

Def $f(x), g(x)$ t.c. $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = \pm \infty$
 $\Downarrow$
 $\mathbb{R}^*$

f è un infinito di ordine superiore risp a g per $x \rightarrow x_0$

se $\lim_{x \rightarrow x_0} \frac{g(x)}{f(x)} = 0$ (cioè $\lim_{x \rightarrow x_0} \left| \frac{f(x)}{g(x)} \right| = +\infty$)

$(g(x) = o(f(x)) \text{ per } x \rightarrow x_0)$

Supponiamo $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$.

$\rightarrow x_0$ Diremo che $f(x)$ è un infinitesimo di ordine superiore
risp. a $g(x)$ per $x \rightarrow x_0$ se

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0 \quad \left(\text{cioè} \quad \lim_{x \rightarrow x_0} \left| \frac{g(x)}{f(x)} \right| = +\infty \right)$$

OSS

OSS

$$\boxed{X \rightarrow X_0}$$

f infinito di ord. sup. risp. a $g(x) \Leftrightarrow g(x) = o(f(x))$
 f infinitesimo " " " " " $\Leftrightarrow f(x) = o(g(x))$

f e g sono infiniti/infinitesimi dello stesso ordine se

$$\lim_{x \rightarrow X_0} \frac{f(x)}{g(x)} = l \in \mathbb{R} \setminus \{0\}, \text{ cioè } f(x) \sim l g(x) \text{ per } x \rightarrow X_0$$

Esempio

x^2 è infinito di ordine inferiore risp. a x^3 per $x \rightarrow -\infty$

x^2 è infinitesimo di ordine inferiore risp. a x^3 per $x \rightarrow 0$

Però:

$$x^2 + x^3 \sim x^3$$

$$x^3 \left(1 + \frac{1}{x} \right)$$

per $x \rightarrow -\infty$

$$x^2 + x^3 \sim x^2$$

$$x^2 (1 + x) = x^2 (1 + o(1))$$

per $x \rightarrow 0$

In generale, se $f(x) = o(g(x))$ per $x \rightarrow x_0$,

si ha $f(x) + g(x) \sim g(x)$ //

$$g(x) \left(1 + \frac{f(x)}{g(x)} \right)$$

↓
0

$$1) \lim_{x \rightarrow +\infty} \frac{b^x}{x^\alpha} = +\infty \quad \forall b > 1 \quad \forall \alpha > 0$$

cioè: b^x è infinito di ord. sup. risp. a x^α $\forall \alpha > 0, \forall b > 1$ per $x \rightarrow +\infty$.
(già visto)

$$2) \lim_{x \rightarrow -\infty} |x|^\alpha b^x = 0 \quad \forall b > 1, \forall \alpha > 0 \quad (\forall \alpha \in \mathbb{R})$$

cioè b^x è un infinitesimo di ordine superiore risp. a $\frac{1}{|x|^\alpha}$ per $x \rightarrow -\infty$, $\forall b > 1, \forall \alpha > 0$

$$3) \lim_{x \rightarrow +\infty} \frac{\log_b x}{x^\alpha} = 0, \quad \forall b > 0, b \neq 1, \forall \alpha > 0 \quad (\forall \alpha \in \mathbb{R})$$

cioè $\log_b x$ è un infinito di ordine inferiore risp. a x^α
 $\forall b > 0, b \neq 1, \forall \alpha > 0$

Infatti abbiamo verificato che, se $a_n \rightarrow +\infty$, allora

$$\lim_{n \rightarrow +\infty} \frac{\log_b a_n}{a_n^\alpha} = 0 \quad + \text{teor. ponte}$$

$$4) \lim_{x \rightarrow 0^+} x^\alpha \log_b x = (0 \cdot (\pm\infty)) = 0$$

$$\forall b > 0, b \neq 1$$

$$\forall \alpha > 0$$

Cioè $\log_b x$ è un infinito di ordine inferiore risp. a $\frac{1}{x^\alpha}$

$$\forall \alpha > 0$$

Dim

$$\lim_{x \rightarrow 0^+} x^\alpha \log_b x = \lim_{y \rightarrow +\infty} \frac{1}{y^\alpha} \log_b \frac{1}{y} = \lim_{y \rightarrow +\infty} \frac{-\log_b y}{y^\alpha} = 0$$

$y = \frac{1}{x} \rightarrow +\infty$

$\log_b \frac{1}{y} = -\log_b y$

(3)

$$4') \lim_{x \rightarrow 0^+} x^\alpha |\log_b x|^c = 0$$

$$\forall b > 0, b \neq 1, \forall \alpha > 0$$

$$\forall c > 0$$

Dim

$$x^\alpha |\log_b x|^c = \left(x^{\frac{\alpha}{c}} |\log_b x| \right)^c \rightarrow 0^c = 0$$


$$\lim_{x \rightarrow 0^+} x^{1/x} = (0^{+\infty}) = 0$$

$$\lim_{x \rightarrow 0^+} x^x = (0^0) = \lim_{x \rightarrow 0^+} e^{\underbrace{x \log x}_0} = 1$$

$$\lim_{x \rightarrow +\infty} \frac{x}{e^{\sqrt{\log x}}} = \left(\frac{+\infty}{+\infty} \right) = \lim_{x \rightarrow +\infty} e^{\underbrace{\log x - \sqrt{\log x}}_{(\log x) \left(1 - \frac{1}{\sqrt{\log x}} \right) \rightarrow +\infty}} = e^{+\infty} = +\infty$$

Limiti notevoli:

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = \left(\frac{0}{0} \right) = 1}$$

Si può scrivere anche così

$$\sin x = x (1 + o(1))$$

$$\sin x \sim x$$

$$\sin x = x + \underbrace{x o(1)}_{o(x)} = x + o(x)$$

per $x \rightarrow 0$
"

Lemma $\forall x \in [0, \frac{\pi}{2})$

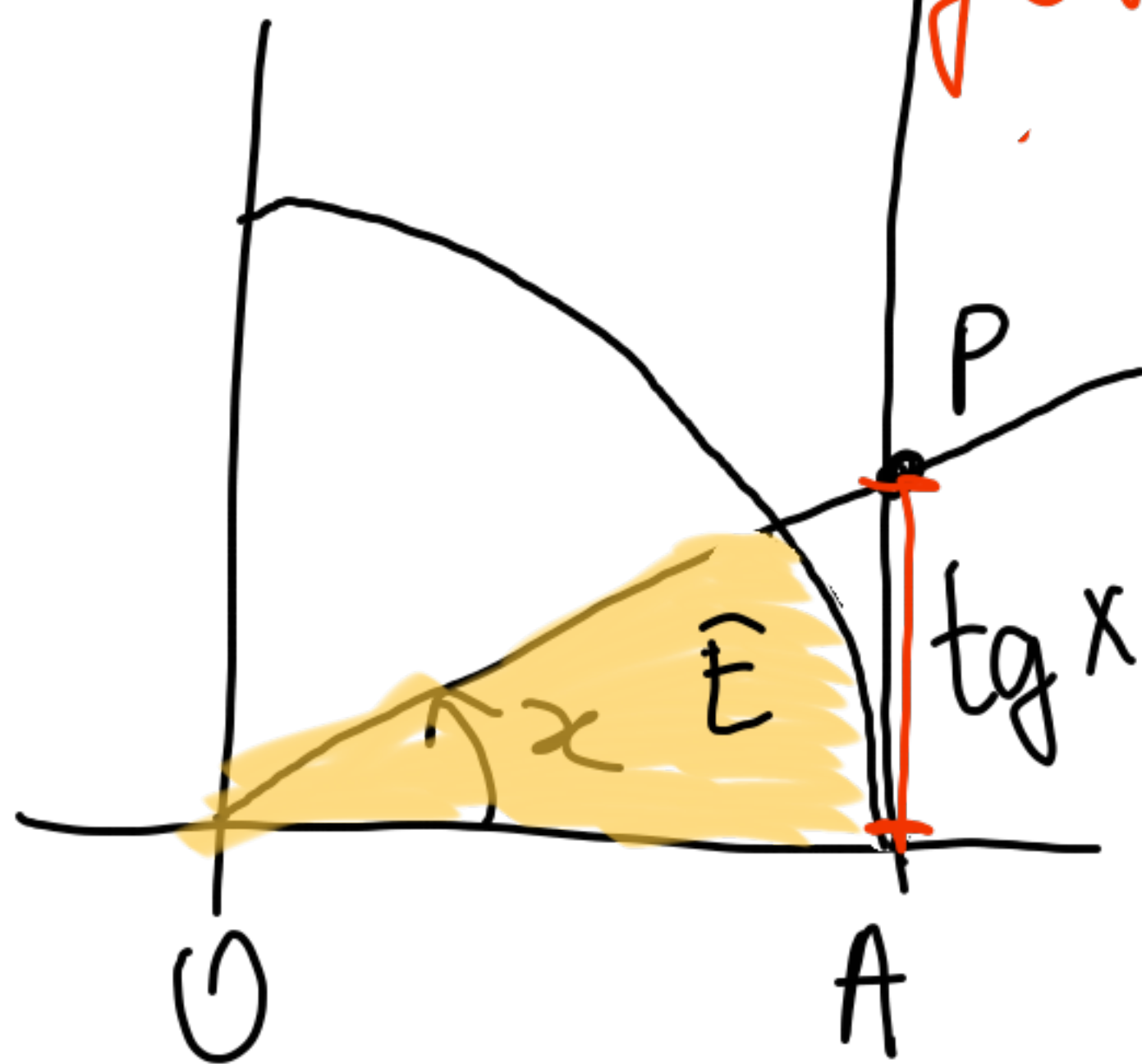
$$\sin x \leq x \leq \tan x$$

già visto

$$\text{area } E \leq \text{area } \triangle OAP$$

$$\frac{x}{2} \leq \frac{\tan x}{2}$$

$$x \leq \tan x$$



$$\text{area } \triangle OAP = \frac{1 \cdot \tan x}{2} = \frac{\tan x}{2}$$

$$\frac{\text{Area } E}{x} = \frac{\pi}{2\pi}$$

$$\Rightarrow \text{Area } E = \frac{x}{2}$$

$$\sin x \leq x \leq \frac{\sin x}{\cos x} \quad \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$1 \geq \frac{x}{\sin x} \geq \frac{1}{\cos x} \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$1 \geq \frac{\sin x}{x} \geq \cos x \Rightarrow \frac{\sin x}{x} \rightarrow 1 \text{ per } x \rightarrow 0^+$$

↓
1

(c) (c)

ma $\frac{\sin x}{x}$ è pari, quindi anche $\frac{\sin x}{x} \rightarrow$ per $x \rightarrow 0^-$

□

$$2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

~~$$\cos x \sim 1 - \frac{x^2}{2}$$~~

Si scrive anche

$$1 - \cos x \sim \frac{x^2}{2} \quad \text{per } x \rightarrow 0$$

$$1 - \cos x = \frac{x^2}{2} (1 + o(1)) \quad //$$

$$1 - \cos x = \frac{x^2}{2} + o(x^2) \quad //$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad //$$

$$x \rightarrow 0$$

$$\frac{1 - \cos x}{x^2} = \left(\frac{0}{0} \right)$$

$$= \frac{\overbrace{(1 - \cos x)(1 + \cos x)}^{1 - \cos^2 x = \sin^2 x}}{x^2 (1 + \cos x)} = \frac{\sin^2 x}{x^2 (1 + \cos x)}$$

$$\rightarrow \frac{1}{2}$$

$$\begin{aligned} & \left(\frac{\sin x}{x} \right)^2 \downarrow 1 \\ & \downarrow 2 \end{aligned}$$

$$3) \lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = 1$$

$$\frac{\operatorname{tg} x}{x} = \left(\frac{\sin x}{x} \right) \left(\frac{1}{\cos x} \right) \rightarrow 1$$

$\downarrow$ $\downarrow$
1 1

$$\operatorname{tg} x \sim x \quad \text{per } x \rightarrow 0$$

$$\operatorname{tg} x = x + o(x) \quad \text{per } x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{x \sin x}{\cos x - 1} = \left(\frac{0}{0} \right) = -2$$

$$\frac{x^2 \sin x}{(\cos x - 1) x}$$

Annotations: The expression is circled in red. Arrows point from the x^2 term to -2 and from the x term in the denominator to 1 .

$$\frac{x \sin x}{\cos x - 1} \sim \frac{x^2}{\frac{x^2}{2}} = -2$$

Annotations: The numerator $x \sin x$ is circled in red with a red arrow pointing to $\sim x$. The denominator $\cos x - 1$ is circled in red with a red arrow pointing to $\sim -\frac{x^2}{2}$.

$$\lim_{x \rightarrow 0} \frac{\text{tg } x - \sin^2 x}{x} = \left(\frac{0}{0} \right) = 1$$

Diagram illustrating the limit calculation using asymptotic equivalence:

$$\overset{\sim x}{\text{tg } x} - \overset{\sim x^2}{\frac{\sin x}{x} \cdot \sin x} \rightarrow 1$$

Annotations: The first term $\text{tg } x$ is circled in purple with an arrow pointing to $\sim x$. The second term is $\frac{\sin x}{x} \cdot \sin x$, where $\frac{\sin x}{x}$ is circled in purple with an arrow pointing to ~ 1 , and $\sin x$ is circled in purple with an arrow pointing to 0 .

$$\text{tg } x - \sin^2 x = x \left(\frac{\text{tg } x}{x} - \frac{\sin^2 x}{x} \right) \sim x$$

Annotation: A red bracket under the term in parentheses is labeled $1 + o(1)$.

$$\lim_{x \rightarrow 0} \frac{2 - 2\cos x}{\sin(x^2) - 3\sin x - x^2} = \left(\frac{0}{0} \right)$$

$$\frac{2 - 2\cos x \sim x^2}{\sin x^2 - 3\sin x - x^2 \sim -3x} \sim \frac{x^2}{-3x} = -\frac{x}{3} \rightarrow 0$$

$$x \left(\underbrace{\frac{\sin(x^2)}{x^2}}_{\downarrow 0} - \underbrace{3 \frac{\sin x}{x}}_{\downarrow -3} - \underbrace{x}_{\downarrow 0} \right)$$

Limiti notevoli che coinvolgono e

$$1) \lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

Vero, perché se $a_n \rightarrow \pm\infty$, $\left(1 + \frac{1}{a_n}\right)^{a_n} \rightarrow e$
+ teor. ponte $\Rightarrow$ la tesi