Sapendo che $P(z) = z^4+4z^3+15z^2+8z+26$ ammette $z=\sqrt{2}i$ come zero, trovare gli altri zeri.

$P(z)$ è a coeff.ti reali, quindi le radici sono reali o complesse coniugate

$\Rightarrow \bar{z} = \overline{(\sqrt{2}i)} = -\sqrt{2}i$ è una radice

In particolare $P(z)$ ammette $(z-\sqrt{2}i)(z+\sqrt{2}i) = z^2+2$ come fattore

Divido $z^4+4z^3+15z^2+8z+26$ per $z^2+2$

$$
\begin{array}{rrrrr|l}
z^4 & +4z^3 & +15z^2 & +8z & +26 & z^2+2 \\
-z^4 & & -2z^2 & & & \hline
/ & 4z^3 & +13z^2 & +8z & +26 & z^2+4z+13 \\
 & -4z^3 & & -8z & & \\
\hline
 & / & 13z^2 & / & +26 & \\
 & & -13z^2 & & -26 & \\
\hline
 & & / & & / &
\end{array}
$$

$$P(z) = (z^2+2)(z^2+4z+13)$$

trovo gli zeri di questo

$$z = -2 \pm \sqrt{4-13} = -2 \pm \sqrt{-9} = -2 \pm 3i$$

Gli zeri sono $\sqrt{2}i$, $-\sqrt{2}i$, $-2-3i$, $-2+3i$.

---

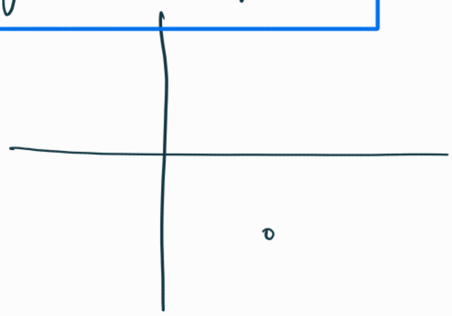
Trovare le radici quinte di $(1-i)^5$ e disegnarle nel piano;

$$1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$$

$$(1-i)^5 = 4\sqrt{2}\, e^{-i\frac{5\pi}{4}} =$$

Trovo le radici $5^e$.

$$z^5 = (1-i)^5$$



$$z_k = \sqrt{2}\, e^{i\theta_k} \qquad \theta_k = \frac{-\frac{5}{4}\pi + 2k\pi}{5} = -\frac{\pi}{4} + \frac{2k\pi}{5}$$

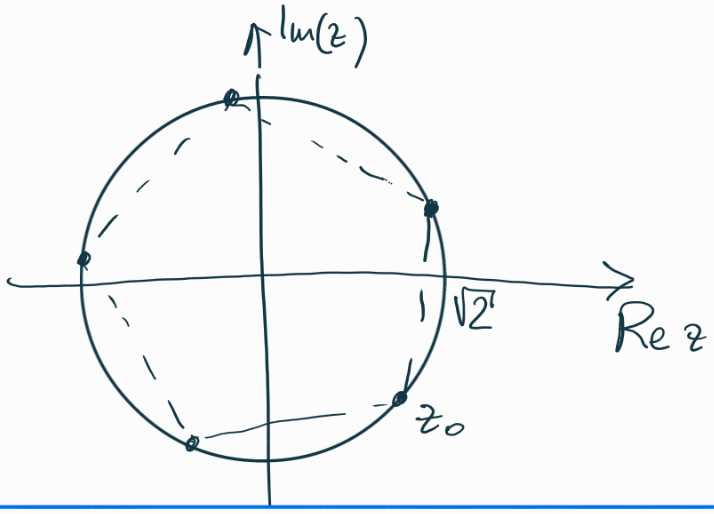
$$z_0 = \sqrt{2}\, e^{-i\frac{\pi}{4}} = 1 - i$$

$$z_1 = \sqrt{2}\, e^{i\left(-\frac{\pi}{4} + \frac{2}{5}\pi\right)} = \sqrt{2}\, e^{i\frac{3}{20}\pi} = \sqrt{2}\left(\cos\left(\frac{3}{20}\pi\right) + i\sin\left(\frac{3}{20}\pi\right)\right)$$

$$z_2 = \sqrt{2}\, e^{i\left(-\frac{\pi}{4} + \frac{4}{5}\pi\right)} = \sqrt{2}\, e^{i\frac{11}{20}\pi} = \sqrt{2}\left(\cos\left(\frac{11}{20}\pi\right) + i\sin\left(\frac{11}{20}\pi\right)\right)$$

$$z_3 = \sqrt{2}\, e^{i\frac{19}{20}\pi} = \ldots$$

$$z_4 = \sqrt{2}\, e^{i\frac{27}{20}\pi} = \_\_\_$$

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$$\boxed{z^3 = -8(z+i)^3} \qquad \text{oss. } z \neq -i$$

$$\Updownarrow$$

$$\frac{z^3}{(z+i)^3} = \underset{8e^{i\pi}}{-8} \qquad \text{Pongo } \boxed{w = \frac{z}{z+i}}$$

$$w^3 = -8 \implies \text{trovo } w_0, w_1, w_2. \implies w_0 = 2e^{i\frac{\pi}{3}} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + \sqrt{3}i$$

Poi risolvo $w = \frac{z}{z+i}$ rispetto a $z$.

$$(z+i)w = z \iff zw - z = -iw$$

$$z(w-1) = -iw$$

$$z = \frac{-iw}{w-1}$$

$$z_0 = \frac{-i\,w_0}{w_0 - 1} = \frac{-i(1+\sqrt{3}i)}{\cancel{1} + \sqrt{3}i - \cancel{1}} = \frac{(-i+\sqrt{3})(-\sqrt{3}i)}{\sqrt{3}i\cdot(-\sqrt{3}i)} = \frac{-\sqrt{3} - 3i}{3} =$$

$$= -\frac{\sqrt{3}}{3} - i$$

---

$$\boxed{z^2 - |z| + \frac{4z}{i} = 0}$$ facile, in coordinate cartesiane.

---

$$\boxed{\left|\frac{w-2}{w+i}\right| = 2.}$$ $w = x+iy.$

$$\frac{|x+iy-2|}{|x+iy+i|} = 2 \iff \frac{\sqrt{(x-2)^2+y^2}}{\sqrt{x^2+(y+1)^2}} = 2.$$

$$\Updownarrow$$

$$(x-2)^2 + y^2 = 4\left(x^2+(y+1)^2\right)$$

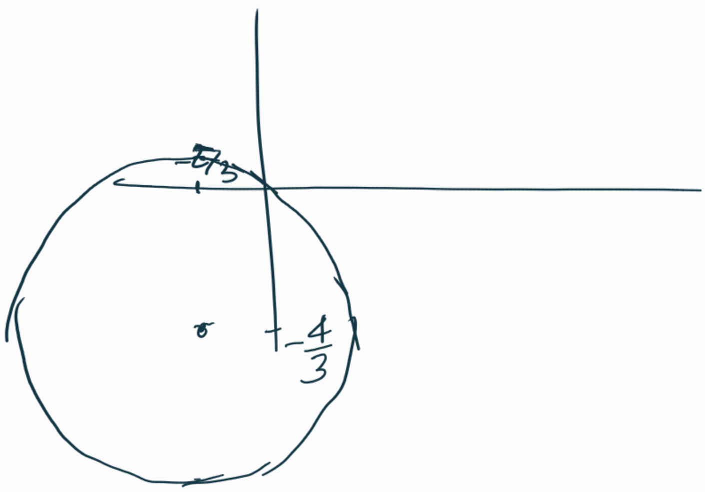
$$x^2 - 4x + \cancel{4} + y^2 = 4x^2 + 4y^2 + 8y + \cancel{4}$$

$$3x^2 + 3y^2 + 4x + 8y = 0$$

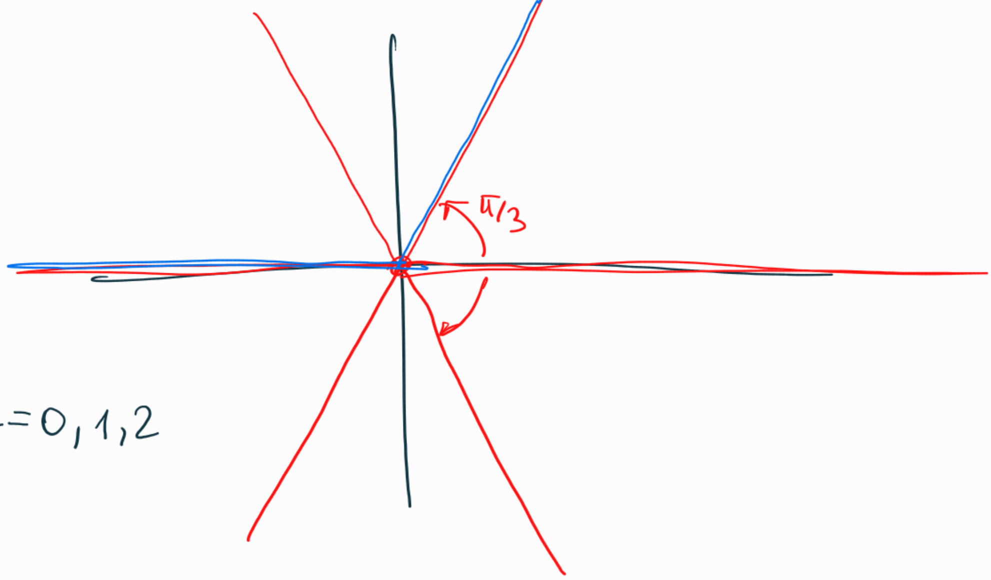
$$3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 3\left(y^2 + \frac{8}{3}y + \frac{16}{9}\right) = \frac{4}{3} + \frac{16}{3} = \frac{20}{3}$$

$$\left(x+\frac{2}{3}\right)^2 + \left(y+\frac{4}{3}\right)^2 = \frac{20}{9}$$

circonf. di centro $\left(-\frac{2}{3}, -\frac{4}{3}\right)$ e raggio $\frac{2\sqrt{5}}{3}$



$$\boxed{w^3 \in \mathbb{R}}$$




$$3(\arg w) = \pi + 2k\pi.$$

$$\arg w = \frac{\pi}{3} + \frac{2k\pi}{3} \qquad k = 0,1,2$$

oppure

$$3(\arg w) = 2k\pi$$

$$\arg w = \frac{2k\pi}{3} \qquad k = 0,1,2$$

Oppure in coord. cartesiane.

---

$$\boxed{i w^3 \in \mathbb{R}.}$$

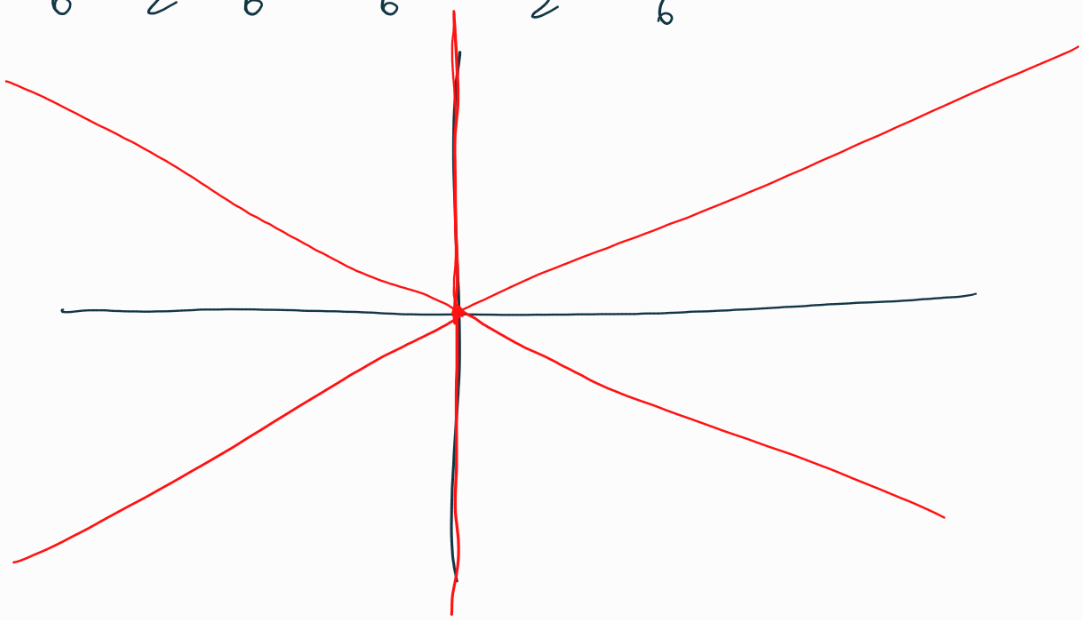
$$w^3 = \frac{\alpha}{i} = -\alpha i \qquad \alpha \in \mathbb{R} \qquad\qquad \frac{1}{i} = \frac{-i}{i(-i)} = -i$$

E' equivalente a richiedere $w^3$ = immaginario puro.

$$\underset{\substack{\| \\ 3 \arg w}}{\arg(w^3)} = \frac{\pi}{2} + k\pi \qquad \Rightarrow \qquad \arg w = \frac{\pi}{6} + \frac{k\pi}{3}$$

$$k = 0,1,2,3,4,5$$

$$\arg w = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{3}{2}\pi, \frac{11}{6}\pi.$$



In coordinate cartesiane $\quad w = x+iy$

$$i w^3 = i\,(x+iy)^3 = i\,(x^3+3x^2yi-3xy^2-iy^3) =$$

$$= ix^3-3x^2y-3xy^2i+y^3 \in \mathbb{R}$$

$$\Updownarrow$$

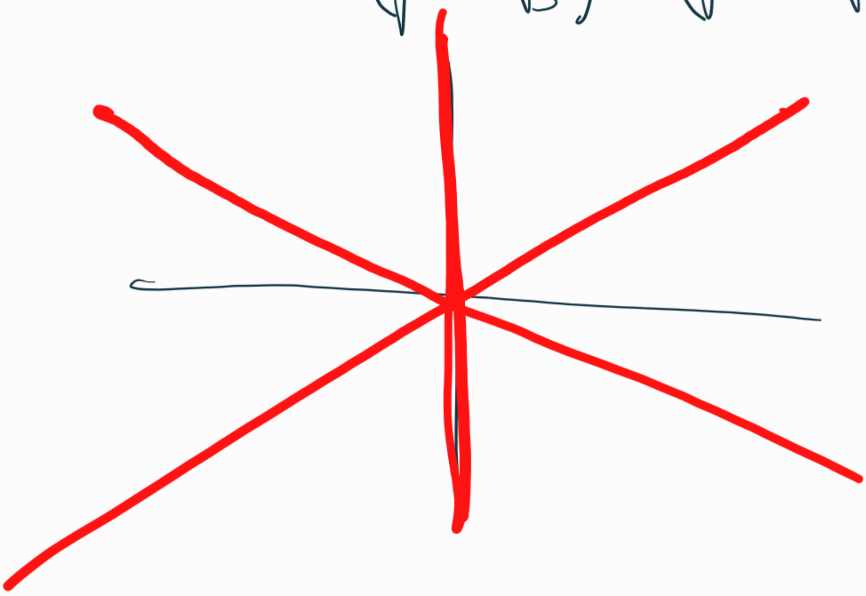
$$x^3-3xy^2=0$$

$$x(x^2-3y^2)=0$$

$$x(x-\sqrt{3}y)(x+\sqrt{3}y)=0$$

$$\Downarrow$$

$$(x=0)\vee\left(y=\frac{x}{\sqrt{3}}\right)\vee\left(y=-\frac{x}{\sqrt{3}}\right)$$



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## Proprietà degli integrali (continua)

Abbiamo visto (proprietà 1) che $\quad a<b$

$$(b-a)\inf_{[a,b]} f \le \int_a^b f(x)\,dx \le (b-a)\sup_{[a,b]} f$$

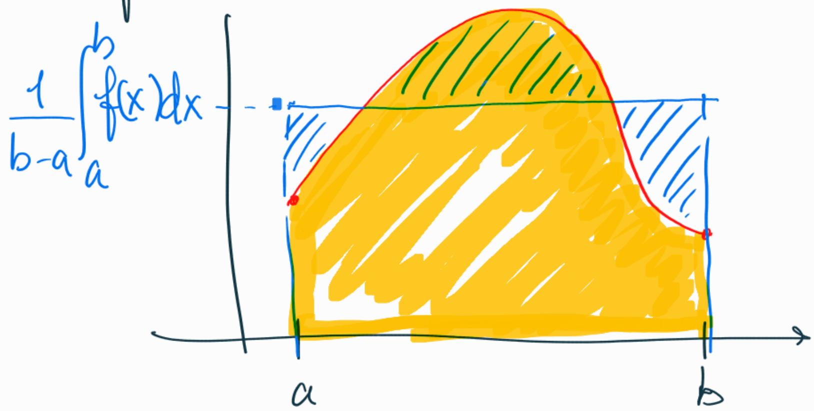
divido per b-a

$$\boxed{\inf_{[a,b]} f \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \sup_{[a,b]} f.} \qquad (*)$$

**Definiamo** $\boxed{\text{valor medio di } f \text{ su } [a,b]}$ la quantità

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

E' il valore costante che, integrato su $[a,b]$, fornisce lo stesso integrale di $f$.




## 10.b Conseguenza:

**Teorema della media integrale.**

Sia $f$ continua in $[a,b]$. ($\Rightarrow f$ integrabile in $[a,b]$)

($\Rightarrow$ vale la $(*)$ $\Rightarrow \min_{[a,b]} f \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \max_{[a,b]} f$ )

(poiché $f$ continua assume tutti i valori compresi tra min e max $\Rightarrow$)

$$\Rightarrow \exists\, c\in[a,b] \text{ t.c. } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

**DEF. Funzioni integrali.**

Sia $f\in R(a,b)$. Sia $c\in[a,b]$.
Definiamo la funzione integrale di $f$ relativa al punto $c$ come

$$F_c(x) = \int_c^x f(t)\,dt$$

# Esempi.

1) $f(x)=x^2, \quad c=0$

$$F_0(x)=\int_0^x t^2\,dt = \frac{x^3}{3} \qquad \forall x\geq 0$$

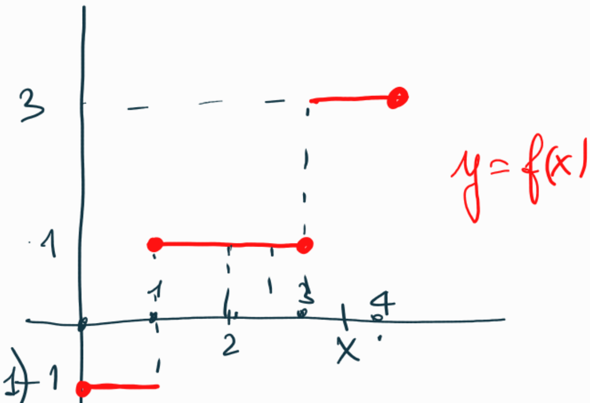
2) $f(x)=x^2 \quad c=1.$

$$F_1(x)=\int_1^x t^2\,dt=\int_0^x t^2\,dt-\int_0^1 t^2\,dt=\frac{x^3}{3}-\frac{1}{3}=$$

$$=\frac{x^3-1}{3}$$

3) Sia $f(x)=\begin{cases}-1 & \text{se } x\in[0,1)\\ 1 & \text{se } x\in[1,3]\\ 3 & \text{se } x\in(3,4]\end{cases}$

$c=2$




$$F_c(x)=\int_2^x f(t)\,dt=\begin{cases}-x & \text{se } x\in[0,1)\\ x-2 & \text{se } x\in[1,3]\\ 3x-8 & \text{se } x\in(3,4]\end{cases}$$

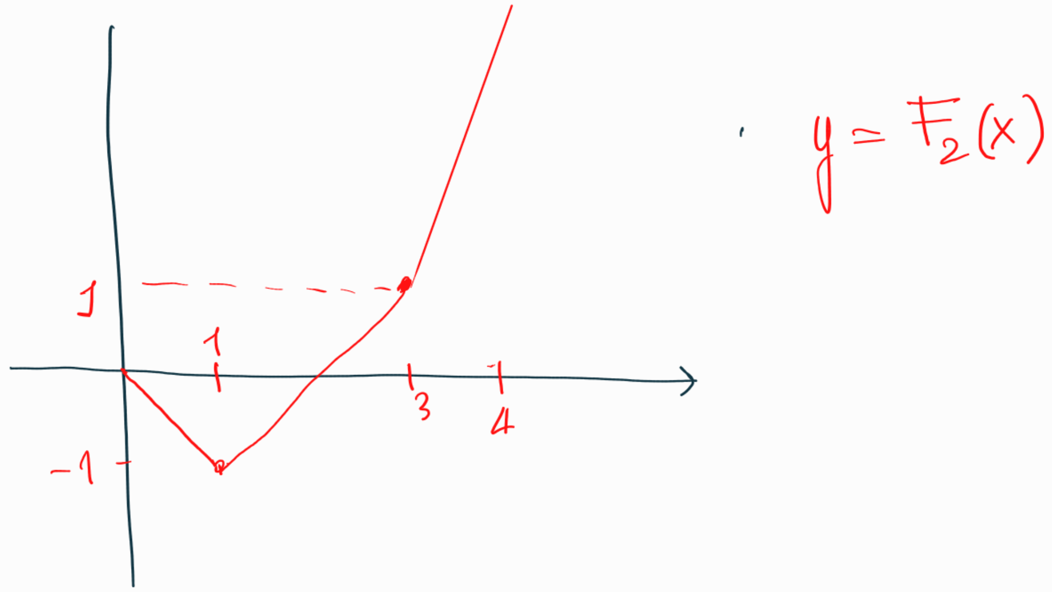
$x\in(3,4] \Rightarrow \displaystyle\int_2^x f(t)\,dt=\underbrace{\int_2^3 1\,dt}_{1(3-2)=1}+\int_3^x 3\,dt=$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3(x-3)=3x-9$

$x\in[0,1) \Rightarrow \displaystyle\int_2^x f(t)\,dt=\underbrace{\int_2^1 \underbrace{f(t)}_{1}\,dt}_{-1}+\int_1^x \underbrace{f(t)}_{-1}\,dt=-x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -1(x-1)=1-x$




In tutti questi esempi:

- $F_c(x)$ è continua.

- Nei punti in cui $f$ è continua, $F_c$ è derivabile,

  e si ha $F_c'(x) = f(x)$

Questi fatti sono <u>generali</u>:

**TEOREMA** (continuità della f. integrale)

Sia $f \in R(a,b)$. Sia $c \in [a,b]$. e sia

$$F_c(x) = \int_c^x f(t)\,dt \qquad \forall x \in [a,b].$$

Allora $F_c$ è continua in $[a,b]$.

<u>Dim</u>. Sia $x_0 \in [a,b]$. Devo provare che

$$\lim_{x \to x_0} F_c(x) = F_c(x_0)$$

$$F_c(x) - F_c(x_0) = \int_c^x f(t)\,dt - \int_c^{x_0} f(t)\,dt = \cancel{\int_c^{x_0} f(t)dt} + \int_{x_0}^x f(t)dt - \cancel{\int_c^{x_0} f(t)dt}$$

$$= \int_{x_0}^{x} f(t)\,dt.$$

$\boxed{\text{Primo caso: } x \to x_0^+}$

$$\Rightarrow \quad 0 \leq |F_c(x) - F_c(x_0)| = \left| \int_{x_0}^{x} f(t)\,dt \right| \leq \int_{x_0}^{x} |f(t)|\,dt \leq$$

dis. triang.

$$\leq \underbrace{\sup_{[a,b]} |f| \quad (x - x_0)}_{\substack{\downarrow \; x \to x_0^+ \\ 0}}$$

se $\quad x \to x_0^-$

$$0 \leq |F_c(x) - F_c(x_0)| = \left| \int_{x_0}^{x} f(t)\,dt \right| = \left| \int_{x}^{x_0} f(t)\,dt \right| \leq$$

dis. triang.

$$\leq \int_{x}^{x_0} |f(t)|\,dt \leq (x_0 - x) \sup_{[a,b]} |f(t)| \xrightarrow[x \to x_0^-]{} 0$$