22/12/2025

SERIE DI POTENZE

$$\sum_{k\geq 0} a_k x^k \qquad \{a_k\}\subseteq \mathbb{R}, \quad x\in\mathbb{R}, \quad k\in\mathbb{N}$$

$$\left[\sum_{k\geq k_0} a_k (t-t_0)^k \longrightarrow \sum_{k\geq k_0} a_k x^k \qquad x=(t-t_0)\right]$$

ESEMPI

① $\sum_{k\geq 0} k^k x^k \qquad k_0=0, \; a_k=k^k$

se $x\neq 0 \qquad (kx)^k \not\to 0$ per $k\to+\infty$

$\Rightarrow$ la serie ① converge solo per $x=0$

② $\sum_{k\geq 0} \frac{1}{k!} x^k$ converge (puntualmente) $\forall x\in\mathbb{R}$

③ $\sum_{k\geq 1} \frac{1}{k} x^k$

$$\left|\frac{1}{k}x^k\right| = \frac{1}{k}|x|^k \qquad \sqrt[k]{\frac{1}{k}|x|^k} = \frac{1}{\sqrt[k]{k}}|x| \xrightarrow{?} L\in[0,1)$$

$\Rightarrow |x|<1$ ho convergenza puntuale per $x\in(-1,1)$

$x=1$ ③ $\Rightarrow \sum_{k\geq 1}\frac{1}{k} = +\infty$

$x=-1$ ③ $\Rightarrow \sum_{k\geq 1}\frac{(-1)^k}{k}$ converge per Leibniz

$\Rightarrow$ ③ converge puntualmente per $x\in[-1,1)$

④ $\sum_{k\geq 0} x^k = \begin{cases} \frac{1}{1-x} & \text{per } x\in(-1,1) \\ \text{non converge} & \text{per } x\in(-\infty,-1]\cup[1,+\infty) \end{cases}$

DEF. $\sum_{k\geq k_0} a_k x^k$ converge

I. **puntualmente** per $x\in(a,b)$ se $\left|\sum_{k\geq k_0} a_k x^k\right| < +\infty \quad \forall x\in(a,b)$

$$\left(\lim_{N\to+\infty} \sum_{k\geq k_0}^{N} a_k x^k = L\in\mathbb{R}\right)$$

II. **assolutamente** per $x\in(a,b)$ se $\sum_{k\geq k_0} |a_k x^k| < +\infty \quad \forall x\in(a,b)$

III. **uniformemente** per $x\in(a,b)$ se

$$\lim_{N\to+\infty} \sup_{x\in(a,b)} \left|\sum_{k\geq N} a_k x^k\right| = 0 \quad \left(\lim_{N\to+\infty} \left\|\sum_{k\geq N} a_k x^k\right\|_{\infty,(a,b)} = 0\right)$$

IV. **totalmente** per $x\in(a,b)$ se $\sum_{k\geq k_0} \|a_k x^k\|_{\infty,(a,b)} < +\infty$

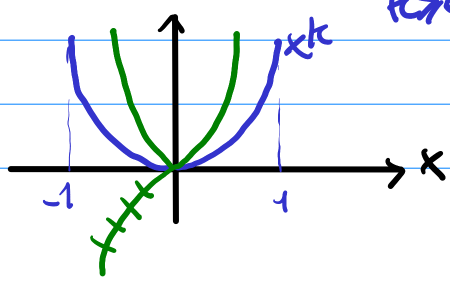
ESEMPIO

$\sum_{k\geq 0} x^k \qquad a_k = 1 \quad \forall k\in\mathbb{N}$

se $x\in(-1,1)$ $\quad \sum_{k\geq 0} x^k = \frac{1}{1-x}$ converge puntualmente

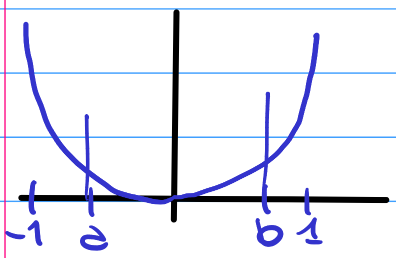
se $x\in(-1,1)$ $\quad \sum_{k\geq 0} |x^k| = \sum_{k\geq 0} |x|^k = \frac{1}{1-|x|}$ converge assolutamente

se $x\in(-1,1)$ $\quad \sum_{k\geq 0} \|x^k\|_{\infty,(-1,1)} = \sum_{k\geq 0} 1 = +\infty$



$\|x^k\|_{\infty,(-1,1)} = \sup_{x\in(-1,1)} |x^k| = \sup_{x\in(-1,1)} |x|^k = 1$

se $x \in [a,b] \subseteq (-1,1)$ $\quad a = -1+\delta_1$, $b = 1-\delta_2$ $\quad \delta_1, \delta_2 > 0$



$$\|x^k\|_{\infty,[a,b]} = \max\{|a|^k, |b|^k\} = |a|^k$$
$|a|, |b| < 1$

$$\sum_{k \geq 0} \|x^k\|_{\infty,[a,b]} \leq \sum_{k\geq 0} |a|^k = \frac{1}{1-|a|} \quad \text{perché } |a| < 1$$

$\Rightarrow$ la serie geometrica converge totalmente in $[a,b] \subseteq (-1,1)$ $\quad (-1 < a < b < 1)$

$$\Big\| \sum_{k \geq N} x^k \Big\|_{\infty,[a,b]} = \sup_{x \in [a,b]} \Big| \sum_{k \geq N} x^k \Big| = \sup_{x\in[a,b]} \Big| \frac{x^N}{1-x} \Big|$$

$$\sum_{k\geq N} x^k = \sum_{j \geq 0} x^{j+N} = x^N \sum_{j\geq 0} x^j = x^N \cdot \frac{1}{1-x} = \frac{x^N}{1-x}$$
$j = k-N$, $k = j+N$

se $|a|, |b| < 1 \Rightarrow \Big\| \sum_{k\geq N} x^k \Big\|_{\infty,[a,b]} \xrightarrow[N\to+\infty]{} 0$

$\Big[\; S_N(x) = \sum_{k \geq k_0}^{N} a_k x^k$

$\forall \varepsilon > 0$: $\quad \|S_N - S_{N+j}\|_{\infty,[a,b]} \leq \varepsilon$ se $N \geq N_0(\varepsilon)$

$$\Big\| \sum_{k\geq N+1}^{N+j} a_k x^k \Big\|_{\infty,[a,b]} \leq \varepsilon \quad \text{se } N \geq N_0(\varepsilon) \quad \forall j \in \mathbb{N}$$

$$\Big\| \sum_{k \geq N+1} a_k x^k \Big\|_{\infty,[a,b]} \leq \varepsilon \;\Big]$$

PROP.

se $\sum_{k\geq k_0} a_k x^k$ converge per $x = \xi \in \mathbb{R}$ allora

$\sum_{k \geq k_0} a_k x^k$ converge per ogni $x \in (-|\xi|, +|\xi|)$

$$\sup_{s\in\mathbb{R}}\left\{ \sum_{k\geq k_0} a_k x^k \text{ converge } \forall x\in(-|s|,+|s|)\right\} = R$$ raggio di convergenza

CRITERIO DI HADAMARD

Data $\sum_{k\geq k_0} a_k x^k$ sia $L := \limsup_{N\to+\infty} \sqrt[N]{|a_N|}$

allora posto $R := \begin{cases} 1/L & \text{se } L\in(0,+\infty) \\ +\infty & \text{se } L=0 \\ 0 & \text{se } L=+\infty \end{cases}$

vale

i. la serie converge semplicemente e assolutamente in $(-R,R)$,
ii. la serie converge uniformemente e totalmente in ogni $K\subseteq(-R,R)$ compatto,
iii. la serie non converge per $x\in(-\infty,-R)\cup(R,+\infty)$.

$$|a_k x^k| = |a_k||x|^k \Rightarrow \sqrt[k]{|a_k||x|^k} = \sqrt[k]{|a_k|}\,|x| \longrightarrow L<1$$

$$\rightsquigarrow |x| < \frac{1}{\lim_{k\to+\infty}\sqrt[k]{|a_k|}}$$

ESEMPIO $\sum_{k\geq 0} a_k x^k$ con $a_k = \begin{cases} 2^k & k \text{ pari} \\ 3^k & k \text{ dispari} \end{cases}$

$$\Rightarrow \sqrt[k]{|a_k|} = \begin{cases} 2 & k \text{ pari} \\ 3 & k \text{ dispari} \end{cases} \Rightarrow \limsup_{k\to+\infty} \sqrt[k]{|a_k|} = 3$$

$$\Rightarrow R = \frac{1}{3}$$

OSSERVAZIONE

se $\exists \lim_{N\to+\infty}\left|\frac{a_{N+1}}{a_N}\right| = L = \lim_{N\to+\infty}\sqrt[N]{|a_N|}$

ESEMPI

$$\sum_{k\geq 0}\frac{1}{2^k}x^k=\sum_{k\geq 0}\Big(\underbrace{\frac{x}{2}}_{t}\Big)^k=\sum_{k\geq 0}t^k=\frac{1}{1-t}=\frac{1}{1-x/2}=\frac{2}{2-x}$$

$$\frac{1}{2^k}=a_k=2^{-k}\longrightarrow \sqrt[k]{|a_k|}=\sqrt[k]{2^{-k}}=(2^{-k})^{\frac{1}{k}}=\frac{1}{2}=2^{-1}\Rightarrow R=2$$

$\boxed{\text{la serie converge P.te A. per } x\in(-2,2)\text{, U.te T. se } x\in[a,b]\subseteq(-2,2)}$

se $x=2\Rightarrow \sum_{k\geq 0}\frac{1}{2^k}2^k=\sum_{k\geq 0}1=+\infty$ (non converge!)

se $x=-2\Rightarrow \sum_{k\geq 0}\frac{1}{2^k}(-2)^k=\sum_{k\geq 0}(-1)^k\cdot\frac{2^k}{2^k}=\sum_{k\geq 0}(-1)^k$ non converge

$$\sum_{k\geq 0}k2^kx^k=?$$

$$a_k=k2^k\Rightarrow \sqrt[k]{k2^k}=2\sqrt[k]{k}\xrightarrow[k\to+\infty]{}2\Rightarrow R=\frac{1}{2}$$

[H] $\Rightarrow$ convergenza P.te A. per $x\in\left(-\frac{1}{2},+\frac{1}{2}\right)$, convergenza U.te T. $x\in[a,b]\subseteq\left(-\frac{1}{2},\frac{1}{2}\right)$

$$\sum_{k\geq 1}\frac{1}{k^2}x^k$$

$$a_k=\frac{1}{k^2},\quad \sqrt[k]{\frac{1}{k^2}}\xrightarrow[k\to+\infty]{}1=L\Rightarrow R=1$$

[H] $\Rightarrow$ convergenza P.te A. $x\in(-1,1)$, convergenza U.te T. $x\in[a,b]\subseteq(-1,1)$

$x=-1$ $\left|\sum_{k\geq 1}\frac{(-1)^k}{k^2}\right|=\left|\sum_{k\geq 1}(-1)^kb_k\right|<+\infty$ $\left(\text{se } \begin{matrix}b_k\longrightarrow 0\\ 0\leq b_{k+1}\leq b_k\end{matrix}\right)$

$x=1$ $\sum_{k\geq 1}\frac{1}{k^2}<+\infty$

convergenza P.te A. $x\in[-1,1]$, conv. U.te T. $x\in[-1,1]$

$$\left\|\frac{1}{k^2}x^k\right\|_{\infty,[-1,1]} = \frac{1}{k^2} \Rightarrow \sum_{k\geq 1}\frac{1}{k^2} < +\infty$$

$$\sum_{k\geq 1}\frac{1}{k^2} = \sum_{k\geq 1}\|a_k x^k\|_{\infty,[-1,1]}$$

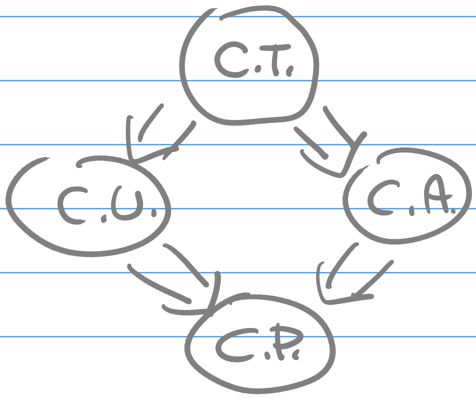
PROPOS.

I. Data $\sum_{k\geq k_0} a_k x^k$ se la serie converge totalmente in $[a,b]$ allora converge uniformemente in $[a,b]$.

II. Data $\sum_{k\geq k_0} a_k x^k$ se la serie converge totalmente in $[a,b]$ allora converge assolutamente in $[a,b]$.

III. Data $\sum_{k\geq k_0} a_k x^k$ se la serie converge uniformemente in $[a,b]$ allora converge puntualmente in $[a,b]$.

IV. Data $\sum_{k\geq k_0} a_k x^k$ se la serie converge assolutamente in $[a,b]$ allora converge puntualmente in $[a,b]$.

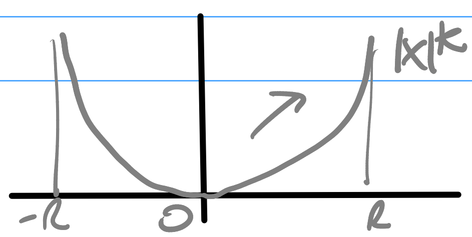


ESEMPIO.

$\sum_{k\geq 0} a_k x^k$ converge assolutamente in $[-R,R]$

$$|a_k x^k| = |a_k||x|^k \leftarrow |a_k||R|^k = |a_k||-R|^k$$

$$\sup_{[-R,R]} |x|^k = R^k$$



$$\text{C.A.} \Rightarrow \sum_{k\geq 0} |a_k| R^k < +\infty$$

$$\sum_{k\geq 0} \|a_k x^k\|_{\infty,[-R,R]} = \sum_{k\geq 0} |a_k| R^k \Rightarrow \text{C.T. in } [-R,R]$$

$\Rightarrow$ se ho convergenza assoluta in un intervallo chiuso $[a,b]$ ho anche convergenza totale in $[a,b]$

ESEMPIO

$$\sum_{k\geq 1} \frac{1}{x^k} \underset{t=1/x}{=} \sum_{k\geq 1} t^k \underset{t\in(-1,1)}{=} \frac{t}{1-t} = \frac{1/x}{1-1/x} = \frac{1}{x-1} = \sum_{k\geq 1} \left[\frac{1}{x}\right]^k$$

$$\left|\frac{1}{x}\right| < 1 \Rightarrow |x| > 1$$

$$\sum_{k\geq 0} x^{2k} = \sum_{j\geq 0} a_j x^j \qquad a_j = \begin{cases} 0 & j = 2k+1 \\ 1 & j = 2k \end{cases}$$

$$\limsup_{j\to+\infty} \sqrt[j]{|a_j|} = \left\{\begin{matrix} 1 & j \text{ pari} \\ 0 & j \text{ dispari} \end{matrix}\right\} = 1 \Rightarrow \text{C.P./A. } x \in (-1,1)$$

$$\sum_{k\geq 0} x^{2k} \underset{x^2=t}{=} \sum_{k\geq 0} t^k = \frac{1}{1-t} = \frac{1}{1-x^2}$$

$$|t| < 1 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1 \quad (|x| < 1)$$

$$\sum_{k\geq 0} e^{kx} \underset{e^x=s}{=} \sum_{k\geq 0} s^k = \frac{1}{1-s} = \frac{1}{1-e^x}$$

$|s| < 1$

$e^x < 1$

$x < 0$

$$\text{C.T. o C.U.} \quad -1 < a \leq s \leq b < 1, \quad s \in [a,b] \subseteq (-1,1)$$

$$(\cancel{a \leq})\, e^x \leq b \Rightarrow x \leq \ln(b) < 0 = \ln(1)$$

$\sum_{k\geq 0} k x^k \qquad a_k = k \qquad \sqrt[k]{|a_k|} = \sqrt[k]{k} \xrightarrow[k\to+\infty]{} 1 \Rightarrow R = 1$

C.A. e P. per $x \in (-1,1)$, C.U. e T. in $[a,b] \subseteq (-1,1)$

$$\sum_{k\geq 0} k x^{k-1} \cdot x = x \sum_{k\geq 0} k x^{k-1} = x \sum_{\substack{k\geq 0 \\ k\geq 1}} \underbrace{(x^k)'}_{\frac{d}{dx}(x^k)} \overset{*}{=} x\left(\sum_{k\geq 1} x^k\right)'$$

$$= x\left(\frac{x}{1-x}\right)' = x \cdot \frac{1-x-x(-1)}{(1-x)^2} = \frac{x}{(1-x)^2} \underset{\forall x \in (-1,1)}{=} \sum_{k\geq 0} k x^k$$

TEOREMA. Sia $\sum_{k\geq k_0} a_k x^k$ con raggio di convergenza $R>0$

posto $F(x) = \sum_{k\geq k_0} a_k x^k$ vale che $F \in C^\infty(-R,R)$ e

$$F^{(j)}(x) = \sum_{k\geq j} a_k \cdot k(k-1)\cdot\ldots\cdot(k-j+1)\, x^{k-j} = \sum a_k (x^k)^{(j)}$$

per $x \in (-R,R)$

$$\left[F^{(j)}(x) = \frac{d^j}{dx^j} F(x)\right]$$

TEOREMA. Sia $\sum_{k\geq k_0} a_k x^k = F(x)$ con $R>0$, allora vale

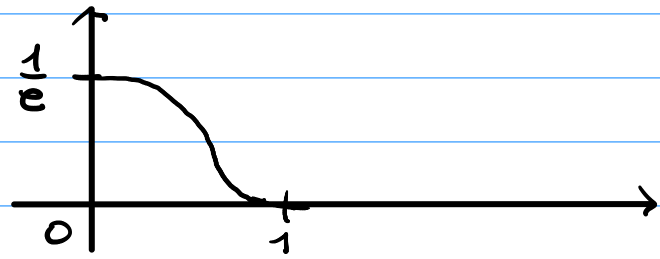
$$\int_a^b F(x)\,dx = \sum_{k\geq k_0} a_k \int_a^b x^k\,dx, \quad [a,b] \subseteq (-R,R)$$

OSSERVAZIONE

L'insieme delle funzioni che "coincidono" con il loro sviluppo in serie di Taylor è detto $C^\omega(-R,R) \leftarrow$ funzioni analitiche

$$C^\omega(-R,R) \subsetneq C^\infty(-R,R)$$

$$f(x)=\begin{cases} e^{-\frac{1}{1-x^2}} & |x|<1 \\ 0 & |x|\geq 1\end{cases}$$



$$f'(x)= e^{-\frac{1}{1-x^2}}\cdot\left(-\frac{-2x}{(1-x^2)^2}\right)$$

$$= \frac{x}{(1-x^2)^2}\, e^{-\frac{1}{1-x^2}} \qquad |x|<1 \quad \Rightarrow \quad \lim_{x\to 1^-} f'(x)=0$$

$$\Rightarrow \quad f\in C^\infty(\mathbb{R}), \qquad f(1)=0$$

$$f'(1)=0 \;\dots\; f^{(j)}(1)=0$$

$$f(x) \underset{\neq}{=} \sum_{k\geq 0} \underset{\overset{\shortparallel}{0}}{a_k}\,(x-1)^k = 0 \qquad a_k = \frac{f^{(k)}(x_0)}{k!}$$