Limiti notevoli che coinvolgono il numero e

2) $$\boxed{\lim_{x\to\pm\infty}\left(1+\frac{a}{x}\right)^x = e^a \qquad \forall a\in\mathbb{R}.}$$

3) $$\boxed{\lim_{x\to 0}(1+x)^{1/x} = \left(1^{\pm\infty}\right) = e}$$

$$\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e$$

$$y=\frac{1}{x}\to+\infty$$

$$\lim_{x\to 0^-}(1+x)^{1/x} = \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y = e$$

$$y=\frac{1}{x}\to-\infty$$

4) $$\lim_{x\to 0}\frac{1}{x}\log(1+x) = \left(\frac{0}{0}\right) = \lim_{x\to 0}\log\left[(1+x)^{1/x}\right] = \log e = 1$$

$$\boxed{\lim_{x\to 0}\frac{1}{x}\log(1+x) = 1}$$ Si può anche scrivere così:

oppure

$$\log(1+x)\sim x \qquad \text{per } x\to 0$$

$$\log(1+x) = x(1+o(1)) \qquad \text{''}$$

$$\log(1+x) = x+o(x) \qquad \text{''}$$

$$\log t\sim t-1 \qquad t\to 1$$

OSS. Se la base è diversa, il risultato cambia!

$$\log_2(1+x) = \frac{\log(1+x)}{\log 2}\sim\frac{x}{\log 2} \qquad x\to 0$$

5) $\lim_{x\to 0} \frac{e^x - 1}{x} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{y}{\underbrace{\log(1+y)}_{\sim y}} \overset{(4)}{=} 1$

$y = e^x - 1 \to 0$

$e^x = 1 + y$

$x = \log(1+y)$

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$$

$$\boxed{\begin{array}{ll} e^x - 1 \sim x & \text{per } x\to 0 \\ e^x - 1 = x(1 + o(1)) & \text{"} \\ e^x - 1 = x + o(x) & \text{"} \\ e^x = 1 + x + o(x) & \text{"} \end{array}}$$

$\cancel{e^x \sim 1 + x}$ non ha lo stesso significato: dice solo che $e^x \to 1$

5') $\lim_{x\to 0} \frac{a^x - 1}{x} = \underbrace{\frac{e^{x\log a} - 1}{x \log a}}_{\to 1} \log a = \log a$

$\forall a > 0$

(Il calcolo vale se $a \neq 0$, ma il caso $a = 0$ è banale)

$$\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \log a \quad \forall a > 0}$$

6) $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \left(\frac{0}{0}\right) =$ $\qquad \alpha \in \mathbb{R}$

$$= \lim_{x\to 0} \frac{\overbrace{e^{\alpha \log(1+x)}}^{\to 0} - 1}{x} =$$

$$= \lim_{x\to 0} \boxed{\frac{e^{\alpha\log(1+x)} - 1}{\alpha \log(1+x)}} \quad \frac{\alpha \boxed{\log(1+x)}}{x} = \alpha \quad \forall \alpha \in \mathbb{R}$$

$t = \alpha \log(1+x) \to 0$ $\qquad \frac{e^t - 1}{t} \to 1 \qquad\qquad 1$

$$6) \quad \boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \forall \alpha \in \mathbb{R}}$$

Questo si scrive anche

$$\boxed{\begin{array}{ll} (1+x)^\alpha - 1 \sim \alpha x & \text{per } x\to 0 \\ (1+x)^\alpha = 1 + \alpha x\,(1 + o(1)) & \quad '' \\ (1+x)^\alpha = 1 + \alpha x + o(x) & \quad '' \end{array}}$$

$\alpha = \frac{1}{2} \qquad \sqrt{1+x} = 1 + \frac{1}{2}x + o(x) \qquad ''$

$\alpha = 2 \qquad (1+x)^2 = 1 + 2x + \underbrace{o(x)}_{x^2} \qquad ''$

$$\lim_{x\to +\infty} \left(\sqrt[3]{x^3 + 2x^2} - x\right) = (+\infty - \infty)$$

1° modo: (già noto) moltiplicando per il "falso quadrato"

Usiamo $\quad A^3 - B^3 = (A-B)(A^2 + AB + B^2) \quad \forall A, B \in \mathbb{R}$

$$\frac{\left(\sqrt[3]{x^3+2x^2}-x\right)\left((x^3+2x^2)^{2/3}+x\sqrt[3]{x^3+2x^2}+x^2\right)}{\left((x^3+2x^2)^{2/3}+x\sqrt[3]{x^3+2x^2}+x^2\right)} =$$

$$= \frac{\cancel{x^3}+2x^2-\cancel{x^3}}{\left(\underbrace{(x^3+2x^2)^{2/3}}_{\sim x^2}+\underbrace{x\sqrt[3]{x^3+2x^2}}_{\sim x^2}+x^2\right)} = \frac{2\cancel{x^2}}{\cancel{x^2}\underbrace{\left[\left(1+\frac{2}{x}\right)^{2/3}+\sqrt[3]{1+\frac{2}{x}}+1\right]}_{\to 3}}$$

$$\to \frac{2}{3} \qquad \text{per } x \to +\infty$$

2° modo (nuovo)

$$\sqrt[3]{x^3+2x^2}-x = x\overbrace{\left[\sqrt[3]{1+\frac{2}{x}}-1\right]}^{\sim \frac{2}{3x}} \sim x\cdot\frac{2}{3x} = \frac{2}{3} \qquad \text{per } x\to+\infty$$

OSS $\sqrt[3]{1+\frac{2}{x}}-1 = \left(1+\frac{2}{x}\right)^{1/3}-1 = (1+t)^{1/3}-1 \overset{(6)}{\sim} \frac{t}{3} = \frac{2}{3x}$

per $x\to+\infty$ $\frac{2}{x}\to 0$ pongo $\frac{2}{x}=t\to 0$

$$\lim_{x\to+\infty}\left(\sqrt[8]{x^8+2x^7}-x\right)$$

1° modo $A^8-B^8=(A-B)(A^7+A^6B+A^5B^2+\dots+AB^6+B^7)$

2° modo

$$\sqrt[8]{x^8+2x^7}-x = x\overbrace{\left(\sqrt[8]{1+\frac{2}{x}}-1\right)}^{\sim \frac{1}{8}\frac{2}{x}=\frac{1}{4x} \quad x\to+\infty} \sim \frac{x}{4x} = \frac{1}{4}$$

$$\sqrt[8]{1+\frac{2}{x}} = \left(1+\boxed{\frac{2}{x}}\right)^{1/8}, \qquad \frac{2}{x}\to 0$$

Trovare $c \in \mathbb{R}$ t.c. $\lim_{x\to -\infty} \left(\frac{x+c}{x-c}\right)^x = 4.$

$$\left(\frac{x+c}{x-c}\right)^x = \left(1^{-\infty}\right) = \left(\frac{x-c+2c}{x-c}\right)^x =$$

$$= \left(1+\frac{2c}{x-c}\right)^x = \underbrace{\left[\left(1+\frac{2c}{x-c}\right)^{x-c}\right]}_{\downarrow \atop e^{2c}} \underbrace{\left(1+\frac{2c}{x-c}\right)^c}_{\downarrow \atop 1} \to e^{2c}$$

Cerco c t.c.

$$e^{2c} = 4 \iff 2c = \log 4 = 2\log 2 \iff \boxed{c = \log 2}$$

In alternativa, per calcolare il limite:

$$\left(\frac{x+c}{x-c}\right)^x = \left(\frac{\cancel{x}\left(1+\frac{c}{x}\right)}{\cancel{x}\left(1-\frac{c}{x}\right)}\right)^x = \frac{\left(1+\frac{c}{x}\right)^x \to e^c}{\left(1-\frac{c}{x}\right)^x \to e^{-c}} \Rightarrow \frac{e^c}{e^{-c}} = e^{2c}$$

oppure

$$\left(\frac{x+c}{x-c}\right)^x = e^{x\log\left(\frac{x+c}{x-c}\right)}$$

$\log(1+t) \sim t$ per $t\to 0$

$$x\log\Big(\underbrace{\frac{x+c}{x-c}}_{\downarrow \atop 1}\Big) = x\overbrace{\log\Big(1+\underbrace{\frac{2c}{x-c}}_{\downarrow \atop 0}\Big)}^{\sim \frac{2c}{x-c}} \sim x\cdot\frac{2c}{x-c} \to 2c$$

$$\lim_{x\to4^+}\frac{\log(5e^{x-4}-x)-\sin(3x-12)}{\underset{\downarrow\,0}{x-4}}=$$

$x-4=y\to0^+$

$$=\lim_{y\to0^+}\frac{\log(5e^y-y-4)-\sin(3y)}{y}=$$

$$=\lim_{y\to0^+}\left[\underbrace{\frac{\log(5e^y-y-4)}{y}}_{\to 4}-\underbrace{\boxed{\frac{\sin(3y)}{y}}}_{\to 3}\right]=4-3=1$$

$\log(5e^y-y-4)\sim4y \qquad \sin(3y)\sim3y$

$$\log(\underbrace{5e^y-y-4}_{\to1})=\log(1+\underbrace{(5e^y-y-5)}_{\to0})\sim$$

$$\sim 5e^y-y-5=5(e^y-1)-y= \qquad \boxed{y\to0}$$

$$=5(y+o(y))-y=$$

$$=4y+o(y)=y\left(4+\underbrace{\frac{o(y)}{y}}_{=o(1)}\right)\sim4y$$

$$\lim_{x\to0}\frac{e^{tg^3x}-1}{x(\cos x-e^{x^2})}=\left(\frac{0}{0}\right)$$

N: $e^{tg^3x}-1\sim tg^3x\sim x^3$ $\qquad tg^3x\to0$ $\qquad x\to0$

$(\operatorname{tg} x)^3 \sim x^3$ per $x \to 0$

D :

$$\cancel{\cos x - e^{x^2} \sim 1 - e^{x^2} \sim -x^2}$$

$$\cancel{\cos x - e^{x^2} \sim \cos x - 1 \sim -\frac{x^2}{2}}$$

ERRATO! perché ho sost. in una somma

$$\cos x - e^{x^2} = \underbrace{(\cos x - 1)}_{\sim -\frac{x^2}{2}} + \underbrace{(1 - e^{x^2})}_{\sim -x^2} =$$

$$= x^2 \left(\frac{\cos x - 1}{x^2} + \frac{1 - e^{x^2}}{x^2} \right) \sim -\frac{3}{2} x^2$$

$\frac{\cos x - 1}{x^2} \to -1/2$, $\frac{1 - e^{x^2}}{x^2} \to -1$, $\to -\frac{3}{2}$

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\frac{\overbrace{e^{\operatorname{tg}^3 x} - 1}^{\sim x^3}}{x \underbrace{(\cos x - e^{x^2})}_{\sim -\frac{3}{2}x^2}} \sim \frac{x^3}{-\frac{3}{2} x^3} = -\frac{2}{3}$$

Altro modo di studiare $\cos x - e^{x^2}$. $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$e^t = 1 + t + o(t) \qquad t \to 0, \quad t = x^2$$

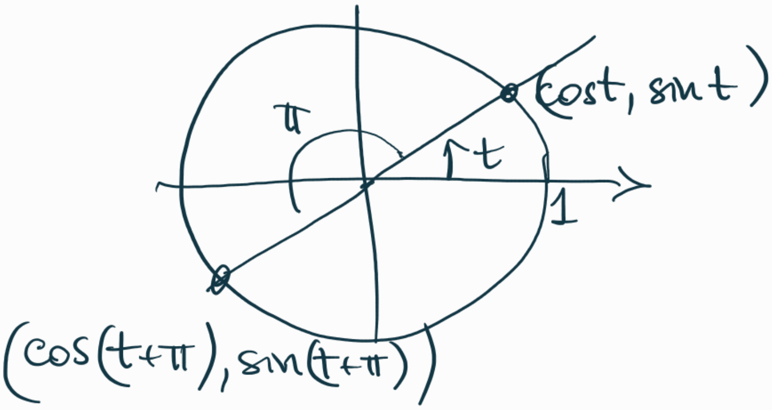
$$e^{x^2} = 1 + x^2 + o(x^2) \qquad x \to 0$$

$$\cos x - e^{x^2} = \cancel{1} - \frac{x^2}{2} + o(x^2) - \cancel{1} - x^2 + o(x^2) =$$

$$= -\frac{3}{2}x^2 + o(x^2) \sim -\frac{3}{2}x^2$$

$$\lim_{x\to\pi} \frac{\sin x}{x-\pi} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{\sin(\pi+y)}{y} = \lim_{y\to 0}\left(-\frac{\sin y}{y}\right) = \\ = -1$$

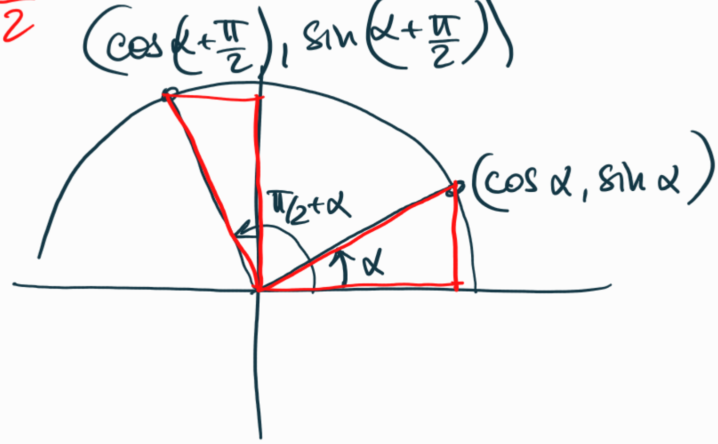
$$y = x - \pi \to 0$$



$$\cos(t+\pi) = -\cos t$$
$$\sin(t+\pi) = -\sin t$$

$$\lim_{n\to+\infty} \underbrace{\overbrace{(2n+5)}^{\sim 2n}}_{\to +\infty} \underbrace{\overbrace{\cos\left(\frac{\pi(n^2+1)}{2n^2+n}\right)}^{\sim \frac{\pi}{4n}}}_{\to \cos\frac{\pi}{2}=0} = \boxed{\frac{\pi}{2}}$$

$$\cos\left(\frac{\pi}{2} + \underset{\to 0}{a_n}\right)$$



$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$$

$$\cos\left(\frac{\pi(n^2+1)}{2n^2+n}\right) = \cos\left(\frac{\pi}{2} + \left(\frac{\pi(n^2+1)}{2n^2+n} - \frac{\pi}{2}\right)\right) =$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi\left(\cancel{2n^2}+2-\cancel{2n^2}-n\right)}{(2n^2+n)2}\right) =$$

$$= -\sin\left(\underbrace{\boxed{\frac{\pi(2-n)}{2(2n^2+n)}}}_{\downarrow\, 0}\right) \sim -\frac{\pi}{2}\,\underbrace{\frac{2-n}{2n^2+n}}_{\sim -\frac{1}{2n}} \sim \frac{\pi}{4n}$$