Integrazione per sostituzione.

Per integrali indefiniti:

se $f: J \to \mathbb{R}$ continua, $\varphi: I \to J$ $C^1(I)$

$$\boxed{\int f(\underbrace{\varphi(t)}_{x})\underbrace{\varphi'(t)\,dt}_{dx} = \left(\int f(x)\,dx\right)_{x=\varphi(t)}}$$

uso questa

Per integrali definiti:

$$\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \left(\int f(\varphi(t))\,\varphi'(t)\,dt\right)\Bigg|_{t=\alpha}^{t=\beta} =$$

$$= \left(\left(\int f(x)\,dx\right)_{x=\varphi(t)}\right)\Bigg|_{t=\alpha}^{t=\beta} = \left(\int f(x)\,dx\right)\Bigg|_{x=\varphi(\alpha)}^{x=\varphi(\beta)} =$$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Cambiano gli estremi.

$x = \varphi(t)$ $\quad dx = \varphi'(t)\,dt$

$t = \alpha \Rightarrow x = \varphi(\alpha)$

$t = \beta \Rightarrow x = \varphi(\beta)$.

Riscrivo la formula:

$$\boxed{\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx}$$

dove φ è di classe $C^1([\alpha,\beta])$, f è continua in $[\varphi(\alpha), \varphi(\beta)]$

$$\int_0^1 \frac{e^t}{1+e^t}\,dt =$$

$x = 1+e^t$

$dx = e^t\,dt$

$t=0 \Rightarrow x = 2$

$t=1 \Rightarrow x = 1+e$

$$= \int_2^{1+e} \frac{dx}{x} = \log x \Big|_2^{1+e} = \log(1+e) - \log 2 = \log\left(\frac{1+e}{2}\right)$$

$$\frac{1}{2}\int_0^{1/\sqrt{2}} \frac{2t}{\sqrt{1-t^4}}\,dt =$$

$t^2 = x \Rightarrow 2t\,dt = dx$

$t=0 \Rightarrow x=0$

$t=\frac{1}{\sqrt{2}} \Rightarrow x = \frac{1}{2}$

$$= \frac{1}{2}\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$= \frac{1}{2} \arcsin x \Big|_0^{1/2} = \frac{1}{2}\arcsin\frac{1}{2} = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$$

A volte si può usare la formula di sostituzione nel verso opposto

$$\int f(x)\,dx \underset{x=\varphi(t)}{=} \int f(\varphi(t))\,\varphi'(t)\,dt$$ per $\int$ indefiniti

dove $\varphi(t) = x$

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$ per $\int$ definiti.

$\varphi : [\alpha,\beta]$ C^1 e invertibile e poniamo $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

$\alpha = \varphi^{-1}(a)$, $\beta = \varphi^{-1}(b)$

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt$$

$$\int \frac{dx}{\sqrt{x}+1} =$$

Poniamo $\sqrt{x} = t \Rightarrow x = t^2$

$\Rightarrow dx = 2t\,dt$

$$= \int \frac{2t}{t+1}\,dt = 2\int \frac{t+1-1}{t+1}\,dt = 2\int \left(1 - \frac{1}{t+1}\right)dt =$$

$$= 2\left(t - \log(t+1)\right) + c = 2\left(\sqrt{x} - \log\left(\sqrt{x}+1\right)\right) + c$$

Integrali di funzioni razionali (rapporti di polinomi).

$$\int \frac{P_n(x)}{Q_m(x)}\,dx \quad \text{dove } P_n \text{ e } Q_m \text{ sono polinomi.}$$

1ª oss Se $n \geq m$, posso sempre fare la divisione tra polinomi

$$P_n(x) = Q_m(x)\,S_{n-m}(x) + R_k(x) \qquad k < m$$

$$\Longrightarrow \frac{P_n(x)}{Q_m(x)} = S_{n-m}(x) + \frac{R_k(x)}{Q_m(x)} \quad \text{con } k < m.$$

$$\int \frac{x^3+2x+1}{x^2-x}\,dx = \int \left(x+1 + \frac{3x+1}{x^2-x}\right) =$$

$$= \frac{x^2}{2} + x + \int \frac{3x+1}{x^2-x}\,dx$$

$$\begin{array}{rl|l}
x^3 & +2x+1 & x^2-x \\ \cline{3-3}
-x^3 & +x^2 & x+1 \quad S_1(x) \\ \cline{1-2}
/ & x^2+2x+1 & \\
 & -x^2+x & \\ \cline{2-2}
/ & 3x+1 \quad R_1(x) &
\end{array}$$

$$\int \frac{3x+1}{x^2-x}\,dx = \int \frac{3x+1}{x(x-1)}\,dx = (*) \qquad x\neq 0,\ x\neq 1$$

Proviamo a scrivere $\dfrac{3x+1}{x(x-1)}$ nella forma

$$\frac{3x+1}{x(x-1)} = \frac{A}{x-1} + \frac{B}{x} = \frac{4}{x-1} - \frac{1}{x}$$

$$\Longleftrightarrow \quad 3x+1 = Ax + B(x-1)$$

grado 1
" 0

$$\begin{cases} 3 = A+B \\ 1 = -B \end{cases} \qquad \begin{aligned} &B = -1 \\ &A = 3-B = 3+1 = 4 \end{aligned}$$

$$(*) = \int \frac{4}{x-1}\,dx - \int \frac{1}{x}\,dx = 4\log|x-1| - \log|x| + c$$

$$= \log\frac{(x-1)^4}{|x|} + c.$$

Da questo momento considereremo $\displaystyle\int \frac{P_n(x)}{Q_m(x)}\,dx$ con $n<m$

(altrimenti: divisione)

$$\int \frac{1}{x^3-3x^2+2x}\,dx.$$

$$x^3-3x^2+2x = x\left(x^2-3x+2\right) = x(x-1)(x-2).$$

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x}$$

$$1 = A(x^2-x) + B(x^2-2x) + C(x^2-3x+2)$$

gr. 2 $\quad 0 = A+B+C \longrightarrow -B-2C=0$

gr. 1 $\quad 0 = -A-2B-3C \nearrow \quad \boxed{B=-2C=-1}$

" 0 $\quad 1 = 2C$

$$\boxed{C=\frac{1}{2}} \qquad \boxed{A=-B-C=1-\frac{1}{2}=\frac{1}{2}}$$

$$\int \frac{dx}{x^3-3x^2+2x} = \frac{1}{2}\int\frac{dx}{x-2} - \int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{x} =$$

$$= \frac{1}{2}\log|x-2| - \log|x-1| + \frac{1}{2}\log|x| + c_1$$

$$= \log\frac{\sqrt{|x-2|\,|x|}}{|x-1|} + c_1 .$$

Cosa succede se il polinomio a denominatore ha radici con molteplicità >1 ?

$$\int \frac{3x+1}{x^3-x^2}\,dx$$

$$x^3-x^2 = x^2(x-1)$$

Si cerca una scomposizione della forma

$$\frac{3x+1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$3x+1 = A(x^2-x) + B(x-1) + Cx^2$$

gr. 2 $\quad 0 = A + C$

$1 \quad 3 = -A + B$

gr. 0 $\quad 1 = -B$

$\boxed{B=-1}$ $\boxed{A=-4}$ $\boxed{C=4}$

$A = B - 3 = -4$

$C = -A$

$$\int \frac{3x+1}{x^3-x^2}\,dx = -4\int \frac{dx}{x} - 1\int \frac{dx}{x^2} + 4\int \frac{dx}{x-1} =$$

$$= -4\log|x| + \frac{1}{x} + 4\log|x-1| + c_1 =$$

$$= 4\log\left|\frac{x-1}{x}\right| + \frac{1}{x} + c_1 .$$

$$\int \frac{3x+1}{x^3(x-1)^2}\,dx$$

Scomposizione $\displaystyle\int \frac{3x+1}{x^3(x-1)^2} = \int \frac{A}{x} + \int \frac{B}{x^2} + \int \frac{C}{x^3} + \int \frac{D}{x-1} + \int \frac{E}{(x-1)^2} =$

Si trovano $A, B, C, D, E \dots$

$$= A\log|x| - \frac{B}{x} - \frac{C}{2x^2} + D\log|x-1| - \frac{E}{x-1} + c_1$$

Cosa succede se ci sono radici complesse coniugate del denom.?

$$\int \frac{3x^2+x}{x^4-1}\,dx$$

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

irriducibile nei reali (radici $= \pm i$)

Si fa una scomposizione del tipo

$$\int \frac{3x^2+x}{x^4-1} = \int \frac{A}{x-1} + \int \frac{B}{x+1} + \int \frac{2Cx+D}{x^2+1} =$$

si trovano A, B, C, D

$$= A\log|x-1| + B\log|x+1| + C\underbrace{\int \frac{2x}{x^2+1}\,dx}_{=} + D\underbrace{\int \frac{dx}{x^2+1}}_{= \operatorname{arctg} x} =$$

$x^2+1=t$
$2x\,dx = dt$

$$\int \frac{dt}{t} = \log|t| = \log(x^2+1)$$

$$= A\log|x-1| + B\log|x+1| + C\log(x^2+1) + D\operatorname{arctg} x + c_1$$

$$\int \frac{1}{(x+1)\underbrace{(x^2+x+1)}_{\text{irriducibile}}}\,dx$$

$\Delta = 1-4<0$

Si scompone così:

$$\int \frac{1}{(x+1)(x^2+x+1)} = \int \frac{A}{x+1} + \int \frac{B(2x+1)+C}{x^2+x+1} = (*)$$

$$1 = A(x^2+x+1) + B(2x+1)(x+1) + C(x+1)$$

$$\begin{array}{l|lll}
\text{gr. } 2 & 0 = & A + 2B & A = -2B \\
\text{" } 1 & 0 = & A + 3B + C & B + C = 0 \\
\text{" } 0 & 1 = & A + B + C & -B + C = 1
\end{array}$$

$$2C = 1 \Rightarrow \boxed{C = \tfrac{1}{2}}$$

$$2B = -1 \quad \boxed{B = -\frac{1}{2}}$$

$$\boxed{A = 1.}$$

$$(*) = \underbrace{A \int \frac{dx}{x+1}}_{A \log|x+1|} + \underbrace{B \int \frac{2x+1}{x^2+x+1}\,dx}_{B \log(x^2+x+1)} + C \int \frac{dx}{x^2+x+1} =$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\underbrace{x^2+x+\frac{1}{4}}_{\left(x+\frac{1}{2}\right)^2}+\frac{3}{4}} = \int \frac{dx}{\left(\frac{2x+1}{2}\right)^2 + \frac{3}{4}} =$$

$$= \frac{4}{3} \int \frac{dx}{\left(\frac{2x+1}{\cancel{2}} \frac{\cancel{2}}{\sqrt{3}}\right)^2 + 1} = \frac{4}{3} \int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} = \frac{4}{3} \int \frac{dx}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1}$$

$$= \frac{4}{3}\,\frac{\sqrt{3}}{2}\ \text{arctg}\,\frac{2x+1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\ \text{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\int \frac{dx}{(3x)^2+1} = \frac{1}{3}\,\text{arctg}\,(3x)$$

$$\int \frac{dx}{(ax+b)^2+1} = \frac{1}{a}\,\text{arctg}\,(ax+b)$$

Caso più complicato ancora: fattore irriducibile con molteplicità >1

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{(1+x^2)-x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{x^2+1} - \frac{1}{2}\int \frac{2x^2}{(x^2+1)^2}\,dx =$$

$$= \text{arctg}\,x - \frac{1}{2}\int x \cdot \frac{2x}{(x^2+1)^2}\,dx = \text{integro per parti}$$

$$f'(x) = \frac{2x}{(x^2+1)^2} \Rightarrow f(x) = \int \frac{2x}{(x^2+1)^2}\,dx \underset{\substack{x^2+1=t \\ 2x\,dx = dt}}{=} \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{x^2+1}$$

$$g(x) = x \Rightarrow g'(x) = 1$$

$$= \text{arctg}\,x + \frac{1}{2}\,\frac{x}{x^2+1} - \frac{1}{2}\underbrace{\int \frac{dx}{x^2+1}}_{\text{arctg}\,x} =$$

$$= \frac{1}{2}\left(\text{arctg}\,x + \frac{x}{x^2+1}\right) + c.$$

$$\boxed{\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\text{arctg}\,x + \frac{x}{x^2+1}\right) + c.}$$

$$\boxed{\int \frac{dx}{(1+x^2)^3}} = \int \frac{(1+x^2)-x^2}{(1+x^2)^3}\,dx = \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\int \frac{x\,2x}{(1+x^2)^3}\,dx$$

$$= \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\left[-\frac{1}{2}\frac{x}{(1+x^2)^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2}\right]$$

$$f'(x) = \frac{2x}{(1+x^2)^3} \Rightarrow f(x) = \int \frac{2x}{(1+x^2)^3}\,dx \underset{1+x^2=t}{=} \int \frac{dt}{t^3} = -\frac{1}{2t^2} =$$

$$g(x) = x \qquad g'(x) = 1 \qquad = -\frac{1}{2(1+x^2)^2}$$

$$= \frac{3}{4}\underbrace{\int \frac{dx}{(1+x^2)^2}}_{\frac{1}{2}\left[\operatorname{arctg} x + \frac{x}{1+x^2}\right]} + \frac{1}{4}\frac{x}{(1+x^2)^2} =$$

$$\boxed{= \frac{3}{8}\operatorname{arctg} x + \frac{3}{8}\frac{x}{1+x^2} + \frac{1}{4}\frac{x}{(1+x^2)^2}}$$

Esercizio per casa: trovare una formula iterativa per

$$I_n(x) = \int \frac{dx}{(1+x^2)^n}$$

$$I_n(x) = \frac{2n-3}{2n-2} I_{n-1}(x) + \frac{1}{2n-2}\frac{x}{(1+x^2)^{n-1}}$$

Esercizio riassuntivo

$$\int \frac{P_n(x)}{(5x-2)(x-1)^3\,(x^2+2x+17)^2}\,dx = (*) \qquad n \leq 7.$$

Si scompone così

$$(*) = \int \frac{A\,dx}{5x-2} + \int \frac{B\,dx}{x-1} + \int \frac{C}{(x-1)^2}\,dx + \int \frac{D}{(x-1)^3}\,dx +$$

$$+ \int \frac{E(2x+2)+F}{x^2+2x+17}\,dx + \int \frac{G(2x+2)+H}{(x^2+2x+17)^2}\,dx = (*)$$

Si trovano $A, B, \dots H$.

$$(*) = \frac{A}{5}\log|5x-2| + B\log|x-1| - \frac{C}{x-1} - \frac{D}{2(x-1)^2} +$$

$$+ E\log(x^2+2x+17) + F\int \frac{dx}{x^2+2x+17} +$$

$$+ G\int \frac{2x+2}{(x^2+2x+17)^2}\,dx + H\int \frac{dx}{(x^2+2x+17)^2}$$

Calcoliamo gli ultimi 3 integrali:

$$\int \frac{dx}{x^2+2x+17} = \int \frac{dx}{\underbrace{(x^2+2x+1)}_{(x+1)^2}+16} = \int \frac{dx}{(x+1)^2+16} =$$

$$= \frac{1}{16}\int \frac{dx}{\left(\frac{x+1}{4}\right)^2+1} = \frac{1}{16}\cdot 4\,\mathrm{arctg}\,\frac{x+1}{4} = \frac{1}{4}\,\mathrm{arctg}\,\frac{x+1}{4}$$

$$\int \frac{(2x+2)\,dx}{(x^2+2x+17)^2} = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{x^2+2x+17}$$

$x^2+2x+17=t$

$(2x+2)\,dx = dt$

$$\int \frac{dx}{(x^2+2x+17)^2} = \int \frac{dx}{((x+1)^2+16)^2} = \frac{1}{256}\int \frac{dx}{\left(\left(\frac{x+1}{4}\right)^2+1\right)^2} =$$

$+1+16$

$$\int \frac{dt}{(1+t^2)^2} = \frac{1}{2}\left[\operatorname{arctg} t + \frac{t}{1+t^2}\right]$$

$$= \frac{1}{256}\cdot 4 \quad \frac{1}{2}\left[\operatorname{arctg}\frac{x+1}{4} + \frac{\frac{x+1}{4}}{1+\frac{(x+1)^2}{16}}\right]$$

$$= \frac{1}{128}\operatorname{arctg}\frac{x+1}{4} + \frac{1}{128}\cdot\frac{1}{4}\,\frac{(x+1)\,16}{x^2+2x+17}$$

$$\int \frac{dx}{x^4+16}$$

Scompore x^4+16.

2 modi:

1) a occhio $\qquad x^4+16 = (x^4+8x^2+16) - 8x^2 =$

$$= (x^2+4)^2 - (2\sqrt{2}x)^2 = (x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4)$$

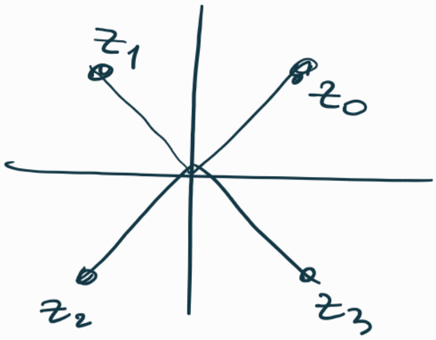
2) con i complessi $\quad z^4 = -16 = 16\, e^{i\pi}$

$$z = 2\, e^{i\theta_k} \qquad \theta_k = \frac{\pi + 2k\pi}{4} = \frac{\pi}{4} + k\frac{\pi}{2}$$

$k = 0, 1, 2, 3.$

$$z_0 = 2e^{i\frac{\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}\,(1+i)$$

$$z_1 = \sqrt{2}\,(-1+i)$$



$$z_2 = \overline{z_1} = \sqrt{2}\,(-1-i)$$

$$z_3 = \overline{z_0} = \sqrt{2}\,(1-i)$$

$$x^4 + 16 = \left((x-\sqrt{2}) - \sqrt{2}i\right)\left((x-\sqrt{2}) + \sqrt{2}i\right)\left(x + \sqrt{2} - \sqrt{2}i\right)\left(x + \sqrt{2} + \sqrt{2}i\right)$$

$$= \left((x-\sqrt{2})^2 - (\sqrt{2}i)^2\right) \qquad \left((x+\sqrt{2})^2 - (\sqrt{2}i)^2\right)$$

$$= \left(x^2 - 2\sqrt{2}x + 2 + 2\right) \qquad \left(x^2 + 2\sqrt{2}x + 2 + 2\right)$$

$$= \left(x^2 - 2\sqrt{2}x + 4\right)\left(x^2 + 2\sqrt{2}x + 4\right)$$

$$\int \frac{dx}{x^4+16} = \int \frac{dx}{\left(x^2 - 2\sqrt{2}x + 4\right)\left(x^2 + 2\sqrt{2}x + 4\right)} =$$

$$= \int \left[\frac{A(2x - 2\sqrt{2}) + B}{x^2 - 2\sqrt{2}x + 4} + \frac{C(2x + 2\sqrt{2}) + D}{x^2 + 2\sqrt{2}x + 4}\right] dx$$

trovo A, B, C, D.

$$= A \log\left(x^2 - 2\sqrt{2}x + 4\right) + B \int \frac{dx}{x^2 - 2\sqrt{2}x + 4} +$$

$$+ C \log\left(x^2+2\sqrt{2}x+4\right) + D \int \frac{dx}{x^2+2\sqrt{2}x+4}$$

$$\int \frac{dx}{x^2-2\sqrt{2}x+4} = \int \frac{dx}{x^2-2\sqrt{2}x+2+2} = \int \frac{dx}{(x-\sqrt{2})^2+2} =$$

$$= \frac{1}{2} \int \frac{dx}{\left(\frac{x-\sqrt{2}}{\sqrt{2}}\right)^2+1} = \frac{1}{2} \sqrt{2} \operatorname{arctg}\left(\frac{x-\sqrt{2}}{\sqrt{2}}\right)$$