cases}$$

$$2\beta - \frac{\gamma}{2} - 5 < 0 \iff -\gamma - 1 - \frac{\gamma}{2} - 5 < 0 \qquad \boxed{\gamma > -4}$$

se $\gamma > -4$, $2\beta + \gamma + 1 = 0 \Rightarrow$ lim vale $-\infty$

$\gamma < -4$ $\quad 2\beta = -1 - \gamma$ $\quad$ " " $+\infty$.

Infine resta il caso in cui $\begin{cases} 2\beta + \gamma + 1 = 0 \\ 2\beta - \frac{\gamma}{2} - 5 = 0 \end{cases} \iff \boxed{\gamma = -4}$

$2\beta = -1 - \gamma = 3$

$$\boxed{\beta = \frac{3}{2}}$$

$$(*) = \lim_{x \to 0^+} \frac{\left(\frac{4}{3}\beta + \frac{\gamma}{3} - 1\right) x^3}{x^3} = \frac{4}{3} \cdot \frac{3}{2} - \frac{4}{3} - 1 = 2 - \frac{4}{3} - 1 = \boxed{= -\frac{1}{3}}$$

$$\lim_{x \to 0^+} \frac{e^{x \cos x} - \log^2(1 + \sqrt{x}) - 1}{\sqrt{\sin x - x \cos x}} = \left(\frac{0}{0}\right) = \sqrt{3}$$

<u>Denom.</u> $\quad \sin x - x \cos x = x - \frac{x^3}{6} + o(x^4) - x\left(1 - \frac{x^2}{2} + o(x^3)\right) =$

$$= x - \frac{x^3}{6} + o(x^4) - x + \frac{x^3}{2} = \frac{x^3}{3} + o(x^4) \sim$$

$$\sim \frac{x^3}{3} \qquad x \to 0^+$$

$$\sqrt{\sin x - x \cos x} \sim \sqrt{\frac{x^3}{3}} = \frac{x^{3/2}}{\sqrt{3}}$$

$\sqrt{\sin x - x\cos x} \sim \sqrt{\frac{x^3}{3}} = \frac{x^{3/2}}{\sqrt{3}}$

Num. $e^{x\cos x} - 1 - \left(\log^2(1+\sqrt{x})\right) =$ ($x\cos x \to 0$)

$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$, $t \to 0$

$= x\cos x + \frac{x^2\cos^2 x}{2} - \left(\sqrt{x} - \frac{x}{2} + \frac{x^{3/2}}{3} + o(x^{3/2})\right)^2$

$t = x\cos x \sim x$

$x\cos x = x - \frac{x^3}{2} + \ldots = x + o(x^{3/2})$; $\frac{x^2\cos^2 x}{2} = o(x^{3/2})$

$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$, $t = \sqrt{x} \to 0$

$= x + o(x^{3/2}) - x + x^{3/2} \sim x^{3/2}$

$f(x) = x - 2\arctg\left(\frac{1+\sin x}{\cos x}\right)$ dove $x \neq \frac{\pi}{2} + k\pi$.

$f(x+2\pi) = f(x) + 2\pi$ basta studiarla in $\left(-\frac{\pi}{2}, \frac{3}{2}\pi\right) \setminus \left\{\frac{\pi}{2}\right\}$

$f'(x) = 1 - 2\frac{1}{1+\frac{(1+\sin x)^2}{\cos^2 x}} \cdot \frac{\cos^2 x - (1+\sin x)(-\sin x)}{\cos^2 x} =$

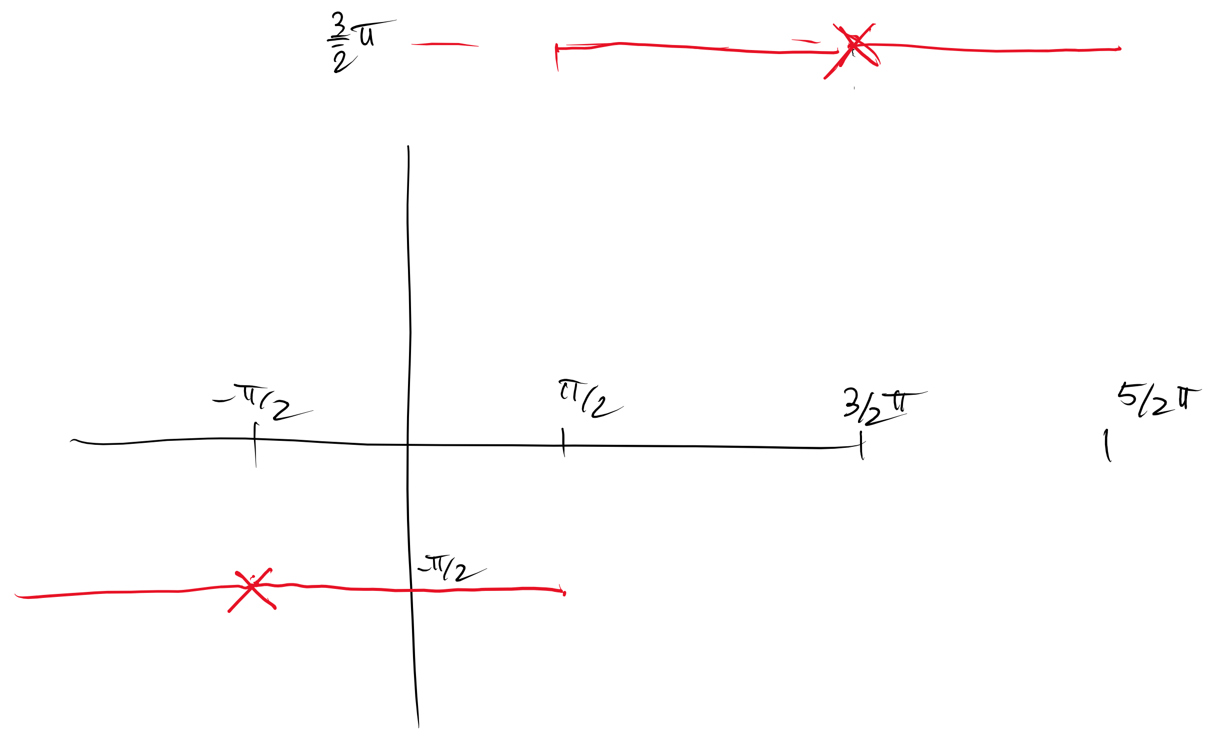
$= 1 - \frac{2(\cos^2 x + \sin x + \sin^2 x)}{\cos^2 x + 1 + \sin^2 x + 2\sin x} =$

$= 1 - \frac{2(1+\sin x)}{2+2\sin x} = 1 - 1 = 0$

$\Rightarrow f$ costante in ogni intervallo della forma $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$.

In $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ la funzione è costante e vale

$$f(0) = 0 - 2\,\mathrm{arctg}\, 1 = -\frac{\pi}{2}.$$



Per $x \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$ $f(x)$ è costante e vale

$$f(\pi) = \pi - 2\,\mathrm{arctg}\left(\frac{1}{-1}\right) = \pi + \frac{\pi}{2} = \frac{3}{2}\pi$$

Trovare il polinomio $P(x)$ di grado minimo t.c.

$$P(x) - \frac{\cos x}{1 - \sin x} = o(x^5) \quad \text{per } x \to 0$$

$1-\sin x$

E' il polinomio di Maclaurin di $f(x) = \frac{\cos x}{1-\sin x}$ di ordine 5

oss. $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \qquad x \to 0$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$

$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + t^5 + o(t^5)$

$t \to 0$, $t = \sin x$

$\frac{1}{1-\sin x} =$

$= 1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \sin^5 x + o(x^5) =$

$= 1 + x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) + \underbrace{\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^2}_{x^2 - \frac{x^4}{3} + o(x^5)} +$

$+ \underbrace{\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right)^3}_{x^3 - \frac{x^5}{2} + o(x^5)} + \underbrace{\left(\qquad\right)^4}_{x^4 + o(x^5)} +$

$+ \underbrace{\left(\qquad\right)^5}_{x^5 + o(x^5)} + o(x^5)$

$= 1 + x + \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) + x^2 - \frac{x^4}{3} + x^3 - \frac{x^5}{2} + x^4 + x^5 =$

$$= 1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + \frac{61}{120}x^5 + o(x^5)$$

$$\frac{\cos x}{1-\sin x} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right)\left(1 + x + x^2 + \frac{5}{6}x^3 + \frac{2}{3}x^4 + \frac{61}{120}x^5 + o(x^5)\right)$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(\frac{5}{6} - \frac{1}{2}\right) + x^4\left(\frac{1}{24} - \frac{1}{2} + \frac{2}{3}\right) +$$

$$+ x^5\left(\frac{2}{3} - \frac{5}{12} + \frac{1}{24}\right) + o(x^5)$$