03/11/2025

ESERCIZIO 3. *Trovare e classificare i punti critici di*

$$f(x_1, x_2) = x_1^3 x_2 - 2x_2^2 + 3x_1^2 x_2$$

Successivamente calcolare massimo e minimo assoluto di f(x, y) *in* $Q = \{(x, y) : |x| \leq 1, |y| \leq 1\}$.

$$Df(x) = (\partial_1 f(x), \partial_2 f(x)) = (3x_1^2 x_2 + 6x_1 x_2 ; x_1^3 - 4x_2 + 3x_1^2)$$

$$¿\ Df(x) = 0 \,? \quad \begin{cases} 3x_1 x_2 (x_1 + 2) = 0 \\ x_1^3 + 3x_1^2 - 4x_2 = 0 \end{cases}$$

$$x_1 = 0 \rightarrow -4x_2 = 0 \Rightarrow x_2 = 0 \Rightarrow O(0,0)$$

$$x_2 = 0 \rightarrow x_1^2(x_1+3) = 0 \begin{cases} \nearrow x_1 = 0 \\ \searrow x_1 = -3 \end{cases} \Rightarrow A(-3,0)$$

$$x_1 = -2 \rightarrow 4(1-x_2) = 0 \rightarrow x_2 = 1 \Rightarrow B(-2,1)$$

$$Hf(x) = \begin{pmatrix} 6x_2(x_1+1) & 3x_1(x_1+2) \\ 3x_1(x_1+2) & -4 \end{pmatrix}$$

$$Hf(O) = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \qquad Hf(A) = \begin{pmatrix} 0 & 9 \\ 9 & -4 \end{pmatrix}$$

$$Hf(B) = \begin{pmatrix} -6 & 0 \\ 0 & -4 \end{pmatrix}$$

$Hf(A) \rightarrow Tr = -4 \quad \det = -81 < 0 \Rightarrow Hf(A)$ indefinita
$\Rightarrow A$ punto di sella

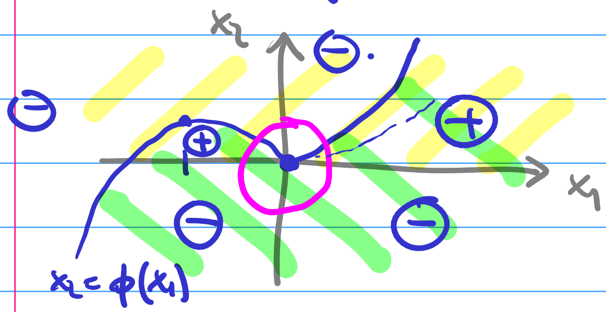
$Hf(B)$ definita negativa $\Rightarrow$ B punto di massimo locale

$Hf(O)$ semidefinita negativa $\Rightarrow$
- O punto di sella
- O punto di massimo locale

?

$f(0)=0$ [studio il segno di $g(x) = f(x) - f(0) = f(x)$]

$f(x) = x_2(x_1^3 - 2x_2 + 3x_1^2) \overset{?}{\gtrless} 0$



$x_2 \geq 0$

$x_2 \leq \frac{1}{2}(x_1^3 + 3x_1^2) = \frac{1}{2}x_1^2(x_1+3)$

$x_2 = \phi(x_1) = \frac{1}{2}x_1^2(x_1+3)$, $\phi'(x_1) = \frac{3}{2}x_1^2 + 3x_1 = \frac{3}{2}x_1(x_1+2)$

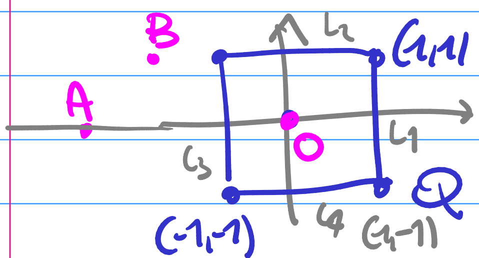
$\Rightarrow O(0,0)$ è un punto di sella

$f(0, x_2) = -2x_2^2 < 0$ se $x_2 \neq 0$

$f\left(\frac{1}{k+1}, \frac{1}{2k}\right) = \frac{1}{2k}\left(\frac{1}{(k+1)^3} - \frac{1}{2k-1} + \frac{3}{(k+1)^2}\right) > 0 \quad \forall k \geq k_0$

$\left(\frac{1}{k+1}, \frac{1}{2k}\right) \longrightarrow (0,0)$

II. $Q = \{ x = (x_1, x_2) : |x_1|, |x_2| \leq 1 \} \subseteq \mathbb{R}^2$

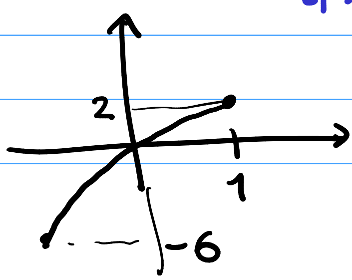


Thm. Weierstrass

$\Downarrow$

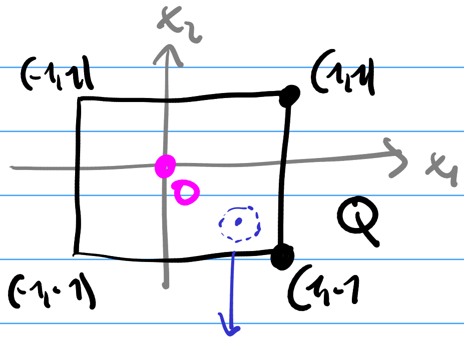
$\exists \max_Q(f), \min_Q(f)$

$L_1 : \{(1,t) : t \in [-1,1]\} \longrightarrow f(1,t) = 2t(2-t) = f_1(t) \; (= f|_{L_1})$
$t \in [-1,1]$



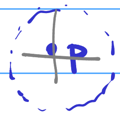
$f_1'(t) = 4 - 4t = 4(1-t) \geq 0$

$\Rightarrow \max(f_1) = 2 = f_1(1) = f(1,1)$
$\min(f_1) = -6 = f_1(-1) = f(1,-1)$



$$f(x_1,x_2)=x_2\left(x_1^3-2x_2+3x_1^2\right)$$

$O(0,0)$ punto di sella



$p+te_1$
$p+te_2$ $\quad t\in[-\varepsilon,\varepsilon]$

se p fosse un pto di massimo per f, allora

$$f(p)-f(p+te_1)\geq 0$$

$$\Rightarrow \frac{f(p+te_1)-f(p)}{t}\ \begin{cases} \leq 0 & \text{se } t>0 \\ \geq 0 & \text{se } t<0 \end{cases}$$

$$\Rightarrow \lim_{t\to 0}\frac{f(p+te_1)-f(p)}{t}=0=\partial_1 f(p)$$

ESEMPIO IMPORTANTE

$$f(x)=\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2 \qquad \lambda_1<\lambda_2<\lambda_3 \quad (\lambda_1<0<\lambda_3)$$

$$x\in\overline{B(0,1)}=\{x_1^2+x_2^2+x_3^2\leq 1\}\subseteq\mathbb{R}^3$$

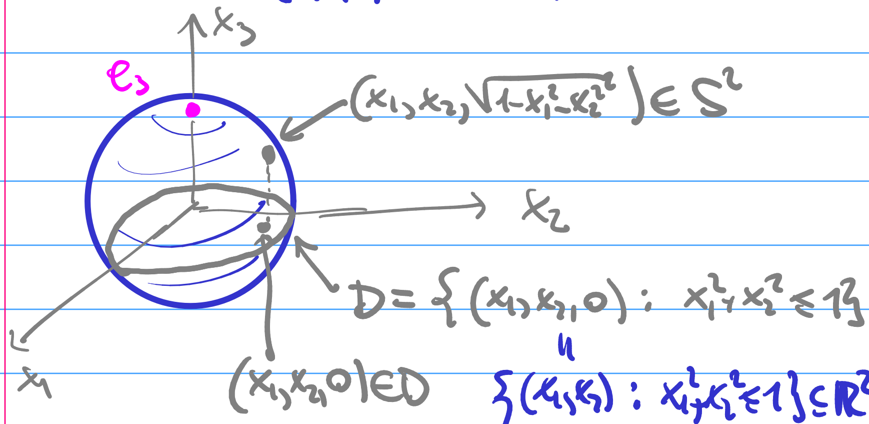
¿ $\max_{\overline{B}}(f)$, $\min_{\overline{B}}(f)$?

$Df(x)=0$, $Df(x)=2(\lambda_1 x_1,\lambda_2 x_2,\lambda_3 x_3)$

$$Hf(x)=2\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} \Rightarrow O \text{ punto di sella}$$

teor. Weierstrass $\Rightarrow \exists\, p, q$ punti di max/min e $p,q\in\partial\overline{B}=S^2=\{x_1^2+x_2^2+x_3^2=1\}$

osservo che $(0,0,1)\in S^2$ è punto di massimo assoluto di $f|_{S}$



$\{(x_1,x_2): x_1^2+x_2^2\le 1\}\subseteq\mathbb{R}^2$

$F: D\to (S^2\cap\{x_3\ge 0\})=S^2_+$

$\underset{\mathbb{R}^2}{(x_1,x_2)} \longrightarrow \underset{\mathbb{R}^3}{(x_1,x_2,\sqrt{\cdots})}$

$$H=f(F(x)): D\to\mathbb{R}$$
$$(x_1,x_2)\to \lambda_1x_1^2+\lambda_2x_2^2+\lambda_3(1-x_1^2-x_2^2)$$

$$D\xrightarrow{F}\mathbb{R}^3\xrightarrow{f}\mathbb{R}$$
$$(x_1,x_2)\longmapsto(x_1,x_2,\sqrt{1-x_1^2-x_2^2})\longmapsto\lambda_1x_1+\lambda_2x_2^2+\lambda_3(1-x_1^2-x_2^2)$$

$$\nabla H(x_1,x_2)=Jf(F(x))\cdot JF(x)=\nabla f(F(x))JF(x)$$

$$\nabla f(F(x))=2(\lambda_1x_1,\lambda_2x_2,\lambda_3\sqrt{1-x_1^2-x_2^2})$$

$$JF(x)=\begin{pmatrix}\partial_1F_1 & \partial_2F_1\\ \partial_1F_2 & \partial_2F_2\\ \partial_1F_3 & \partial_2F_3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\\ -\frac{x_1}{\sqrt{1-x_1^2-x_2^2}} & -\frac{x_2}{\sqrt{1-x_1^2-x_2^2}}\end{pmatrix}$$

$$\nabla H(0,0)=2(0,0,\lambda_3)\cdot\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}=(0,0)$$

i vettori del tipo $(0,0,t)$ sono il nucleo di $JF(0,0)$!

$G(x):=g(F(x))$ con $g(x)=x_1^2+x_2^2+x_3^2-1$

$$\Rightarrow G(\underset{\substack{\uparrow \\ (x_1,x_2)\in D}}{x}) = g(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) = x_1^2 + x_2^2 + (\sqrt{1-x_1^2-x_2^2})^2 - 1 \equiv 0$$

$$(0,0) = \nabla G(x) = \nabla g(F(x)) \cdot JF(x)$$

$$(0,0) = \nabla G(0,0) = \nabla g(0,0,1) \cdot JF(0,0)$$

$$= 2(0,0,1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\nabla H(0,0) = 2(0,0,\lambda_3) \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{JF(0,0)} = (0,0)$$

$$\underset{\nabla f(0,0,1)}{\uparrow}$$

$$\Rightarrow \nabla g(0,0,1) \,//\, \nabla f(0,0,1)$$

$$\nabla f(0,0,1) = c \nabla g(0,0,1)$$

$L(x,c) = f(x) - c\,g(x)$ ← funzione di Lagrange

$$\nabla L = (\partial_1 f(x) - c\,\partial_1 g(x), \dots, \partial_n f(x) - c\,\partial_n g(x), -g(x))$$

Esercizio 10. *Determinare massimo e minimo assoluti (se esistono) della funzione*

$$f(x_1, x_2) = (x_1 - x_2)^2 - x_1^2 x_2^2$$

nel dominio $D = \{x_1^2 + x_2^2 \leq 1\} \subseteq \mathbb{R}^2$.

osservo che D è compatto ($D = \overline{B(0,1)} \subseteq \mathbb{R}^2$), $f \in C^\infty$, quindi $\exists \max_D(f), \min_D(f)$

punti interni a D: $\nabla f(x_1,x_2) = (2x_1 - 2x_2 - 2x_1x_2^2, 2x_2 - 2x_1 - 2x_1^2 x_2)$

$$\begin{cases} x_1 - x_2 = x_1 x_2^2 \\ x_1 - x_2 = -x_1^2 x_2 \end{cases} \Rightarrow x_1 x_2 (x_1 + x_2) = 0$$

se $x_1=0 \rightarrow \nabla f(x) = (-2x_2, 2x_2) = (0,0)$ se $x_2=0$ $(0,0)$

se $x_2=0 \rightarrow \nabla f(x) = (2x_1, -2x_1) = (0,0)$ se $x_1=0$

se $x_1+x_2=0 \rightarrow \nabla f(x) = (4t-2t^3, -4t+2t^3) = (0,0)$

$x_1=t$ $\qquad 2t(2-t^2)=0$

$x_2=-t$

$\rightarrow t=0,\ t=\pm\sqrt{2}$ $\quad (\sqrt{2},-\sqrt{2})$, $(0,0)$, $(-\sqrt{2},\sqrt{2})$

solo $(0,0)\in D$!

$$Hf(x) = \begin{pmatrix} 2-2x_2^2 & -2-4x_1x_2 \\ -2-4x_1x_2 & 2-2x_1^2 \end{pmatrix} \qquad Hf(0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$Hf(0)$ è semidefinita positiva perché $Tr = 4 > 0$

$\det = 0$

$$\begin{aligned} f(x_1,x_2) = f(r,\theta) &= (r\cos(\theta) - r\operatorname{sen}(\theta))^2 - r^4\operatorname{sen}^2(\theta)\cos^2(\theta) \\ &= r^2 - 2r^2\operatorname{sen}(\theta)\cos(\theta) - r^4\operatorname{sen}^2(\theta)\cos^2(\theta) \\ &= r^2 - r^2[2\operatorname{sen}(\theta)\cos(\theta) + r^2\operatorname{sen}^2\cos^2(\theta)] \\ &= r^2(1-\operatorname{sen}(2\theta) - r^2\operatorname{sen}^2(\theta)\cos^2(\theta)) \end{aligned}$$

$$\left.\begin{aligned} &\theta = \tfrac{\pi}{4} \rightarrow f(r,\tfrac{\pi}{4}) = -\tfrac{1}{4}r^4 < 0 \quad \forall r \\ &\theta = 0 \rightarrow f(r,0) = r^2 > 0 \quad \forall r \end{aligned}\right\} f(0,0)=0$$

$\Rightarrow (0,0)$ punto di sella

① $\partial D = \{ (\cos(\theta), \sin(\theta)) ; \theta \in [0, 2\pi] \} \Rightarrow f(\cos(\theta), \sin(\theta))$

$$L(x,c) = f(x_1,x_2) - c\,g(x_1,x_2)$$
$$= (x_1-x_2)^2 - x_1^2 x_2^2 - c(x_1^2+x_2^2-1)$$

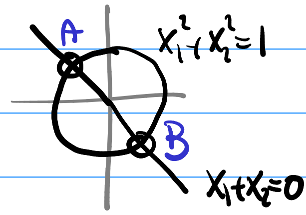
$$\begin{cases} \partial_1 L(x,c) = 2(x_1-x_2) - 2x_1x_2^2 - 2cx_1 = 0 \\ \partial_2 L(x,c) = 2(x_2-x_1) - 2x_1^2x_2 - 2cx_2 = 0 \\ \partial_3 L(x,c) = -(x_1^2+x_2^2-1) = 0 \end{cases}$$

$$\boxed{\begin{array}{l} f(A) = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2}\cdot\frac{1}{2} = \frac{7}{4} \\ f(B) = \frac{7}{4} \\ f(C) = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)^2 - \frac{1}{4} = -\frac{1}{4} = f(D) \end{array}}$$

$$\begin{cases} x_1 - x_2 = x_1x_2^2 + cx_1 \\ x_1 - x_2 = -x_1^2x_2 - cx_2 \\ x_1^2 + x_2^2 = 1 \end{cases}$$

$$x_1x_2^2 + cx_1 = -x_1^2x_2 - cx_2$$
$$x_1x_2(x_2+x_1) = -c(x_1+x_2)$$

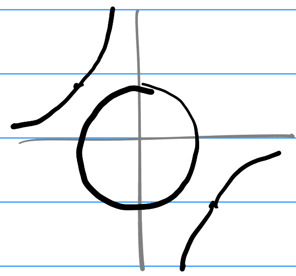
$$(x_1+x_2)(x_1x_2+c) = 0 \quad \begin{array}{l} \nearrow x_1 = -x_2 \\ \searrow x_1x_2 = -c \end{array}$$

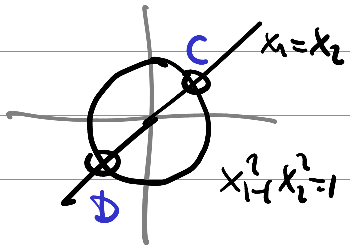


$A\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -B$

$$(x_1-x_2)(1-c) = 0 \leftarrow \begin{cases} x_1 - x_2 = -c\,x_2 + cx_1 \\ x_1 - x_2 = c\,x_1 - c\,x_2 \end{cases}$$

$\swarrow x_1 = x_2$ $\qquad \searrow c = 1 \longrightarrow$





$C = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -D$

A,B pti di max

C,D pt. di min.

ESEMPIO.

$$f(x) = (Mx)\cdot x \qquad x \in S^{2} = \{ \|x\|_2 = 1 \}$$
$$M \in \mathcal{M}_{n,n}(\mathbb{R})$$

S^{n} è chiuso e limitato $\Rightarrow$ compatto
f è un polinomio

Weierstrass $\exists\, p,q \in S^{2}$: $f(p) = \max_{S^2}(f)$, $f(q) = \min_{S^2}(f)$

$$\nabla f(p) = \lambda \nabla g(p) \Rightarrow (M + M^{T})p = 2\lambda p$$

$$\begin{cases} g(x) = x_1^2 + \dots + x_n^2 - 1 \Rightarrow \nabla g(x) = 2x \\ \nabla f(x) = (M + M^{T})x \end{cases}$$

ESERCIZIO 7. *Determinare, se esistono, massimo e minimo in* $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq x_1\}$ *della funzione*

$$f(x_1, x_2) = x_1 x_2^2 + x_1^2 x_2 - x_1 x_2 - x_1^2 x_2^2$$

Determinare l'estremo superiore e inferiore di f.

$$f(x) = x_1 x_2^2 + x_1^2 x_2 - x_1 x_2 - x_1^2 x_2^2 \in C^{\infty}(D)$$
$$D = \{ x_1^2 + x_2^2 - x_1 \leq 0 \} = \overline{B\left(\left(+\tfrac{1}{2}, 0\right), \tfrac{1}{2}\right)} \qquad \text{compatto}$$

$$\nabla f(x) = \left(x_2^2 + 2x_1x_2 - x_2 - 2x_1x_2^2 ,\ 2x_1x_2 + x_1^2 - x_1 - 2x_1^2 x_2 \right)$$

$$\nabla f(x) \overset{?}{=} (0,0) \qquad \begin{cases} x_2(x_2 + 2x_1 - 1 - 2x_1x_2) = 0 \\ x_1(2x_2 + x_1 - 1 - 2x_1x_2) = 0 \end{cases}$$

$$O(0,0) \qquad \begin{matrix} \text{se } x_2 = 0 \rightarrow x_1(x_1 - 1) = 0 \Rightarrow A(1,0) \\ \text{se } x_1 = 0 \rightarrow x_2(x_2 - 1) = 0 \Rightarrow B(0,1) \end{matrix}$$

c'è $\exists$ pti critici t.c. $\begin{cases} x_2 + 2x_1 - 1 - 2x_1x_2 = 0 \\ 2x_2 + x_1 - 1 - 2x_1x_2 = 0 \end{cases}$?

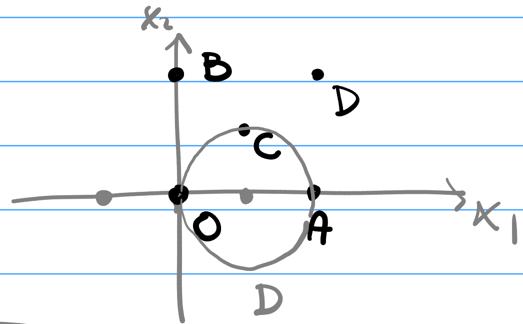
sottraggo $\longrightarrow x_2 + 2x_1 - 1 - 2x_2 - x_1 + 1 = 0$

$$x_1 - x_2 = 0 \Longrightarrow x_1 = x_2$$

$$t + 2t - 1 - 2t^2 = 0$$

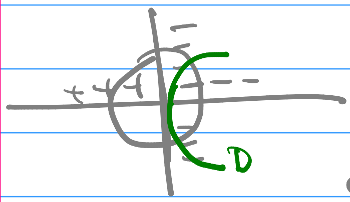
$$2t^2 - 3t + 1 = 0 \qquad t_{1,2} = \frac{3 \mp \sqrt{9-8}}{4} \begin{cases} \frac{1}{2} \\ 1 \end{cases}$$

$C\left(\frac{1}{2}, \frac{1}{2}\right) \qquad E(1,1)$



$f(x) = -x_1^3 - x_2^2$

$(0,0)$ pto critico $\quad Df(x) = (-3x_1^2, -2x_2)$



$$Hf(x) = \begin{pmatrix} -6x_1 & 0 \\ 0 & -2 \end{pmatrix} \qquad Hf(0) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

0 sarebbe una sella per f ma è un massimo assoluto per $f|_D$

$$(x_1(\theta), x_2(\theta)) = \left(\frac{1}{2} + \frac{1}{2}\cos(\theta), \frac{1}{2}\operatorname{sen}(\theta)\right) \qquad \theta \in [0, 2\pi]$$

$$h(\theta) = f(x_1(\theta), x_2(\theta)) = \left(\frac{1}{2} + \frac{1}{2}\cos(\theta)\right)^2 \cdot \frac{1}{2}\operatorname{sen}(\theta) + \left(\frac{1}{2} + \frac{1}{2}\cos(\theta)\right) \cdot \frac{1}{4}\operatorname{sen}^2(\theta)$$

$$- \frac{1}{4}(1+\cos(\theta))\operatorname{sen}(\theta) - \frac{1}{16}(1+\cos(\theta))^2\operatorname{sen}^2(\theta)$$

$$= \frac{1}{4}\Big[\frac{1}{2}\operatorname{sen}(\theta)(1+\cos(\theta))^2 + \frac{1}{2}\operatorname{sen}^2(\theta)(1+\cos(\theta))^2 - \operatorname{sen}(\theta)(1+\cos(\theta))$$

$$- \frac{1}{4}\operatorname{sen}^2(\theta)(1+\cos(\theta))^2\Big]$$

[Vedi foglio soluzioni]

TEOR.
MOLTIPLICATORI di LAGRANGE

$f, g : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ di classe C^1
A aperto

$M = \{x \in A : g(x) = 0\}$, $x_0 \in M$ tale che $Dg(x_0) \neq 0$

allora x_0 è un punto critico di f vincolato su M se è un punto critico di $L(x, \lambda) := f(x) - \lambda g(x)$, $(x, \lambda) \in A \times \mathbb{R}$.

TEOR. (BIS)

$f, g_i : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ di classe C^1 $(i = 1, \dots, k < n)$
A aperto

$M = \{x \in A : g_i(x) = 0 \ \forall i = 1, \dots, k\}$, $x_0 \in M$, $Jg(x_0)$ di rango massimo

allora x_0 è un punto critico di f vincolato su M se è un punto critico della funzione di Lagrange

$$L(x, \Lambda) = f(x) - \Lambda \cdot g(x)$$
$$= f(x) - (\lambda_1 g_1(x) + \lambda_2 g_2(x) + \dots + \lambda_k g_k(x))$$
$$(x, \Lambda) \in A \times \mathbb{R}^k$$

ESERCIZIO 51. *Date due sfere $\gamma_1, \gamma_2 \subseteq \mathbb{R}^3$ disgiunte si classifichino tutti i punti critici della funzione* $f(p, q) = \|p - q\|_2$ *con* $p \in \gamma_1$ *e* $q \in \gamma_2$.

ESERCIZIO 2.47. *Assegnato il seguente vincolo*

$$T = \left\{\left[2 - (x^2 + y^2)^{1/2}\right]^2 + z^2 - 1 = 0\right\} \subseteq \mathbf{R}^3$$

i. si provi che T è chiuso e limitato,
ii. si trovino tutti i punti critici (vincolati su T) della funzione $f(x, y, z) = y$,
iii. si scriva il polinomio di Taylor del secondo ordine della funzione implicitamente definita da T intorno ai punti critici trovati in ii.

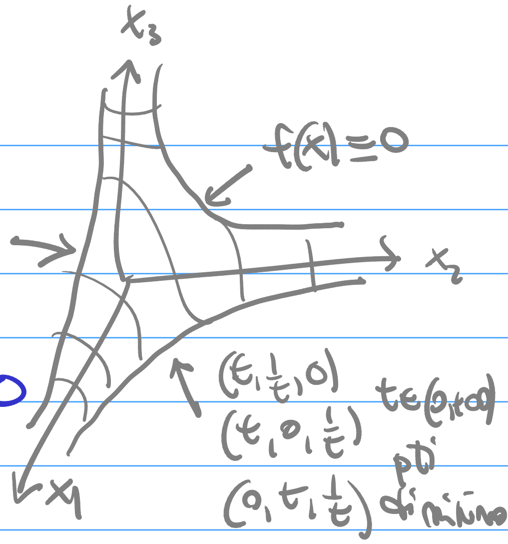
ESERCIZIO $f(x) = x_1 x_2 x_3$ $(x \in Q = \{x_1, x_2, x_3 \geq 0\})$

$x \in M = \{x_1 x_2 + x_1 x_3 + x_2 x_3 = 1\} \cap Q$

$$L(x, c) = x_1 x_2 x_3 - c\,(x_1 x_2 + x_1 x_3 + x_2 x_3 - 1)$$

cerco i punti critici di L in $\mathbb{R}^4$

$$\begin{cases} \partial_1 L(x,c) = x_2x_3 - c(x_2+x_3) = 0 \\ \partial_2 L(x,c) = x_1x_3 - c(x_1+x_3) = 0 \\ \partial_3 L(x,c) = x_1x_2 - c(x_1+x_2) = 0 \\ \partial_4 L(x,c) = -(x_1x_2 + x_1x_3 + x_2x_3 - 1) = 0 \end{cases}$$



osservo che $c \neq 0$

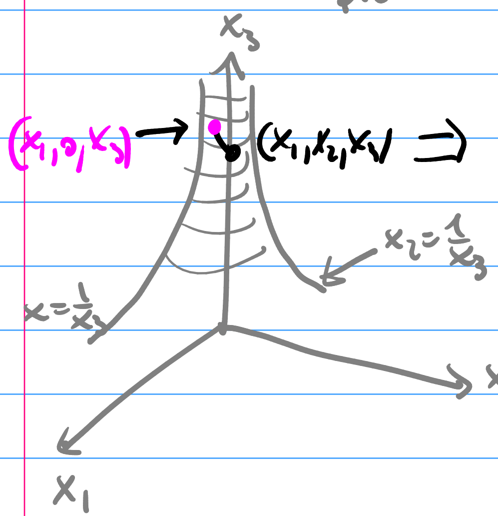
considero $x \cdot (\partial_1 L, \partial_2 L, \partial_3 L) = 3x_1x_2x_3 - c(2x_1x_2 + 2x_1x_3 + 2x_2x_3)$

$$\Rightarrow 3x_1x_2x_3 - 2c = 0 \Rightarrow x_1x_2x_3 = \frac{2}{3}c$$

$$c(\underbrace{x_1x_2 + x_1x_3}_{1-x_2x_3}) = c(\underbrace{x_1x_2 + x_2x_3}_{1-x_1x_3}) = c(\underbrace{x_1x_3 + x_2x_3}_{1-x_1x_2})$$

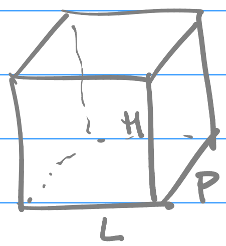
$$\Rightarrow x_1x_2 = x_2x_3 = x_1x_3 = \frac{1}{3} \Rightarrow x_1 = x_2 = x_3 = \frac{\sqrt{3}}{3}$$

$$f\Big(\underbrace{\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}}_{\text{pto di massimo assoluto}}\Big) = \frac{\sqrt{3}}{9} = \frac{1}{3\sqrt{3}} \leftarrow \max(f|_M)$$



$\Rightarrow$ $0 \leq x_1 \leq \frac{1}{x_3}$, $0 \leq x_2 \leq$ $\Rightarrow f(x) = x_1x_2x_3 \leq \frac{1}{x_3} \cdot \frac{1}{x_3} \cdot x_3 = \frac{1}{x_3}$

$$x_1x_2x_3 \leq \frac{1}{3\sqrt{3}} \quad \forall x \in M$$



$V = PLH$ $\qquad x := \lambda L$

$S = 2(PL + PH + HL)$ $\qquad y := \lambda P$

$PL + PH + HL = \frac{S}{2}$ $\qquad z := \lambda H$

se $\lambda^2 = \frac{2}{S}$ allora $xy + yz + xz = 1$

poiché $xy+yz+xz=1 \Rightarrow (x,y,z)\in M$

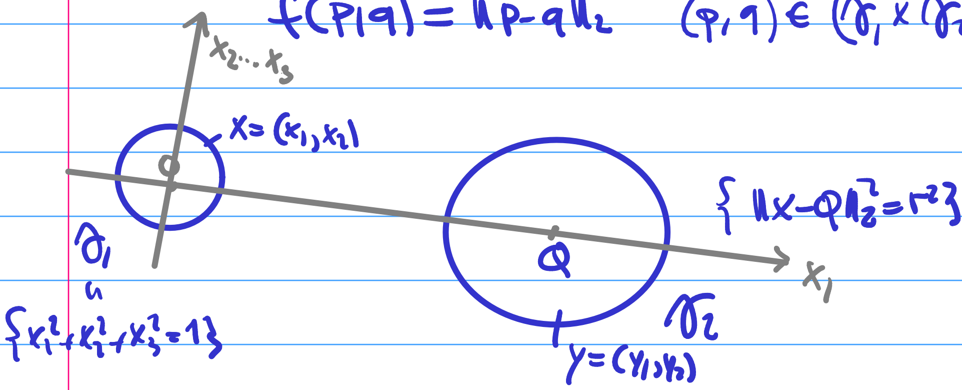
$$\Rightarrow f(x)=xyz \le \frac{1}{3\sqrt{3}}$$

$$\lambda^3 \ LPH \le \frac{1}{3\sqrt{3}}$$

$$V \le \left[\frac{S}{6}\right]^{3/2} \Rightarrow V^2 \le \frac{1}{6^3} S^3$$

γ_1 , γ_2 sfere disgiunte in $\mathbb{R}^3$

$$f(p,q)=\|p-q\|_2 \qquad (p,q)\in(\gamma_1\times\gamma_2)$$



$Q = ce_1$, con $c > 1+r$

$$\bar{f}(x,y)=\|x-y\|_2^2=(x_1-y_1)^2+(x_2-y_2)^2$$

$$\gamma_1=\{x_1^2+x_2^2=1\} \quad ; \quad \gamma_2=\{(y_1-c)^2+y_2^2=r^2\}$$

$$L(x_1,x_2,y_1,y_2,\lambda_1,\lambda_2)=(x_1-y_1)^2+(x_2-y_2)^2-\lambda_1(x_1^2+x_2^2-1)-\lambda_2((y_1-c)^2+y_2^2-r^2)$$

$$=(x-y)\cdot(x-y)-\lambda_1(x\cdot x-1)-\lambda_2((y-ce_1)\cdot(y-ce_1)-r^2)$$

$(y-ce_1)\cdot(y-ce_1)=(y_1-c,y_2)\cdot(y_1-c,y_2)$

$$\begin{cases} \left.\begin{array}{l}\partial_1 L\\ \partial_2 L\end{array}\right\} = 2(x-y)-2\lambda_1 x = 0 \\ \left.\begin{array}{l}\partial_3 L\\ \partial_4 L\end{array}\right\} = 2(y-x)-2\lambda_2(y-ce_1)=0 \\ \partial_5 L = 1-x\cdot x=0 \\ \partial_6 L = r^2-(y-ce_1)\cdot(y-ce_1)=0 \end{cases}$$

$\lambda_1 \neq 0$ (altrimenti $x=y$)

$(1-\lambda_1)x=y \Rightarrow x \parallel y$

$2\lambda_1 x-2\lambda_2(1-\lambda_1)x+2\lambda_2 ce_1=0$

$\Rightarrow x \parallel e_1$

$\Rightarrow x=(x_1,0) \quad y=(y_1,0)$

$$\Rightarrow x=(\pm 1,0) \ \text{ e } \ y=(c\pm r,0)$$

$\max_{\gamma_1 \times \gamma_2} (\bar{f})$

PUNTI CRITICI: ① $\begin{cases} x = (-1, 0) \\ y = (c-r, 0) \end{cases}$ ② $\begin{cases} x = (-1, 0) \\ y = (c+r, 0) \end{cases}$ $\bar{f}(②) = (c+r+1)^2$

$\bar{f}(①) = (c-r+1)^2$

③ $\begin{cases} x = (1, 0) \\ y = (c-r, 0) \end{cases}$ ④ $\begin{cases} x = (1, 0) \\ y = (c+r, 0) \end{cases}$ $\bar{f}(④) = (c+r-1)^2$

$\bar{f}(③) = (c-(r+1))^2 \leftarrow \min_{\gamma_1 \times \gamma_2}(\bar{f})$

$$T = \{ x : [2-(x_1^2+x_2^2)^{1/2}]^2 + x_3^2 = 1 \} \subseteq \mathbb{R}^3$$

$$F(x) = x_2$$

T è chiuso perché controimmagine di un chiuso attraverso una funzione continua

$$x_3^2 \le [2-(x_1^2+x_2^2)^{1/2}]^2 + x_3^2 = 1 \Rightarrow -1 \le x_3 \le 1$$

$$[2-(x_1^2+x_2^2)^{1/2}]^2 \le 1 \Rightarrow -1 \le 2-(x_1^2+x_2^2)^{1/2} \le 1$$

$$-3 \le -(x_1^2+x_2^2)^{1/2} \le -1$$

$$1 \le (x_1^2+x_2^2)^{1/2} \le 3$$

$\Rightarrow T \subseteq [-3,3]^2 \times [-1,1]$

quindi T è limitato

$$L(x, c) = x_2 - c\left[[2-(x_1^2+x_2^2)^{1/2}]^2 + x_3^2 - 1 \right]$$

$$\partial_1 L = 0 - c \cdot 2[2-(x_1^2+x_2^2)^{1/2}] \cdot \left(-\frac{x_1}{(x_1^2+x_2^2)^{1/2}}\right) = 0$$

$$\partial_2 L = 1 - c \quad '' \quad \cdot \left(-\frac{x_2}{(x_1^2+x_2^2)^{1/2}}\right) = 0$$

$$\partial_3 L = 0 - c \cdot 2x_3 = 0$$

$$1 + 2c\,\frac{2-|x_2|}{|x_2|} \cdot x_2 = 0 + \text{eq vincolo} \Rightarrow \boxed{x_2 = \pm 1, \pm 3} \ldots$$