AM1 2025-12-22 Esercitazione

Thursday, April 10, 2025 17:00

$$\int \frac{2\cos x + 6 - 3\sin^2 x}{(2\cos x + \sin x + 2)\sin^2 x}\, dx = (*)$$

$$\boxed{t = \operatorname{tg}\frac{x}{2}} \Longrightarrow \frac{x}{2} = \operatorname{arctg} t + k\pi \qquad x \neq \pi + 2k\pi \quad k \in \mathbb{Z}$$

$$\Rightarrow x = 2\operatorname{arctg} t + 2k\pi \Rightarrow dx = \frac{2\,dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2} \quad , \quad \sin x = \frac{2t}{1+t^2}$$

$$(*) = \int \frac{2\frac{(1-t^2)}{1+t^2} + 6 - \frac{3\cdot 4t^2}{(1+t^2)^2}}{\left(2\frac{(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} + 2\right)\frac{4t^2}{(1+t^2)^{\cancel{2}}}} \; \frac{2\,dt}{\cancel{1+t^2}} =$$

$$= \frac{1}{2}\int \frac{(1-t^2)(1+t^2) + 3(1+t^2)^2 - 6t^2}{[(1-\cancel{t^2}) + t + 1 + \cancel{t^2}]\; t^2}\, dt =$$

$$= \frac{1}{2}\int \frac{1 - t^4 + 3 + 3t^4 + \cancel{6t^2} - \cancel{6t^2}}{(t+2)\,t^2}\, dt =$$

$$= \frac{1}{2}\int \frac{2t^4 + 4}{(t+2)t^2}\, dt = \int \frac{t^4+2}{(t+2)t^2}\, dt =$$

Faccio la divisione

$$\begin{array}{lll|l} t^4 & & +2 & t^3 + 2t^2 \\ \hline -t^4 & -2t^3 & & t-2 \end{array}$$

$$
\begin{array}{l|l}
\underline{-t^4 \quad -2t^3} & \overline{t-2} \\
/ \quad -2t^3 \qquad +2 & \\
\underline{\quad 2t^3 + 4t^2} & \\
/ \qquad 4t^2+2 &
\end{array}
$$

$$= \int \frac{(t^4+2)\,dt}{(t+2)t^2} = \int \left(t-2 + \frac{4t^2+2}{(t+2)t^2}\right) dt =$$

$$= \frac{t^2}{2} - 2t + \int \left(\frac{A}{t+2} + \frac{B}{t} + \frac{C}{t^2}\right) dt = (*)$$

Cerco A, B, C:

$$\frac{4t^2+2}{(t+2)t^2} = \frac{A}{t+2} + \frac{B}{t} + \frac{C}{t^2}$$

$$4t^2+2 = At^2 + B(t^2+2t) + C(t+2)$$

$$\begin{cases} 4 = A + B \\ 0 = 2B + C \\ 2 = 2C \end{cases} \Rightarrow \boxed{C = 1}$$

$$2B = -C = -1 \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\boxed{A = 4 - B = 4 + \frac{1}{2} = \frac{9}{2}}$$

$$(*) = \frac{t^2}{2} - 2t + \frac{9}{2}\int \frac{dt}{t+2} - \frac{1}{2}\int \frac{dt}{t} + \int \frac{dt}{t^2} =$$

$$= \left(\frac{t^2}{2} - 2t + \frac{9}{2}\log|t+2| - \frac{1}{2}\log|t| - \frac{1}{t} + c_1 \right)\Bigg|_{t=\operatorname{tg}\frac{x}{2}}$$

Data $$f(x) = \begin{cases} \dfrac{\sin(2x)}{x} & \text{se} \quad x<0 \\[2ex] a(x^5+1)+bx^2 & \underline{\text{se} \quad x\geq 0,} \end{cases}$$

Trovare tutti i valori $a, b \in \mathbb{R}$ t.c.

a) f è continua in $\mathbb{R}$

b) f è derivabile in $\mathbb{R}$

c) f def^{te} crescente $\underline{\text{per } x \to +\infty}$

d) f ha un flesso in $x_0 = 1$

e) f ammette max. locale in $x_0 = 1$.

a) Pb. di continuità solo per $x_0 = 0$.

Ovviamente f è continua da destra in $x_0 = 0$.

Per la continuità da sinistra deve essere

$$\lim_{x\to 0^-} f(x) = \underset{\underset{a}{\parallel}}{f(0)}$$

$$\parallel$$

$$\lim_{x\to 0^-} \underset{\underset{1}{\downarrow}}{\boxed{\frac{\sin(2x)}{2x}}} \cdot 2 \quad \underset{2}{\parallel}$$

f continua $\Leftrightarrow$ $a=2$, b qualsiasi $\square$

b) f derivabile? Pb. solo in $x_0=0$.

f deve essere continua $\Rightarrow \boxed{a=2}$

Per $x \geq 0$ $\quad f(x) = 2(x^5+1) + b x^2$

$$f'(x) = 10x^4 + 2bx \Rightarrow f'_+(0) = 0$$

Quanto vale $f'_-(0)$

1° modo $\quad f'(x) = \left(\frac{\sin(2x)}{x}\right)' = \frac{2\cos(2x)x - \sin(2x)}{x^2}$ per $x<0$

Siccome f è continua, posso fare $\lim_{x \to 0^-} f'(x)$

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \frac{2\cos(2x)x - \sin(2x)}{x^2} =$$

$$= \lim_{x \to 0^-} \frac{2x(\cancel{1}+o(x)) - \cancel{2x} + o(x^2)}{x^2} = \lim_{x \to 0^-} \frac{o(x^2)}{x^2} = 0$$

$$\cos t = 1 - \frac{t^2}{2} + o(t^2) \qquad t \to 0$$

$$\cos(2x) = 1 - \frac{4x^2}{2} + o(x^2) = 1 + o(x) \qquad x \to 0$$

$$\sin(2x) = 2x + o(x^2) \qquad "$$

$f'_-(0) = 0 = f'_+(0) \Rightarrow$ f derivabile e $f'(0)=0$

Risp. $\quad a=2$, b qualsiasi.

In alternativa

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} =$$

$$= \lim_{x\to 0^-} \frac{\frac{\sin(2x)}{x} - 2}{x} = \lim_{x\to 0^-} \frac{\sin(2x)-2x}{x^2} =$$

$$\sin(2x) = 2x + o(x^2)$$

$$= \lim_{x\to 0^-} \frac{\cancel{2x} + o(x^2) \cancel{-2x}}{x^2} = 0 \Rightarrow$$ stessa conclusione.

c) f def^{te} crescente per $x \to +\infty$

$f(x) = a(x^5+1) + b\,x^2$ def^{te} crescente per $x\to+\infty$

$f'(x) = 5ax^4 + 2bx \geq 0$ def^{te} per $x \to +\infty$

Se $a > 0$. $\Rightarrow \lim_{x\to+\infty} f'(x) = +\infty \Rightarrow f'(x) > 0$ def^{te} per $x\to+\infty$

$\Rightarrow f$ def^{te} crescente per $x \to +\infty$

Se $a<0 \Rightarrow \lim_{x\to+\infty} f'(x) = -\infty \Rightarrow f'(x) < 0$ def^{te} per $x\to+\infty$

non va bene

se $a = 0 \Rightarrow f'(x) = 2bx \geq 0$ def^{te} per $x \to +\infty$

se e solo se $b \geq 0$

Risp. alla domanda c) $a>0$, b qualsiasi

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oppure $a=0$, $b\geq 0$.

d) f ha un flesso in $x_0=1$.

Intorno a $x_0=1$ $f(x)=a(x^5+1)+bx^2$

$$f'(x)=5ax^4+2bx$$

$$f''(x)=20ax^3+2b$$

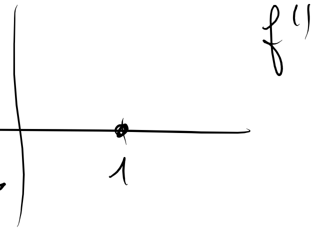
Impongo $f''(1)=0 \iff 20a+2b=0 \iff \boxed{b=-10a}$

$$f(x)=a(x^5+1)-10ax^2$$

Questo non basta a dire che $x_0=1$ è un flesso.

$f'''(x)=60ax^2 \Rightarrow f'''(1)=60a$

Se $a>0$ $\left.\begin{matrix} f''(1)=0 \\ f'''(1)>0 \end{matrix}\right| \Rightarrow$ f'' cambia segno attraversando $x_0=1$



$\Rightarrow$ flesso.

se $a<0$ $\left.\begin{matrix} f''(1)=0 \\ f'''(1)<0 \end{matrix}\right| \Rightarrow$ stessa conclusione

se $a=0$, $b=-10a \Rightarrow a=b=0 \Rightarrow f\equiv 0$ vicino a 1.

$\Rightarrow$ tutti i pti sono flessi.

Risposta alla domanda d) a qualsiasi, $b=-10a$.

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In alternativa, dopo aver imposto $f''(1)=0 \iff b=-10a$

$$f''(x) = 20\,a\,x^3 - 20\,a = 20\,a\,(x^3-1) =$$

$$= 20\,a\,\underbrace{(x-1)}_{\text{cambia segno attraversando } x=1}\underbrace{(x^2+x+1)}_{\text{sempre } \underset{0}{\vee}}$$

cambia segno attraversando 1 a parte il caso $a=b=0$ in cui f è costante.

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e) f ammette max locale in $x_0=1$.

$f(x) = a(x^5+1)+bx^2$ vicino a $x_0=1$

$f'(x) = 5ax^4+2bx$

Fermat $\Rightarrow$ deve essere $\underset{5a+2b}{\overset{}{\underset{\shortparallel}{f'(1)}}}=0$

$$\boxed{b=-\frac{5}{2}a}$$

$f''(x) = 20\,a\,x^3-5a \qquad f''(1) = 20\,a-5a = 15a.$

Se $a>0 \Rightarrow \left.\begin{array}{l} f'(1)=0, \\ f''(1)>0 \end{array}\right| \Rightarrow$ $x=1$ punto di min locale stretto.

se $a<0 \Rightarrow \left.\begin{array}{l} f'(1)=0 \\ f''(1)<0 \end{array}\right| \Rightarrow$ max. locale stretto

$$f''(1)<0 \quad |$$

Se $a=0 \Rightarrow b=0 \Rightarrow f\equiv 0$ per $x>0$

tutti i pti sono max

Solut^{ne} $\boxed{a\le 0\,,\quad b=-\frac{5}{2}a}$

$$\int \operatorname{arctg}(4\sqrt{x}-1)\,dx =$$

$4\sqrt{x}-1=t$

$4\sqrt{x}=t+1$

$16x=(t+1)^2$

$x=\frac{(t+1)^2}{16}$

$dx=\frac{2}{16}(t+1)dt=\frac{t+1}{8}dt$

$$=\frac{1}{8}\int (t+1)\operatorname{arctg} t\,dt =$$

per parti

$f'(t)=t+1 \Rightarrow f(t)=\frac{(t+1)^2}{2}$

$g(t)=\operatorname{arctg} t \Rightarrow g'(t)=\frac{1}{1+t^2}$

$$=\frac{1}{8}\left[\frac{(t+1)^2}{2}\operatorname{arctg} t-\frac{1}{2}\int\frac{(t+1)^2}{1+t^2}dt\right]=$$

$$=\frac{1}{16}\left[(t+1)^2\operatorname{arctg} t-\int\frac{t^2+1+2t}{1+t^2}dt\right]=$$

$$= \frac{1}{16}\left[\quad '' \quad - \int \left(1 + \frac{2t}{1+t^2}\right) dt \right] =$$

$$= \frac{1}{16}\left[(t+1)^2 \operatorname{arctg} t - t - \log(1+t^2) + c \right] \Bigg|_{t=4\sqrt{x}-1} =$$

$$= \frac{1}{16}\left[16x \operatorname{arctg}(4\sqrt{x}-1) - 4\sqrt{x} - \log\left(1+(4\sqrt{x}-1)^2\right)\right] + c_1$$

In alternativa, integrare subito per parti

$$\int \operatorname{arctg}(4\sqrt{x}-1)\,dx = x \operatorname{arctg}(4\sqrt{x}-1) - \underbrace{\int \frac{x}{1+(4\sqrt{x}-1)^2}\,\frac{2}{\sqrt{x}}\,dx}$$

$$\sqrt{x} = t$$

$$\text{oppure } 4\sqrt{x}-1 = t$$

$$\int \arcsin\left(\sqrt[3]{2x+1}\right) dx =$$

$$\sqrt[3]{2x+1} = t$$

$$2x+1 = t^3$$

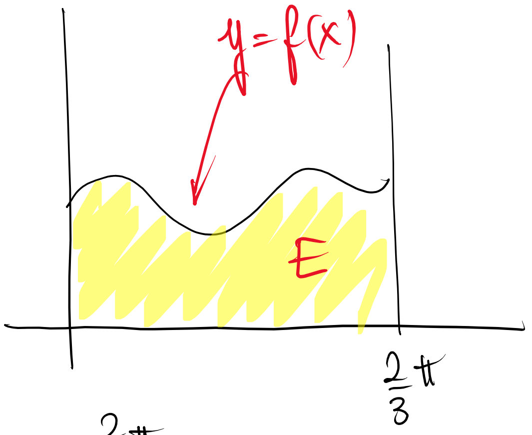
$$x = \frac{t^3-1}{2}$$

$$dx = \frac{3}{2} t^2\,dt$$

$$= \frac{3}{2}\int t^2 \arcsin t \, dt = \dots \text{ per parti}$$

ee

Calcolare l'area della regione limitata del 1° quadrante delimitata dagli assi coordinati, dalla retta $x=\frac{2\pi}{3}$ e dal grafico di $f(x)=\frac{1}{3-\cos x+2\sin x}$



$$\text{Area } E=\int_0^{\frac{2}{3}\pi}\frac{dx}{3-\cos x+2\sin x}=$$

$$t=\operatorname{tg}\frac{x}{2}\qquad \boxed{x\neq\pi(2k+1)}$$

$$dx=\frac{2\,dt}{1+t^2}$$

ok per il nostro $\int$

$$\cos x=\frac{1-t^2}{1+t^2},\quad \sin x=\frac{2t}{1+t^2}$$

$x=0 \qquad t=\operatorname{tg}0=0$

$x=\frac{2}{3}\pi \qquad t=\operatorname{tg}\frac{\pi}{3}=\sqrt{3}$

$$=\int_0^{\sqrt3}\frac{2\,dt}{3-\frac{1-t^2}{1+t^2}+\frac{4t}{1+t^2}}\quad\frac{\qquad}{1+t^2}=2\int_0^{\sqrt3}\frac{dt}{3+3t^2-1+t^2+4t}=$$

$$=2\int_0^{\sqrt3}\frac{dt}{4t^2+4t+2}=2\int_0^{\sqrt3}\frac{dt}{\underbrace{(4t^2+4t+1)}_{(2t+1)^2}+1}=$$

$$=\frac{\cancel{2}}{\cancel{2}}\operatorname{arctg}(2t+1)\Big|_0^{\sqrt3}=\operatorname{arctg}(2\sqrt3+1)-\frac{\pi}{4}$$

$$= \frac{\cancel{2}}{\cancel{2}} \quad \operatorname{arctg}(2t+1)\Big|_0 = \operatorname{arctg}(2\sqrt{3}+1) - \frac{\pi}{4}$$

$$\int \frac{dt}{2t^2+2t+1}$$

$$\parallel$$

$$\underset{\substack{\parallel \\ (\sqrt{2}t)^2}}{2t^2} + \underset{\substack{\parallel \\ 2(\sqrt{2}t)\frac{1}{\sqrt{2}}}}{2t} + \frac{1}{2} + \frac{1}{2} = \left(\sqrt{2}t + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} =$$

$$= \left(\frac{2t+1}{\sqrt{2}}\right)^2 + \frac{1}{2} =$$

$$= \frac{1}{2}\left[(2t+1)^2 + 1\right]$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \underbrace{\cos^4 x}_{\text{pari}} \, dx = 2\int_0^{\pi/2} \cos^4 x \, dx =$$

1° formule di bisezione $\quad \cos^2 x = \frac{1+\cos(2x)}{2}$

$$= 2\int_0^{\pi/2} \frac{(1+\cos(2x))^2}{4}\,dx = \frac{1}{2}\int_0^{\pi/2}\left(1+\cos^2(2x) + \underbrace{\cancel{2\cos(2x)}}_{\text{dà contributo nullo}}\right)dx$$

non dà contributo

$$= \frac{1}{2}\int_0^{\pi/2}\left(1 + \frac{1+\boxed{\cos(4x)}}{2}\right)dx = \frac{1}{2}\cdot\frac{3}{2}\,\frac{\pi}{2} = \frac{3}{8}\pi.$$

2° per parti

$$= 2\int^{\pi/2} \cos^3 \qquad = 2\left(\cancel{\cos^3 x \sin x}\Big|^{\pi/2} + 2\int^{\pi/2}\cos^2 x \sin^2 x\,dx\right) =$$

$$=2\underbrace{\int_0^{\pi/2}\cos^3 x\ \cos x\,dx} = 2\left(\cancel{\cos^3 x\sin x\Big|_0^{\pi/2}} + 3\int_0^{\pi/2}\cos^2 x\sin^2 x\,dx\right) =$$

$$f'(x)=\cos x \quad\Rightarrow\quad f(x)=\sin x$$

$$g(x)=\cos^3 x \quad\Rightarrow\quad g'(x)=-3\cos^2 x\sin x$$

$$=6\int^{\pi/2}\cos^2 x\,(1-\cos^2 x)\,dx = 6\int_0^{\pi/2}\cos^2 x\,dx \,-6\int_0^{\pi/2}\cos^4 x\,dx$$

$$\Rightarrow 8\int_0^{\pi/2}\cos^4 x\,dx = 6\int_0^{\pi/2}\cos^2 x\,dx$$

$$\Rightarrow 2\int_0^{\pi/2}\cos^4 x\,dx = \frac{3}{2}\boxed{\int_0^{\pi/2}\cos^2 x\,dx}$$

già fatto varie volte, anche per parti

$$\lim_{x\to 0}\frac{\overbrace{e^x-e^{\sin x}}^{\sim\frac{x^3}{6}}}{\underbrace{\operatorname{tg}x-x}_{\sim\frac{x^3}{3}}} = \left(\frac{0}{0}\right) = \frac{1/6}{1/3} = \frac{1}{2}$$

<u>den</u>.

$$\operatorname{tg}x-x = \left(\cancel{x}+\frac{x^3}{3}+o(x^3)\right)-\cancel{x} = \frac{x^3}{3}+o(x^3)\sim\frac{x^3}{3}$$

<u>Num</u>.

$$e^x-e^{\overset{\to 0}{\sin x}} = \cancel{1}+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3)\ +$$

$$\left.-\left(\cancel{1}+\sin x+\frac{\sin^2 x}{2}+\frac{\sin^3 x}{6}+\underset{o(x^3)}{\underset{\parallel}{o(\sin^3 x)}}\right)\right| =$$

$$\sin x = x-\frac{x^3}{6}+o(x^3)$$

$$= \cancel{x} + \overset{6}{\cancel{\frac{x^2}{2}}} + \cancel{\frac{x^3}{6}} + o(x^3) - \cancel{x} + \frac{x^3}{6} - \cancel{\frac{1}{2}}\cancel{(x^2)} - \cancel{\frac{1}{6}}x^3 =$$

$$= \frac{x^3}{6} + o(x^3) \sim \frac{x^3}{6}$$

In alternativa:

$$e^x - e^{\sin x} = \underbrace{e^x}_{\downarrow \, 1}\left(1 - e^{\sin x - x}\right) \sim$$

$$\sim 1 - e^{\sin x - x} = -\left(e^{\overset{\to 0}{\boxed{\sin x - x}}} - 1\right) \sim$$

$e^t - 1 \sim t$ per $t \to 0$

$$\sim -(\sin x - x) = x - \sin x = \cancel{x} - \left(\cancel{x} - \frac{x^3}{6} + o(x^3)\right) \sim$$

$$\sim \frac{x^3}{6}$$

Ordine di infinitesimo di

$$\left(\frac{1-\sqrt{1-x}}{-1+\sqrt{1+x}}\right)^{\alpha} - 1 \qquad \text{per } x \to 0^+ \quad (\alpha > 0)$$

OSS $$\frac{1-\sqrt{1-x}}{-1+\sqrt{1+x}} = \frac{(1-\sqrt{1-x})(1+\sqrt{1+x})}{(-1+\sqrt{1+x})(1+\sqrt{1+x})} = \frac{\qquad}{\cancel{1}+x-\cancel{1}}$$

$$= \frac{1 - \sqrt{1-x} + \sqrt{1+x} - \sqrt{(1-x^2)}}{x} =$$

$$= \frac{1 - \sqrt{1-x} + \sqrt{1+x} - \sqrt{(1-x^2)}}{x} =$$

$$= \frac{\cancel{1} - \left(\cancel{1} - \frac{x}{2} - \cancel{\frac{x^2}{8}} + o(x^2)\right) + \left(\cancel{1} + \frac{x}{2} - \cancel{\frac{x^2}{8}}\right) - \left(\cancel{1} - \frac{x^2}{2}\right)}{x} =$$

$$\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + o(t^2) \qquad t \to 0$$

$$= \frac{x + \frac{x^2}{2} + o(x^2)}{x} = 1 + \frac{x}{2} + o(x)$$

$$\left(\frac{1 - \sqrt{1-x}}{-1 + \sqrt{1+x}}\right)^{\alpha} - 1 = \Big(1 + \underbrace{\frac{x}{2} + o(x)}_{\downarrow\, 0}\Big)^{\alpha} - 1 \sim$$

$(1+t)^{\alpha} - 1 \sim \alpha t$ per $t \to 0$

$$\sim \alpha\left(\frac{x}{2} + o(x)\right) = \frac{\alpha x}{2} + o(x) \sim \frac{\alpha x}{2}$$

infinitesimo di ordine 1.