$$\int \frac{2}{\sqrt{2-4x^2}}\,dx = \frac{2}{\sqrt{2}}\int \frac{dx}{\sqrt{1-2x^2}} = \frac{\sqrt{2}}{\sqrt{2}}\arcsin\left(\sqrt{2}x\right) =$$

$$= \arcsin\left(\sqrt{2}\,x\right)$$

5) $$\int R\left(x, \sqrt{x^2+c}\right) dx \qquad c \in \mathbb{R}$$

Varie sost. possibili

$\boxed{\sqrt{x^2+c} = x+t}$, cioè $t = \sqrt{x^2+c} - x$

$$\cancel{x^2}+c = \cancel{x^2}+2xt+t^2$$

$$\boxed{x = \frac{c-t^2}{2t}} \Rightarrow \boxed{dx = \frac{-2t^2-(c-t^2)}{2t^2}dt = -\frac{t^2+c}{2t^2}dt}$$

$$\boxed{\sqrt{x^2+c} = \frac{c-t^2}{2t}+t = \frac{c+t^2}{2t}}$$

$$\int \frac{dx}{x+\sqrt{x^2+3}} = (*)$$

$$\sqrt{x^2+3} = x+t \Rightarrow \cancel{x^2}+3 = \cancel{x^2}+2xt+t^2$$

$$x = \frac{3-t^2}{2t} \qquad dx = \frac{-2t^2-3+t^2}{2\ t^2}dt =$$

$$= -\frac{t^2+3}{2t^2}dt$$

$$\sqrt{x^2+3} = \frac{3-t^2}{2t}+t = \frac{3+t^2}{2t}$$

$$(*) = \int \frac{\left(-\frac{t^2+3}{\cancel{2}t^{\cancel{2}}}\right)}{\frac{3-\cancel{t^2}}{\cancel{2t}}+\frac{3+\cancel{t^2}}{\cancel{2t}}}dt = -\int \frac{t^2+3}{6t}dt =$$

$$= -\frac{1}{6}\int\left(t+\frac{3}{t}\right)dt = -\frac{1}{6}\left(\frac{t^2}{2}+3\log t\right)$$

$$= -\frac{1}{6}\left[\frac{\left(\sqrt{x^2+3}-x\right)^2}{2} + 3\log\left(\sqrt{x^2+3}-x\right)\right] + C$$

$$\int \frac{dx}{x+\sqrt{x^2+3}} =$$

La presenza di $\sqrt{x^2+3}$ suggerisce un'altra sostituz.

$$x = \sqrt{3}\,\sinh t$$

$$\sqrt{x^2+3} = \sqrt{3\underbrace{(\sinh^2 t+1)}_{\cosh^2 t}} = \sqrt{3}\,\cosh t$$

$$dx = \sqrt{3}\,\cosh t\, dt$$

$$= \int \frac{\cancel{\sqrt{3}}\,\cosh t\, dt}{\cancel{\sqrt{3}}\,\sinh t+\cancel{\sqrt{3}}\,\cosh t} = \int \frac{e^t+e^{-t}}{e^t-\cancel{e^{-t}}+e^t+\cancel{e^{-t}}}\, dt =$$

$$= \int \left(\frac{e^t + e^{-t}}{2e^t} \right) dt = \int \left(\frac{1}{2} + \frac{e^{-2t}}{2} \right) dt =$$

$$= \frac{1}{2} t - \frac{1}{4} e^{-2t} + c =$$

$$= \frac{1}{2} \log\left(\frac{x + \sqrt{x^2+3}}{\cancel{\sqrt{3}}} \right) - \frac{1}{4} \frac{3}{\left(x + \sqrt{x^2+3}\right)^2} + c$$

$$t = \operatorname{settsinh} \frac{x}{\sqrt{3}} =$$

$$= \log\left(\frac{x}{\sqrt{3}} + \sqrt{\frac{x^2}{3} + 1} \right) =$$

$$= \log \frac{x + \sqrt{x^2+3}}{\sqrt{3}}$$

Integrale definito

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx =$$

$$= \int_{-1}^{\sqrt{3}-2} \frac{t^2-1}{2\cancel{t}}\left(+\frac{\cancel{2t}}{1+t^2}\right)\frac{t^2-1}{2t^2}\,dt$$

$$= \int_{-1}^{\sqrt{3}-2} \frac{(t^2-1)^2}{2t^2(1+t^2)}$$

divisione fratti semplici

1° modo

$$\sqrt{x^2-1}-x=t$$

$$-1 = 2xt+t^2$$

$$x = -\frac{1+t^2}{2t}$$

$$\sqrt{x^2-1} = t - \frac{1+t^2}{2t} = \frac{t^2-1}{2t}$$

$$dx = -\frac{2t^2-(1+t^2)2t}{2t^2}\,dt =$$

$$= -\frac{t^2-1}{2t^2}\,dt$$

2° modo

$$x = \cosh t$$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx =$$

$$\sqrt{x^2-1} = \sqrt{\cosh^2 t - 1} = \sinh t$$

$$dx = \sinh t\,dt$$

$$= \int \frac{(\sinh t)^2}{\cosh t}\,dt =$$

$$\int R(e^t)$$

$$\int \frac{(\sinh t)^2}{(\cosh t)^2}\cosh t\,dt =$$

$$\int\left(1 - \frac{1}{1+s^2}\right)ds$$

$$\sinh t = s$$

$$= \int \frac{\sinh^2 t}{(1+\sinh^2 t)}\underbrace{\cosh t\,dt}_{ds} = \int \frac{s^2}{(1+s^2)}\,ds$$

OSS se fosse stato

$$\int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x}\,dx$$

avremmo posto

$$x = -\cosh t$$

3° modo

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{2}\int_1^2 \frac{\sqrt{x^2-1}}{x^2}\,2x\,dx =$$

$x^2 = t \quad 2x\,dx = dt$

$$= \frac{1}{2}\int_1^4 \frac{\sqrt{t-1}}{t}\,dt = \int_0^{\sqrt{3}} \frac{s^2+1-1}{s^2+1}\,ds =$$

$\sqrt{t-1} = s$

$t = s^2+1$

$dt = 2s\,ds$

$$\int_0^{\sqrt{3}} \left(1 - \frac{1}{s^2+1}\right) = \sqrt{3} - \operatorname{arctg}\sqrt{3} = \sqrt{3} - \frac{\pi}{3}$$

<u>OSS</u> Attenzione: ci sono funzioni, anche molto semplici, le cui primitive non si possono scrivere in termini delle funz. elementari già note. Es.

$$\int e^{x^2}\,dx, \quad \int e^{-x^2}\,dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \sin(x^2)\,dx,$$

$$\int \frac{e^x}{x}\,dx$$

Queste primitive esistono. Una primitiva di e^{x^2} è $F(x) = \int_0^x e^{t^2}\,dt$ ma non si esprime in termini di altre funzioni elementari

Esempio

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

(error function)