3 Vector-Valued Functions

Introduction

The primary focus of Chapter 2 was on scalar-valued functions, although general mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$ were considered occasionally. This chapter concerns vector-valued functions of two special types:

1. Continuous mappings of one variable (i.e., functions $\mathbf{x}: I \subseteq \mathbf{R} \to \mathbf{R}^n$, where I is an interval, called **paths** in $\mathbf{R}^n$).
2. Mappings from (subsets of) $\mathbf{R}^n$ to itself (called **vector fields**).

An understanding of both concepts is required later, when we discuss line and surface integrals.

3.1 Parametrized Curves and Kepler's Laws

Paths in $\mathbf{R}^n$

We begin with a simple definition. Let I denote any interval in $\mathbf{R}$. (So I can be of the form $[a, b]$, (a, b), $[a, b)$, $(a, b]$, $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$, or $(-\infty, \infty) = \mathbf{R}$.)

DEFINITION 1.1 A **path in $\mathbf{R}^n$** is a continuous function $\mathbf{x}: I \to \mathbf{R}^n$. If $I = [a, b]$ for some numbers $a < b$, then the points $\mathbf{x}(a)$ and $\mathbf{x}(b)$ are called the **endpoints** of the path $\mathbf{x}$. (Similar definitions apply if $I = [a, b)$, $[a, \infty)$, etc.)

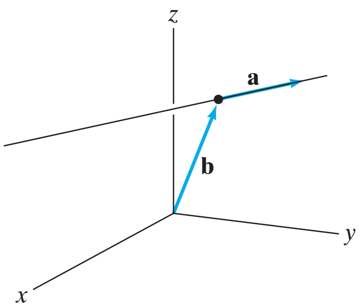

Figure 3.1 The path **x** of Example 1.

EXAMPLE 1 Let **a** and **b** be vectors in $\mathbf{R}^3$ with $\mathbf{a} \neq \mathbf{0}$. Then the function $\mathbf{x}: (-\infty, \infty) \to \mathbf{R}^3$ given by

$$\mathbf{x}(t) = \mathbf{b} + t\mathbf{a}$$

defines the path along the straight line parallel to **a** and passing through the endpoint of the position vector of **b** as in Figure 3.1. (See formula (1) of §1.2.) ◆

EXAMPLE 2 The path $\mathbf{y}: [0, 2\pi) \to \mathbf{R}^2$ given by

$$\mathbf{y}(t) = (3\cos t, 3\sin t)$$

can be thought of as the path of a particle that travels once, counterclockwise, around a circle of radius 3 (Figure 3.2). ◆

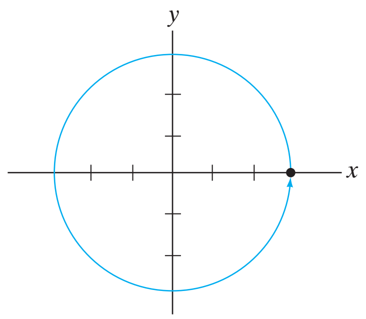

Figure 3.2 The path **y** of Example 2.

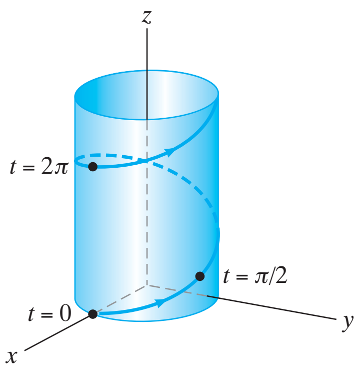

Figure 3.3 The path **z** of Example 3.

EXAMPLE 3 The map $\mathbf{z}\colon \mathbf{R} \to \mathbf{R}^3$ defined by

$$\mathbf{z}(t) = (a\cos t, a\sin t, bt), \qquad a, b \text{ constants } (a > 0)$$

is called a **circular helix**, so named because its projection in the xy-plane is a circle of radius a. The helix itself lies in the right circular cylinder $x^2 + y^2 = a^2$ (Figure 3.3). The value of b determines how tightly the helix twists. ◆

We distinguish between a path $\mathbf{x}$ and its range or image set $\mathbf{x}(I)$, the latter being a curve in $\mathbf{R}^n$. By definition, a path is a function, a dynamic object (at least when we imagine the independent variable t to represent time), whereas a curve is a static figure in space. With such a point of view, it is natural for us to consider the derivative $D\mathbf{x}(t)$, which we also write as $\mathbf{x}'(t)$ or $\mathbf{v}(t)$, to be the **velocity** vector of the path. We can readily justify such terminology. Since

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

is a function of just one variable,

$$\mathbf{v}(t) = \mathbf{x}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t}.$$

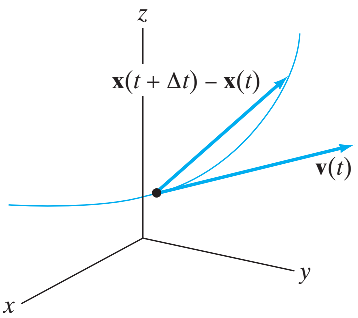

Figure 3.4 The path **x** and its velocity vector **v**.

Thus, $\mathbf{v}(t)$ is the instantaneous rate of change of position $\mathbf{x}(t)$ with respect to t (time), so it can appropriately be called velocity. Figure 3.4 provides an indication as to why we draw $\mathbf{v}(t)$ as a vector tangent to the path at $\mathbf{x}(t)$. Continuing in this vein, we introduce the following terminology:

DEFINITION 1.2 Let $\mathbf{x}\colon I \to \mathbf{R}^n$ be a differentiable path. Then the velocity $\mathbf{v}(t) = \mathbf{x}'(t)$ exists, and we define the **speed** of $\mathbf{x}$ to be the magnitude of velocity; that is,

$$\text{Speed} = \|\mathbf{v}(t)\|.$$

If $\mathbf{v}$ is itself differentiable, then we call $\mathbf{v}'(t) = \mathbf{x}''(t)$ the **acceleration** of $\mathbf{x}$ and denote it by $\mathbf{a}(t)$.

EXAMPLE 4 The helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$ has

$$\mathbf{v}(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k} \quad \text{and} \quad \mathbf{a}(t) = -a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j}.$$

Thus, the acceleration vector is parallel to the xy-plane (i.e., is horizontal). The speed of this helical path is

$$\|\mathbf{v}(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2},$$

which is constant. ◆

The velocity vector $\mathbf{v}$ is important for another reason, namely, for finding equations of tangent lines to paths. The **tangent line** to a differentiable path $\mathbf{x}$, at the point $\mathbf{x}_0 = \mathbf{x}(t_0)$, is the line through $\mathbf{x}_0$ that is parallel to any (nonzero) tangent vector to $\mathbf{x}$ at $\mathbf{x}_0$. Since $\mathbf{v}(t)$, when nonzero, is always tangent to $\mathbf{x}(t)$, we may use equation (1) of §1.2 to obtain the following vector parametric equation for the tangent line:

$$\mathbf{l}(s) = \mathbf{x}_0 + s\mathbf{v}_0. \tag{1}$$

Here $\mathbf{v}_0 = \mathbf{v}(t_0)$ and s may be any real number.

In equation (1), we have $\mathbf{l}(0) = \mathbf{x}_0$. To relate the new parameter s to the original parameter t for the path, we set $s = t - t_0$ and establish the following result:

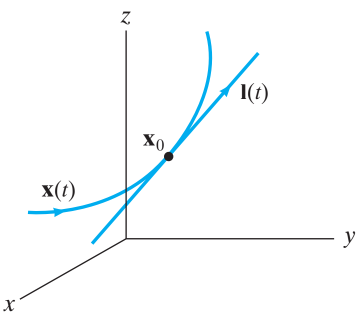

Figure 3.5 The path of the line tangent to $\mathbf{x}(t)$ at the point $\mathbf{x}_0$.

PROPOSITION 1.3 Let $\mathbf{x}$ be a differentiable path and assume that $\mathbf{v}_0 = \mathbf{v}(t_0) \neq \mathbf{0}$. Then a vector parametric equation for the line tangent to $\mathbf{x}$ at $\mathbf{x}_0 = \mathbf{x}(t_0)$ is either

$$\mathbf{l}(s) = \mathbf{x}_0 + s\mathbf{v}_0 \tag{2}$$

or

$$\mathbf{l}(t) = \mathbf{x}_0 + (t - t_0)\mathbf{v}_0. \tag{3}$$

(See Figure 3.5.)

EXAMPLE 5 If $\mathbf{x}(t) = (3t + 2, t^2 - 7, t - t^2)$, we find parametric equations for the line tangent to $\mathbf{x}$ at $(5, -6, 0) = \mathbf{x}(1)$.

For this path, $\mathbf{v}(t) = \mathbf{x}'(t) = 3\mathbf{i} + 2t\mathbf{j} + (1 - 2t)\mathbf{k}$, so that

$$\mathbf{v}_0 = \mathbf{v}(1) = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Thus, by formula (3),

$$\mathbf{l}(t) = (5\mathbf{i} - 6\mathbf{j}) + (t - 1)(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

Taking components, we read off the parametric equations for the coordinates of the tangent line as

$$\begin{cases} x = 3t + 2 \\ y = 2t - 8 \\ z = 1 - t \end{cases}.$$

◆

The physical significance of the tangent line is this: Suppose a particle of mass m travels along a path $\mathbf{x}$. If, suddenly, at $t = t_0$, all forces cease to act on the particle (so that, by Newton's second law of motion $\mathbf{F} = m\mathbf{a}$, we have $\mathbf{a}(t) \equiv \mathbf{0}$ for $t \geq t_0$), then the particle will follow the tangent line path of equation (3).

EXAMPLE 6 If Roger Ramjet is fired from a cannon, then we can use vectors to describe his trajectory. (See Figure 3.6.)

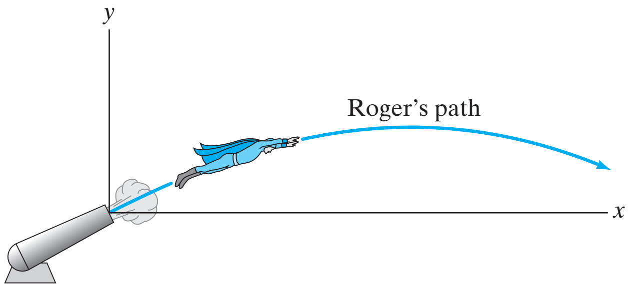

Figure 3.6 Roger Ramjet's path.

We'll assume that Roger is given an initial velocity vector $\mathbf{v}_0$ by virtue of the firing of the cannon and that thereafter the only force acting on Roger is due to gravity (so, in particular, we neglect any air resistance). Let us choose coordinates so that Roger is initially at the origin, and throughout our calculations we'll neglect the height of the cannon. Let $\mathbf{x}(t) = (x(t), y(t))$ denote Roger's path. Then the information we have is

$$\mathbf{a}(t) = \mathbf{x}''(t) = -g\,\mathbf{j}$$

(i.e., the acceleration due to gravity is constant and points downward); hence,

$$\mathbf{v}(0) = \mathbf{x}'(0) = \mathbf{v}_0$$

and

$$\mathbf{x}(0) = \mathbf{0}.$$

Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we simply integrate the expression for acceleration componentwise to find the velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\mathbf{j}\,dt = -gt\,\mathbf{j} + \mathbf{c}.$$

Here $\mathbf{c}$ is an arbitrary constant *vector* (the "constant of integration"). Since $\mathbf{v}(0) = \mathbf{v}_0$, we must have $\mathbf{c} = \mathbf{v}_0$, so that

$$\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{v}_0.$$

Integrating again to find the path,

$$\mathbf{x}(t) = \int \mathbf{v}(t)\,dt = \int (-gt\,\mathbf{j} + \mathbf{v}_0)\,dt = -\frac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{d},$$

where $\mathbf{d}$ is another arbitrary constant vector. From the remaining fact that $\mathbf{x}(0) = \mathbf{0}$, we conclude that

$$\mathbf{x}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 \qquad (4)$$

describes Roger's path.

To understand equation (4) better, we write $\mathbf{v}_0$ in terms of its components:

$$\mathbf{v}_0 = v_0 \cos\theta\,\mathbf{i} + v_0 \sin\theta\,\mathbf{j}.$$

Here $v_0 = \|\mathbf{v}_0\|$ is the initial speed. (We're really doing nothing more than expressing the rectangular components of $\mathbf{v}_0$ in terms of polar coordinates.

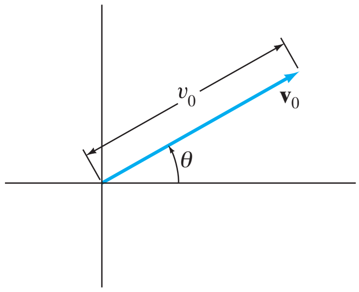

Figure 3.7 Roger's initial velocity.

See Figure 3.7.) Thus,

$$\mathbf{x}(t) = -\tfrac{1}{2}gt^2\mathbf{j} + t(v_0\cos\theta\,\mathbf{i} + v_0\sin\theta\,\mathbf{j})$$

$$= (v_0\cos\theta)t\,\mathbf{i} + \left((v_0\sin\theta)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

From this, we may read off the parametric equations:

$$\begin{cases} x = (v_0\cos\theta)t \\ y = (v_0\sin\theta)t - \dfrac{1}{2}gt^2 \end{cases},$$

from which it is not difficult to check that Roger's path traces a parabola. ◆

Here are two practical questions concerning the set-up of Example 6: First, for a given initial velocity, how far does Roger travel horizontally? Second, for a given initial speed, how should the cannon be aimed so that Roger travels (horizontally) as far as possible? To find the range of the cannon shot and thereby answer the first question, we need to know when $y = 0$ (i.e., when Roger hits the ground). Thus, we solve

$$(v_0\sin\theta)t - \tfrac{1}{2}gt^2 = t(v_0\sin\theta - \tfrac{1}{2}gt) = 0$$

for t. Hence, $y = 0$ when $t = 0$ (which is when Roger blasts off) and when $t = (2v_0\sin\theta)/g$. At this later time,

$$x = (v_0\cos\theta)\cdot\left(\frac{2v_0\sin\theta}{g}\right) = \frac{v_0^2\sin 2\theta}{g}. \tag{5}$$

Formula (5) is Roger's horizontal range for a given initial velocity. To maximize the range for a given initial speed v_0, we must choose θ so that $(v_0^2\sin 2\theta)/g$ is as large as possible. Clearly, this happens when $\sin 2\theta = 1$ (i.e., when $\theta = \pi/4$).

Kepler's Laws of Planetary Motion (optional)

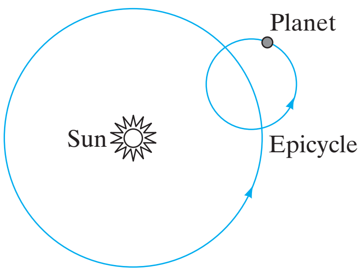

Figure 3.8 An epicycle.

Since classical antiquity, individuals have sought to understand the motions of the planets and stars. The majority of the ancient astronomers, using a combination of crude observation and faith, believed all heavenly bodies revolved around the earth. Fortunately, the heliocentric (or "sun-centered") theory of Nicholas Copernicus (1473–1543) did eventually gain favor as observational techniques improved. However, it was still believed that the planets traveled in circular orbits around the sun. This circular orbit theory did not correctly predict planetary positions, so astronomers postulated the existence of **epicycles**, smaller circular orbits traveling along the major circular arc, an example of which is shown in Figure 3.8. Although positional calculations with epicycles yielded results closer to the observed data, they still were not correct. Attempts at further improvements were made using second- and third-order epicycles, but any gains in predictive power were made at a cost of considerable calculational complexity. A new idea was needed. Such inspiration came from Johannes Kepler (1571–1630), son of a saloonkeeper and assistant to the Danish astronomer Tycho Brahe. The classical astronomers were "stuck on circles" for they believed the circle to be a perfect form and that God would use only such perfect figures for planetary motion. Kepler, however, considered the other conic sections to be as elegant as the circle and so hypothesized the simple theory that planetary orbits are elliptical. Empirical evidence bore out this theory.

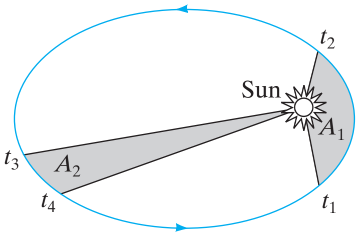

Figure 3.9 Kepler's second law of planetary motion: If $t_2 - t_1 = t_4 - t_3$, then $A_1 = A_2$, where A_1 and A_2 are the areas of the shaded regions.

Kepler's three laws of planetary motion are

1. The orbit of a planet is elliptical, with the sun at a focus of the ellipse.
2. During equal periods of time, a planet sweeps through equal areas with respect to the sun. (See Figure 3.9.)
3. The square of the period of one elliptical orbit is proportional to the cube of the length of the semimajor axis of the ellipse.

Kepler's laws changed the face of astronomy. We emphasize, however, that they were discovered empirically, not analytically derived from general physical laws. The first analytic derivation is frequently credited to Newton, who claimed to have established Kepler's laws (at least the first and third laws) in Book I of his *Philosophiae Naturalis Principia Mathematica* (1687). However, a number of scientists and historians of science now consider Newton's proof of Kepler's first law to be flawed and that Johann Bernoulli (1667–1748) offered the first rigorous derivation in 1710.[1] In the discussion that follows, Newton's law of universal gravitation is used to prove all three of Kepler's laws.

In our work below, we assume that the only physical effects are those between the sun and a single planet—the so-called two-body problem. (The n-body problem, where $n \geq 3$ is, by contrast, an important area of current mathematical research.) To set the stage for our calculations, we take the sun to be fixed at the origin O in $\mathbf{R}^3$ and the planet to be at the moving position P. We also need the following two "vector product rules," whose proofs we leave to you:

PROPOSITION 1.4

1. If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^n$, then
$$\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) = \mathbf{y} \cdot \frac{d\mathbf{x}}{dt} + \mathbf{x} \cdot \frac{d\mathbf{y}}{dt}.$$
2. If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^3$, then
$$\frac{d}{dt}(\mathbf{x} \times \mathbf{y}) = \frac{d\mathbf{x}}{dt} \times \mathbf{y} + \mathbf{x} \times \frac{d\mathbf{y}}{dt}.$$

First, we establish the following preliminary result:

PROPOSITION 1.5 The motion of the planet is planar, and the sun lies in the planet's plane of motion.

PROOF Let $\mathbf{r} = \overrightarrow{OP}$. Then $\mathbf{r}$ is a vector whose representative arrow has its tail fixed at O. (Note that $\mathbf{r} = \mathbf{r}(t)$; that is, $\mathbf{r}$ is a function of time.) If $\mathbf{v} = \mathbf{r}'(t)$, we will show that $\mathbf{r} \times \mathbf{v}$ is a constant vector $\mathbf{c}$. This result, in turn, implies that $\mathbf{r}$ must always be perpendicular to $\mathbf{c}$ and, hence, that $\mathbf{r}$ always lies in a plane with $\mathbf{c}$ as normal vector.

To show that $\mathbf{r} \times \mathbf{v}$ is constant, we show that its derivative is zero. By part 2 of Proposition 1.4,
$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a},$$

[1] For an indication of the more recent controversy surrounding Newton's mathematical accomplishments, see R. Weinstock, "Isaac Newton: Credit where credit won't do," *The College Mathematics Journal,* **25** (1994), no. 3, 179–192, and C. Wilson, "Newton's orbit problem: A historian's response," *Ibid.,* 193–200, and related papers.

by the definitions of velocity and acceleration. We know that $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ (why?), so

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \mathbf{a}. \tag{6}$$

Now we use Newton's laws. Newton's law of gravitation tells us that the planet is attracted to the sun with a force

$$\mathbf{F} = -\frac{GMm}{r^2}\,\mathbf{u}, \tag{7}$$

where G is Newton's gravitational constant ($= 6.6720 \times 10^{-11}$ Nm^2/kg^2), M is the mass of the sun, m is the mass of the planet (in kilograms), $r = \|\mathbf{r}\|$, and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ (distances in meters). On the other hand, Newton's second law of motion states that, for the planet,

$$\mathbf{F} = m\mathbf{a}.$$

Thus,

$$m\mathbf{a} = -\frac{GMm}{r^2}\,\mathbf{u},$$

or

$$\mathbf{a} = -\frac{GM}{r^3}\,\mathbf{r}. \tag{8}$$

Therefore, $\mathbf{a}$ is just a scalar multiple of $\mathbf{r}$ and hence is always parallel to $\mathbf{r}$. In view of equations (6) and (8), we conclude that

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{r} \times \mathbf{a} = \mathbf{0}$$

(i.e., that $\mathbf{r} \times \mathbf{v}$ is constant). ■

THEOREM 1.6 (KEPLER'S FIRST LAW) In a two-body system consisting of one sun and one planet, the planet's orbit is an ellipse and the sun lies at one focus of that ellipse.

PROOF We will eventually find a polar equation for the planet's orbit and see that this equation defines an ellipse as described. We retain the notation from the proof of Proposition 1.5 and take coordinates for $\mathbf{R}^3$ so that the sun is at the origin, and the path of the planet lies in the xy-plane. Then the constant vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$ used in the proof of Proposition 1.5 may be written as $c\mathbf{k}$, where c is some nonzero real number. This set-up is shown in Figure 3.10.

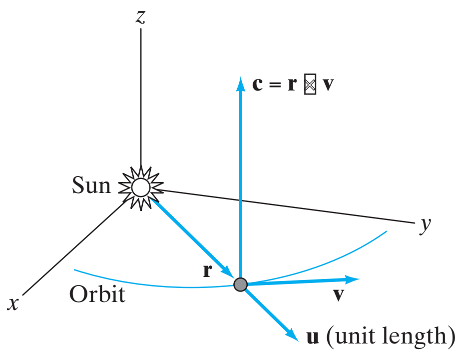

Figure 3.10 Establishing Kepler's laws.

Step 1. We find another expression for $\mathbf{c}$. By definition of $\mathbf{u}$ in formula (7), $\mathbf{r} = r\mathbf{u}$, so that, by the product rule,

$$\mathbf{v} = \frac{d}{dt}(r\mathbf{u}) = r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}.$$

Hence,

$$\mathbf{c} = \mathbf{r} \times \mathbf{v} = (r\mathbf{u}) \times \left(r\frac{d\mathbf{u}}{dt} + \frac{dr}{dt}\mathbf{u}\right) = r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) + r\frac{dr}{dt}(\mathbf{u} \times \mathbf{u}).$$

Since $\mathbf{u} \times \mathbf{u}$ must be zero, we conclude that

$$\mathbf{c} = r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right). \tag{9}$$

Step 2. We derive the polar equation for the orbit. Before doing so, however, note the following result, whose proof is left to you as an exercise:

PROPOSITION 1.7 If $\mathbf{x}(t)$ has constant length (i.e., $\|\mathbf{x}(t)\|$ is constant for all t), then $\mathbf{x}$ is perpendicular to its derivative $d\mathbf{x}/dt$.

Continuing now with the main argument, note that the vector $\mathbf{r}(t)$ is defined so that its magnitude is precisely the polar coordinate r of the planet's position. Using equations (8) and (9), we find that

$$\begin{aligned}
\mathbf{a} \times \mathbf{c} &= \left(-\frac{GM}{r^2}\mathbf{u}\right) \times r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \\
&= -GM\left[\mathbf{u} \times \left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right)\right] \\
&= GM\left[\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \times \mathbf{u}\right] \\
&= GM\left[(\mathbf{u} \cdot \mathbf{u})\frac{d\mathbf{u}}{dt} - \left(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}\right)\mathbf{u}\right] && \text{(see Exercise 27 of §1.4)} \\
&= GM\left[1\frac{d\mathbf{u}}{dt} - 0\mathbf{u}\right] && \text{(by Proposition 1.7)} \\
&= \frac{d}{dt}(GM\mathbf{u}),
\end{aligned}$$

since G and M are constant. On the other hand, we can "reverse" the product rule to find that

$$\begin{aligned}
\mathbf{a} \times \mathbf{c} &= \frac{d\mathbf{v}}{dt} \times \mathbf{c} \\
&= \frac{d\mathbf{v}}{dt} \times \mathbf{c} + \mathbf{v} \times \frac{d\mathbf{c}}{dt} && \text{(since } \mathbf{c} \text{ is constant)} \\
&= \frac{d}{dt}(\mathbf{v} \times \mathbf{c}).
\end{aligned}$$

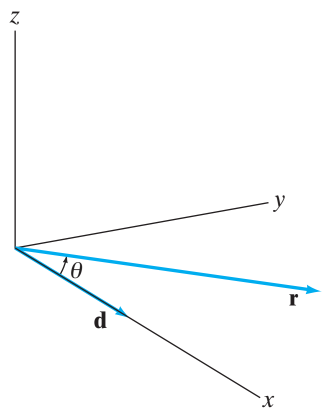

Figure 3.11 The angle θ is the angle between $\mathbf{r}$ and $\mathbf{d}$.

Thus,

$$\mathbf{a} \times \mathbf{c} = \frac{d}{dt}(GM\mathbf{u}) = \frac{d}{dt}(\mathbf{v} \times \mathbf{c}),$$

and, hence,

$$\mathbf{v} \times \mathbf{c} = GM\mathbf{u} + \mathbf{d}, \tag{10}$$

where $\mathbf{d}$ is an arbitrary constant vector. Because both $\mathbf{v} \times \mathbf{c}$ and $\mathbf{u}$ lie in the xy-plane, so must $\mathbf{d}$.

Let us adjust coordinates, if necessary, so that $\mathbf{d}$ points in the $\mathbf{i}$-direction (i.e., so that $\mathbf{d} = d\mathbf{i}$ for some $d \in \mathbf{R}$). This can be accomplished by rotating the whole set-up about the z-axis, which does not lift anything lying in the xy-plane out of that plane. Then the angle between $\mathbf{r}$ (and hence $\mathbf{u}$) and $\mathbf{d}$ is the polar angle θ as shown in Figure 3.11.

By Theorem 3.3 of Chapter 1,

$$\mathbf{u} \cdot \mathbf{d} = \|\mathbf{u}\| \, \|\mathbf{d}\| \cos\theta = d\cos\theta. \tag{11}$$

Since $c = \|\mathbf{c}\|$,

$$\begin{aligned} c^2 &= \mathbf{c} \cdot \mathbf{c} \\ &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} \\ &= \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) && \text{(Why? See formula (4) of §1.4.)} \\ &= r\mathbf{u} \cdot (GM\mathbf{u} + \mathbf{d}) && \text{by equation (10).} \end{aligned}$$

Hence,

$$c^2 = GMr + rd\cos\theta$$

by equation (11). We can readily solve this equation for r to obtain

$$r = \frac{c^2}{GM + d\cos\theta}, \tag{12}$$

the polar equation for the planet's orbit.

Step 3. We now check that equation (12) really does define an ellipse by converting to Cartesian coordinates. First, we'll rewrite the equation as

$$r = \frac{c^2}{GM + d\cos\theta} = \frac{(c^2/GM)}{1 + (d/GM)\cos\theta},$$

and then let $p = c^2/GM, e = d/GM$ for convenience. (Note that $p > 0$.) Hence, equation (12) becomes

$$r = \frac{p}{1 + e\cos\theta}. \tag{13}$$

A little algebra provides the equivalent equation,

$$r = p - er\cos\theta. \tag{14}$$

Now $r\cos\theta = x$ (x being the usual Cartesian coordinate), so that equation (14) is equivalent to

$$r = p - ex.$$

To complete the conversion, we square both sides and find, by virtue of the fact that $r^2 = x^2 + y^2$,

$$x^2 + y^2 = p^2 - 2pex + e^2x^2.$$

A little more algebra reveals that

$$(1-e^2)x^2 + 2pex + y^2 = p^2. \tag{15}$$

Therefore, the curve described by the preceding equation is an ellipse if $0 < |e| < 1$, a parabola if $e = \pm 1$, and a hyperbola if $|e| > 1$. Analytically, there is no way to eliminate the last two possibilities. Indeed, "uncaptured" objects such as comets or expendable deep space probes can have hyperbolic or parabolic orbits. However, to have a *closed* orbit (so that the planet repeats its transit across the sky), we are forced to conclude that the orbit must be elliptical.

More can be said about the elliptical orbit. Dividing equation (15) by $1 - e^2$ and completing the square in x, we have

$$\left(x + \frac{pe}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{p^2}{(1-e^2)^2}.$$

This is equivalent to the rather awkward-looking equation

$$\frac{\left(x + pe/(1-e^2)\right)^2}{p^2/(1-e^2)^2} + \frac{y^2}{p^2/(1-e^2)} = 1. \tag{16}$$

From equation (16), we see that the ellipse is centered at the point $(-pe/(1-e^2), 0)$, that its semimajor axis has length $a = p/(1-e^2)$, and that its semiminor axis has length $b = p/\sqrt{1-e^2}$. The foci of the ellipse are at a distance

$$\sqrt{a^2 - b^2} = \sqrt{\frac{p^2}{(1-e^2)^2} - \frac{p^2}{1-e^2}} = \frac{p|e|}{1-e^2}$$

from the center. (See Figure 3.12.) Hence, we see that one focus must be at the origin, the location of the sun. Our proof is, therefore, complete. ■

Fortunately, all the toil involved in proving the first law will pay off in proofs of the second and third laws, which are considerably shorter. Again, we retain all the notation we already introduced.

THEOREM 1.8 (KEPLER'S SECOND LAW) During equal intervals of time, a planet sweeps through equal areas with respect to the sun.

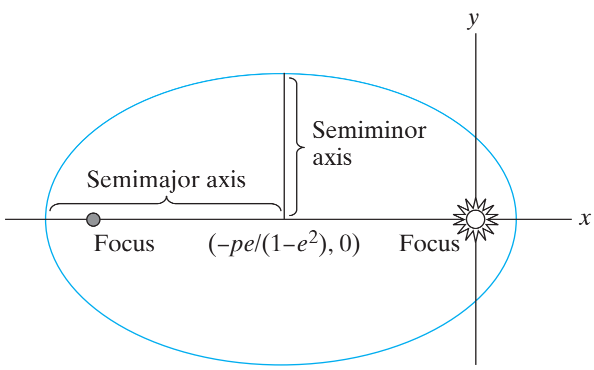

Figure 3.12 The ellipse of equation (16).

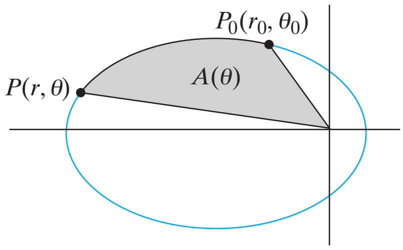

Figure 3.13 The shaded area $A(\theta)$ is given by $\int_{\theta_0}^{\theta} \frac{1}{2} r^2\, d\varphi$.

PROOF Fix one point P_0 on the planet's orbit. Then the area A swept between P_0 and a second (moving) point P on the orbit is given by the polar area integral

$$A(\theta) = \int_{\theta_0}^{\theta} \frac{1}{2} r^2\, d\varphi.$$

(See Figure 3.13.) Thus, we may reformulate Kepler's law to say that dA/dt is constant. We establish this reformulation by relating dA/dt to a known constant, namely, the vector $\mathbf{c} = \mathbf{r} \times \mathbf{v}$.

By the chain rule (in one variable),

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt}.$$

By the fundamental theorem of calculus,

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\int_{\theta_0}^{\theta} \frac{1}{2} r^2\, d\varphi = \frac{1}{2}[r(\theta)]^2.$$

Hence,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}. \tag{17}$$

Now, we relate $\mathbf{c}$ to $d\theta/dt$ by means of equation (9). Therefore, we compute $\mathbf{u} \times d\mathbf{u}/dt$ in terms of θ. Recall that $\mathbf{u} = \dfrac{1}{r}\mathbf{r}$ and $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$. Thus,

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$\frac{d\mathbf{u}}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}.$$

Hence, it follows by direct calculation of the cross product that

$$\mathbf{c} = r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) = r^2\frac{d\theta}{dt}\mathbf{k},$$

so $c = \|\mathbf{c}\| = r^2 d\theta/dt$, and equation (17) implies that

$$\frac{dA}{dt} = \frac{1}{2}c, \tag{18}$$

a constant. ■

THEOREM 1.9 (KEPLER'S THIRD LAW) If T is the length of time for one planetary orbit, and a is the length of the semimajor axis of this orbit, then $T^2 = Ka^3$ for some constant K.

PROOF We focus on the total area enclosed by the elliptical orbit. The area of an ellipse whose semimajor and semiminor axes have lengths a and b, respectively, is πab. This area must also be that swept by the planet in the time interval $[0, T]$. Thus, we have

$$\begin{aligned}\pi ab &= \int_0^T \frac{dA}{dt}\,dt \\ &= \int_0^T \frac{1}{2}c\,dt \qquad \text{by equation (18)} \\ &= \frac{1}{2}cT.\end{aligned}$$

Hence,

$$T = \frac{2\pi ab}{c}, \quad \text{so} \quad T^2 = \frac{4\pi^2 a^2 b^2}{c^2}. \tag{19}$$

Now, b and c are related to a, so these quantities must be replaced before we are done. In particular, from equation (16), $b^2 = p^2/(1 - e^2)$, so

$$b^2 = pa.$$

Also

$$p = \frac{c^2}{GM}.$$

(See equations (12) and (13).) With these substitutions, the result in (19) becomes

$$T^2 = \frac{4\pi^2 a^2 (pa)}{pGM} = \left(\frac{4\pi^2}{GM}\right) a^3.$$

This last equation shows that T^2 is proportional to a^3, but it says even more: The constant of proportionality $4\pi^2/GM$ depends entirely on the mass of the sun—the constant is the same for *any* planet that might revolve around the sun. ■

3.1 Exercises

In Exercises 1–6, sketch the images of the following paths, using arrows to indicate the direction in which the parameter increases:

1. $\begin{cases} x = 2t - 1 \\ y = 3 - t \end{cases}, \quad -1 \le t \le 1$

2. $\mathbf{x}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$

3. $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}, \quad -6\pi \le t \le 6\pi$

4. $\begin{cases} x = 3\cos t \\ y = 2\sin 2t \end{cases}, \quad 0 \le t \le 2\pi$

5. $\mathbf{x}(t) = (t, 3t^2 + 1, 0)$

6. $\mathbf{x}(t) = (t, t^2, t^3)$

Calculate the velocity, speed, and acceleration of the paths given in Exercises 7–10.

7. $\mathbf{x}(t) = (3t - 5)\mathbf{i} + (2t + 7)\mathbf{j}$

8. $\mathbf{x}(t) = 5\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

9. $\mathbf{x}(t) = (t\sin t, t\cos t, t^2)$

10. $\mathbf{x}(t) = (e^t, e^{2t}, 2e^t)$

In Exercises 11–14, (a) use a computer to give a plot of the given path $\mathbf{x}$ *over the indicated interval for* t*; identify the direction in which* t *increases. (b) Show that the path lies on the given surface* S*.*

T **11.** $\mathbf{x}(t) = (3\cos \pi t, 4\sin \pi t, 2t)$, $-4 \le t \le 4$; S is elliptical cylinder $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

T **12.** $\mathbf{x}(t) = (t\cos t, t\sin t, t)$, $-20 \le t \le 20$; S is cone $z^2 = x^2 + y^2$.

T **13.** $\mathbf{x}(t) = (t\sin 2t, t\cos 2t, t^2)$, $-6 \le t \le 6$; S is paraboloid $z = x^2 + y^2$.

14. $\mathbf{x}(t) = (2\cos t, 2\sin t, 3\sin 8t)$, $0 \le t \le 2\pi$; S is cylinder $x^2 + y^2 = 4$.

In Exercises 15–18, find an equation for the line tangent to the given path at the indicated value for the parameter.

15. $\mathbf{x}(t) = te^{-t}\,\mathbf{i} + e^{3t}\,\mathbf{j}$, $t = 0$

16. $\mathbf{x}(t) = 4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j} + 5t\,\mathbf{k}$, $t = \pi/3$

17. $\mathbf{x}(t) = (t^2, t^3, t^5)$, $t = 2$

18. $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$, $t = 1$

19. (a) Sketch the path $\mathbf{x}(t) = (t, t^3 - 2t + 1)$.

(b) Calculate the line tangent to $\mathbf{x}$ when $t = 2$.

(c) Describe the image of $\mathbf{x}$ by an equation of the form $y = f(x)$ by eliminating t.

(d) Verify your answer in part (b) by recalculating the tangent line, using your result in part (c).

Exercises 20–23 concern Roger Ramjet and his trajectory when he is shot from a cannon as in Example 6 of this section.

20. Verify that Roger Ramjet's path in Example 6 is indeed a parabola.

21. Suppose that Roger is fired from the cannon with an angle of inclination θ of 60° and an initial speed v_0 of 100 ft/sec. What is the maximum height Roger attains?

22. Suppose that Roger is fired from the cannon with an angle of inclination θ of 60° and that he hits the ground $1/2$ mile from the cannon. What, then, was Roger's initial speed?

23. If Roger is fired from the cannon with an initial speed of 250 ft/sec, what angle of inclination θ should be used so that Roger hits the ground 1500 ft from the cannon?

24. Gertrude is aiming a Super Drencher water pistol at Egbert, who is 1.6 m tall and is standing 5 m away. Gertrude holds the water gun 1 m above ground at an angle α of elevation. (See Figure 3.14.)

(a) If the water pistol fires with an initial speed of 7 m/sec and an elevation angle of 45°, does Egbert get wet?

(b) If the water pistol fires with an initial speed of 8 m/sec, what possible angles of elevation will cause Egbert to get wet? (Note: You will want to use a computer algebra system or a graphics calculator for this part.)

25. A malfunctioning rocket is traveling according to the path $\mathbf{x}(t) = \left(e^{2t}, 3t^3 - 2t, t - \frac{1}{t}\right)$ in the hope of reaching a repair station at the point $(7e^4, 35, 5)$. (Here t represents time in minutes and spatial coordinates are measured in miles.) At $t = 2$, the rocket's engines suddenly cease. Will the rocket coast into the repair station?

26. Two billiard balls are moving on a (coordinatized) pool table according to the respective paths $\mathbf{x}(t) = \left(t^2 - 2, \frac{t^2}{2} - 1\right)$ and $\mathbf{y}(t) = (t, 5 - t^2)$, where t represents time measured in seconds.

(a) When and where do the balls collide?

(b) What is the angle formed by the paths of the balls at the collision point?

27. Establish part 1 of Proposition 1.4 in this section: If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^n$, show that

$$\frac{d}{dt}(\mathbf{x}\cdot\mathbf{y}) = \mathbf{y}\cdot\frac{d\mathbf{x}}{dt} + \mathbf{x}\cdot\frac{d\mathbf{y}}{dt}.$$

28. Establish part 2 of Proposition 1.4 in this section: If $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in $\mathbf{R}^3$, show that

$$\frac{d}{dt}(\mathbf{x}\times\mathbf{y}) = \frac{d\mathbf{x}}{dt}\times\mathbf{y} + \mathbf{x}\times\frac{d\mathbf{y}}{dt}.$$

29. Prove Proposition 1.7.

30. (a) Show that the path $\mathbf{x}(t) = (\cos t, \cos t\sin t, \sin^2 t)$ lies on a unit sphere.

(b) Verify that $\mathbf{x}(t)$ is always perpendicular to the velocity vector $\mathbf{v}(t)$.

(c) Use Proposition 1.7 to show that if a differentiable path lies on a sphere centered at the origin, then its position vector is always perpendicular to its velocity vector.

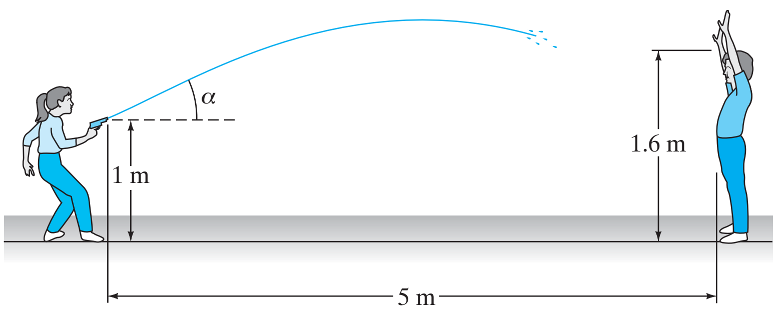

Figure 3.14 Figure for Exercise 24.

31. Consider the path

$$\begin{cases} x = (a + b\cos\omega t)\cos t \\ y = (a + b\cos\omega t)\sin t\,, \\ z = b\sin\omega t \end{cases}$$

where a, b, and ω are positive constants and $a > b$.

(a) Use a computer to plot this path when

i. $a = 3$, $b = 1$, and $\omega = 15$.
ii. $a = 5$, $b = 1$, and $\omega = 15$.
iii. $a = 5$, $b = 1$, and $\omega = 25$.

Comment on how the values of a, b, and ω affect the shapes of the image curves.

(b) Show that the image curve lies on the torus

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2.$$

(A **torus** is the surface of a doughnut.)

32. For the path $\mathbf{x}(t) = (e^t\cos t, e^t\sin t)$, show that the angle between $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ remains constant. What is the angle?

33. Consider the path $\mathbf{x}\colon \mathbf{R} \to \mathbf{R}^2$, $\mathbf{x}(t) = (t^2, t^3 - t)$.

(a) Show that this path intersects itself, that is, that there are numbers t_1 and t_2 such that $\mathbf{x}(t_1) = \mathbf{x}(t_2)$.

(b) At the point where the path intersects itself, it makes sense to say that the image curve has *two* tangent lines. What is the angle between these tangent lines?

34. Although the path $\mathbf{x} : [0, 2\pi] \to \mathbf{R}^2$, $\mathbf{x}(t) = (\cos t, \sin t)$ may be the most familiar way to give a parametric description of a unit circle, in this problem you will develop a different set of parametric equations that gives the x- and y-coordinates of a point on the circle in terms of rational functions of the parameter. (This particular parametrization turns out to be useful in the branch of mathematics known as number theory.)

To set things up, begin with the unit circle $x^2 + y^2 = 1$ and consider all lines through the point $(-1, 0)$. (See Figure 3.15.) Note that every line other than the vertical line $x = -1$ intersects the circle at a point (x, y) other than $(-1, 0)$. Let the parameter t be the slope of the line joining $(-1, 0)$ and a point (x, y) on the circle.

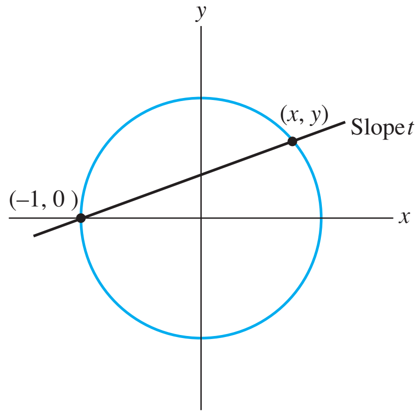

Figure 3.15 Figure for Exercise 34.

(a) Give an equation for the line of slope t joining $(-1, 0)$ and (x, y). (Your answer should involve x, y, and t.)

(b) Use your answer in part (a) to write y in terms of x and t. Then substitute this expression for y into the equation for the unit circle. Solve the resulting equations for x in terms of t. Your answer(s) for x will give the points of intersection of the line and the circle.

(c) Use your result in part (b) to give a set of parametric equations for points (x, y) on the unit circle.

(d) Does your parametrization in part (c) cover the entire circle? Which, if any, points are missed?

35. Let $\mathbf{x}(t)$ be a path of class C^1 that does not pass through the origin in $\mathbf{R}^3$. If $\mathbf{x}(t_0)$ is the point on the image of $\mathbf{x}$ closest to the origin and $\mathbf{x}'(t_0) \neq \mathbf{0}$, show that the position vector $\mathbf{x}(t_0)$ is orthogonal to the velocity vector $\mathbf{x}'(t_0)$.

3.2 Arclength and Differential Geometry

In this section, we continue our general study of parametrized curves in $\mathbf{R}^3$, considering how to measure such geometric properties as length and curvature. This can be done by defining three mutually perpendicular unit vectors that form the so-called moving frame specially adapted to a path $\mathbf{x}$. Our study takes us briefly into the branch of mathematics called **differential geometry**, an area where calculus and analysis are used to understand the geometry of curves, surfaces, and certain higher-dimensional objects (called **manifolds**).

Length of a Path

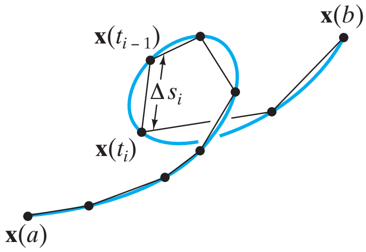

Figure 3.16 Approximating the length of a C^1 path.

For now, let $\mathbf{x}\colon [a, b] \to \mathbf{R}^3$ be a C^1 path in $\mathbf{R}^3$. Then we can approximate the length L of $\mathbf{x}$ as follows: First, partition the interval $[a, b]$ into n subintervals. That is, choose numbers $t_0, t_1, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$. If, for $i = 1, \ldots, n$, we let Δs_i denote the distance between the points $\mathbf{x}(t_{i-1})$ and $\mathbf{x}(t_i)$ on the path, then

$$L \approx \sum_{i=1}^{n} \Delta s_i. \tag{1}$$

(See Figure 3.16.) We have $\mathbf{x}(t) = (x(t), y(t), z(t))$, so that the distance formula (i.e., the Pythagorean theorem) implies

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2},$$

where $\Delta x_i = x(t_i) - x(t_{i-1})$, $\Delta y_i = y(t_i) - y(t_{i-1})$, and $\Delta z_i = z(t_i) - z(t_{i-1})$. It is entirely reasonable to hope that the approximation in (1) improves as the Δt_i's become closer to zero. Hence, we *define* the length L of $\mathbf{x}$ to be

$$L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i{}^2 + \Delta y_i{}^2 + \Delta z_i{}^2}. \tag{2}$$

Now, we find a way to rewrite equation (2) as an integral. On each subinterval $[t_{i-1}, t_i]$, apply the mean value theorem (three times) to conclude the following:

1. There must be some number t_i^* in $[t_{i-1}, t_i]$ such that

$$x(t_i) - x(t_{i-1}) = x'(t_i^*)(t_i - t_{i-1});$$

that is, $\Delta x_i = x'(t_i^*)\Delta t_i$.

2. There must be another number t_i^{**} in $[t_{i-1}, t_i]$ such that

$$\Delta y_i = y'(t_i^{**})\Delta t_i.$$

3. There must be a third number t_i^{***} in $[t_{i-1}, t_i]$ such that

$$\Delta z_i = z'(t_i^{***})\Delta t_i.$$

Therefore, with a little algebra, equation (2) becomes

$$L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{x'(t_i^*)^2 + y'(t_i^{**})^2 + z'(t_i^{***})^2}\, \Delta t_i. \tag{3}$$

When the limit appearing in equation (3) is finite, it gives the value of the definite integral

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Note that the integrand is precisely $\|\mathbf{x}'(t)\|$, the speed of the path. (This makes perfect sense, of course. Speed measures the rate of distance traveled per unit time, so integrating the speed over the elapsed time interval should give the total distance traveled.) Moreover, it's not hard to see how we should go about defining the length of a path in $\mathbf{R}^n$ for arbitrary n.

DEFINITION 2.1 The **length** $L(\mathbf{x})$ of a C^1 path $\mathbf{x}\colon [a, b] \to \mathbf{R}^n$ is found by integrating its speed:

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\|\, dt.$$

EXAMPLE 1 To check our definition in a well-known situation, we compute the length of the path

$$\mathbf{x}\colon [0, 2\pi] \to \mathbf{R}^2, \qquad \mathbf{x}(t) = (a\cos t, a\sin t), \qquad a > 0.$$

We have

$$\mathbf{x}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j},$$

so

$$\|\mathbf{x}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a.$$

Thus, Definition 2.1 gives

$$L(\mathbf{x}) = \int_0^{2\pi} a\, dt = 2\pi a.$$

Since the path traces a circle of radius a once, the length integral works out to be the circumference of the circle, as it should. ◆

EXAMPLE 2 For the helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$, we have

$$\mathbf{x}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k},$$

so that $\|\mathbf{x}'(t)\| = \sqrt{a^2 + b^2}$, and

$$L(\mathbf{x}) = \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = 2\pi\sqrt{a^2 + b^2}.$$

When $b = 0$, the helix reverts to a circle and the length integral agrees with the previous example. ◆

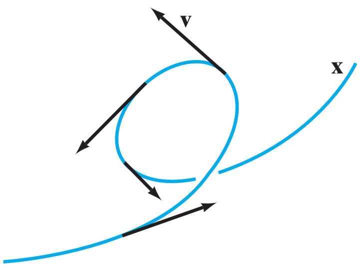

Figure 3.17 A C^1 path.

Although we have defined the length integral only for C^1 (or "smooth-looking") paths, there is no problem with extending our definition to the **piecewise** C^1 case. By definition, a C^1 path is one with a continuously varying velocity vector, and so it typically looks like the path in Figure 3.17. A piecewise C^1 path is one that may not be C^1 but instead consists of finitely many C^1 chunks. A continuous, piecewise C^1 path that is *not* C^1 typically looks like the path in Figure 3.18. Each of the three portions of the path defined for (i) $a \le t \le t_1$, (ii) $t_1 \le t \le t_2$, and (iii) $t_2 \le t \le b$ is of class C^1, but the velocity, if nonzero, would be discontinuous at $t = t_1$ and $t = t_2$. To define the length of a piecewise C^1 path, all we need do is break up the path into its C^1 pieces, calculate the length of each piece, and add to get the total length. For the piecewise C^1 path shown in Figure 3.18, this means we would take

$$\int_a^{t_1} \|\mathbf{x}'(t)\|\, dt + \int_{t_1}^{t_2} \|\mathbf{x}'(t)\|\, dt + \int_{t_2}^{b} \|\mathbf{x}'(t)\|\, dt$$

to be the length.

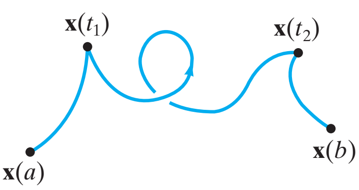

Figure 3.18 A piecewise C^1 path $\mathbf{x}\colon [a, b] \to \mathbf{R}^3$.

WARNING Even if a path is continuous, the definite integral in Definition 2.1 may fail to exist. An example of such an unfortunate situation is furnished by the path $\mathbf{x}\colon [0, 1] \to \mathbf{R}^2$,

$$\mathbf{x}(t) = (t, y(t)), \quad \text{where} \quad y(t) = \begin{cases} t \sin \dfrac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Such a path is called **nonrectifiable**. It is a fact that any C^1 path with endpoints is rectifiable, which is why we made such a condition part of Definition 2.1.

The Arclength Parameter

The calculation of the length of a path is not only useful (and moderately interesting) in itself, but it also provides a way for us to **reparametrize** the path with a parameter that depends solely on the geometry of the curve traced by the path, not on the way in which the curve is traced.

Let $\mathbf{x}$ be any C^1 path and assume that the velocity $\mathbf{x}'$ is never zero. Fix a point P_0 on the path and let a be such that $\mathbf{x}(a) = P_0$. We define a one-variable function s of the given parameter t that measures the length of the path from P_0 to any other (moving) point P by

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau. \tag{4}$$

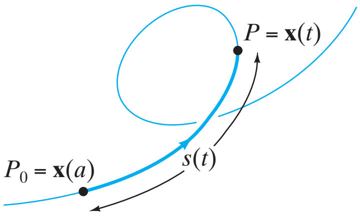

Figure 3.19 The arclength reparametrization.

(See Figure 3.19. The Greek letter tau, τ, is used purely as a dummy variable—the standard convention is never to have the same variable appearing in both the integrand and either of the limits of integration.) If t happens to be less than a, then the value of s in formula (4) will be negative. This is nothing more than a consequence of how the "base point" P_0 is chosen.

Here's how to get the new parameter: From formula (4) and from the fundamental theorem of calculus,

$$\frac{ds}{dt} = \frac{d}{dt}\int_a^t \|\mathbf{x}'(\tau)\|\, d\tau = \|\mathbf{x}'(t)\| = \text{speed}. \tag{5}$$

Since we have assumed that $\mathbf{x}'(t) \neq \mathbf{0}$, it follows that ds/dt is nonzero. Hence, ds/dt is always positive, so s is a strictly increasing function of t. Thus, s is, in fact, an invertible function; that is, it is at least theoretically possible to solve the equation $s = s(t)$ for t in terms of s. If we imagine doing this, then we can reparametrize the path $\mathbf{x}$, using the arclength parameter s as independent variable.

EXAMPLE 3 For the helix $\mathbf{x}(t) = (a \cos t, a \sin t, bt)$, if we choose the "base point" P_0 to be $\mathbf{x}(0) = (a, 0, 0)$, then we have

$$s(t) = \int_0^t \|\mathbf{x}'(\tau)\|\, d\tau = \int_0^t \sqrt{a^2 + b^2}\, d\tau = \sqrt{a^2 + b^2}\, t,$$

so that

$$s = \sqrt{a^2 + b^2}\, t,$$

or

$$t = \frac{s}{\sqrt{a^2 + b^2}}.$$

(What the preceding tells us is that this reparametrization just rescales the time variable.) Hence, we can rewrite the helical path as

$$\mathbf{x}(s) = \left(a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right). \qquad \blacklozenge$$

EXAMPLE 4 The explicit determination of the arclength parameter for a given parametrized path is a delicate matter. Consider the path

$$\mathbf{x}(t) = \left(t, \frac{\sqrt{2}}{2} t^2, \frac{1}{3} t^3 \right).$$

Then $\mathbf{x}'(t) = (1, \sqrt{2}t, t^2)$ and, if we take the base point to be $\mathbf{x}(0) = (0, 0, 0)$, then

$$\begin{aligned} s(t) &= \int_0^t \sqrt{1 + 2\tau^2 + \tau^4}\, d\tau \\ &= \int_0^t \sqrt{(1+\tau^2)^2}\, d\tau = \int_0^t (1+\tau^2)\, d\tau = t + \frac{t^3}{3}. \end{aligned}$$

On the other hand, the path $\mathbf{y}(t) = (t, t^2, t^3)$ is quite similar to $\mathbf{x}$, yet it has no readily calculable arclength parameter. In this case, $\mathbf{y}'(t) = (1, 2t, 3t^2)$ and the resulting integral for $s(t)$ is

$$s(t) = \int_0^t \sqrt{1 + 4\tau^2 + 9\tau^4}\, d\tau.$$

It can be shown that this integral has no "closed form" formula (i.e., a formula that involves only finitely many algebraic and transcendental functions). ◆

The significance of the arclength parameter s is that it is an **intrinsic** parameter; it depends only on how the curve itself bends, not on how fast (or slowly) the curve is traced. To see more precisely what this means, we resort to the chain rule. Consider s as an intermediate variable and t as a final variable. Then we have

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{x}'(s)\frac{ds}{dt} && \text{by the chain rule,} \\ &= \mathbf{x}'(s)\|\mathbf{x}'(t)\| && \text{by (5).} \end{aligned}$$

Since $\mathbf{x}'(t) \neq \mathbf{0}$, we can solve for $\mathbf{x}'(s)$ to find

$$\mathbf{x}'(s) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}. \tag{6}$$

Therefore, $\mathbf{x}'(s)$ is precisely the normalization of the original velocity vector, and so it is a unit vector. Hence, the reparametrized path $\mathbf{x}(s)$ has **unit speed**, regardless of the speed of the original path $\mathbf{x}(t)$. (This result makes good geometric sense, too. If arclength, rather than time, is the parameter, then speed is measured in units of "length per length," which necessarily must be one.)

The only unfortunate note to our story is that the integral in formula (4) is usually impossible to compute exactly, thus making it impossible to compute s as a simple function of t. (The case of the helix is a convenient and rather special exception.) One generally prefers to work indirectly, letting the chain rule come to the rescue. We shall see this indirect approach next.

The Unit Tangent Vector and Curvature

Let $\mathbf{x}: I \subseteq \mathbf{R} \to \mathbf{R}^3$ be a C^3 path and assume that $\mathbf{x}'$ is never zero.

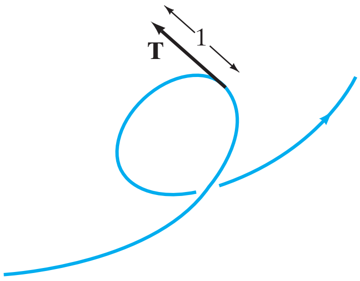

Figure 3.20 A unit tangent vector.

DEFINITION 2.2 The **unit tangent vector T** of the path $\mathbf{x}$ is the normalization of the velocity vector; that is,

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

We see from Definition 2.2 that the unit tangent vector is undefined when the speed of the path is zero. Also note that, from equation (6), $\mathbf{T}$ is $d\mathbf{x}/ds$, where s is the arclength parameter. Geometrically, $\mathbf{T}$ is the tangent vector of unit length that points in the direction of increasing arclength, as suggested by Figure 3.20.

EXAMPLE 5 For the helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, we have

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2+b^2}}.$$

On the other hand, if we parametrize the helix using arclength so that

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right),$$

then

$$\mathbf{T}(s) = \mathbf{x}'(s) = \frac{-a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}.$$

This agrees (as it should) with the first expression for $\mathbf{T}$, since $s = \sqrt{a^2+b^2}\,t$, as shown in Example 3. ◆

Using the unit tangent vector, we can define a quantity that measures how much a path bends as we travel along it. To do so, note the following key facts:

PROPOSITION 2.3 Assume that the path $\mathbf{x}$ always has nonzero speed. Then

1. $d\mathbf{T}/dt$ is perpendicular to $\mathbf{T}$ for all t in I (the domain of the path $\mathbf{x}$).
2. $\|d\mathbf{T}/dt\|\,|_{t=t_0}$ equals the angular rate of change (as t increases) of the direction of $\mathbf{T}$ when $t = t_0$.

PROOF (You can omit reading this proof for the moment if you are interested in the main flow of ideas.) To prove part 1, we have

$$\mathbf{T}(t)\cdot\mathbf{T}(t) = 1,$$

since $\mathbf{T}$ is a unit vector. Hence,

$$\frac{d}{dt}(\mathbf{T}\cdot\mathbf{T}) = 0,$$

because the derivative of a constant is zero. Also we have

$$\frac{d}{dt}(\mathbf{T}\cdot\mathbf{T}) = \mathbf{T}\cdot\frac{d\mathbf{T}}{dt} + \frac{d\mathbf{T}}{dt}\cdot\mathbf{T},$$

by the product rule (Proposition 1.4). Thus,

$$2\mathbf{T}\cdot\frac{d\mathbf{T}}{dt} = 0.$$

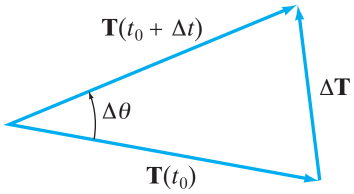

Figure 3.21 The vector triangle used in the proof of Proposition 2.3.

Therefore, $\mathbf{T}$ is always perpendicular to $d\mathbf{T}/dt$. (See Proposition 1.7.)

Now we prove part 2. Because $\mathbf{T}$ is a unit vector for all t, only its direction can change as t increases. This angular rate of change of $\mathbf{T}$ is precisely

$$\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\Delta t},$$

where $\Delta\theta$ comes from the vector triangle shown in Figure 3.21. To make the argument technically simpler, we shall assume that $\Delta\mathbf{T}\neq\mathbf{0}$. We claim that

$$\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|} = 1. \tag{7}$$

Then, from equation (7),

$$\begin{aligned}\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\Delta t} &= \lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|}\frac{\|\Delta\mathbf{T}\|}{\Delta t} = \lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|}\lim_{\Delta t\to 0^+}\frac{\|\Delta\mathbf{T}\|}{\Delta t}\\ &= 1\cdot\lim_{\Delta t\to 0^+}\frac{\|\Delta\mathbf{T}\|}{\Delta t}.\end{aligned}$$

Since Δt is assumed to be positive in the limit, we may conclude that

$$\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\Delta t} = \lim_{\Delta t\to 0^+}\left\|\frac{\Delta\mathbf{T}}{\Delta t}\right\| = \left\|\frac{d\mathbf{T}}{dt}\right\|,$$

as desired.

To establish equation (7), the law of cosines applied to the vector triangle in Figure 3.21 implies

$$\begin{aligned}\|\Delta\mathbf{T}\|^2 &= \|\mathbf{T}(t+\Delta t)\|^2 + \|\mathbf{T}(t)\|^2 - 2\|\mathbf{T}(t+\Delta t)\|\,\|\mathbf{T}(t)\|\cos\Delta\theta\\ &= 2 - 2\cos\Delta\theta,\end{aligned}$$

because $\mathbf{T}$ is always a unit vector. Thus,

$$\begin{aligned}\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|} &= \lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\sqrt{2-2\cos\Delta\theta}}\\ &= \lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\sqrt{2\cdot 2(\sin^2(\Delta\theta/2))}}\end{aligned}$$

from the half-angle formula, and so

$$\lim_{\Delta t\to 0^+}\frac{\Delta\theta}{\|\Delta\mathbf{T}\|} = \lim_{\Delta t\to 0^+}\frac{\Delta\theta/2}{\sin(\Delta\theta/2)} = 1,$$

from the well-known trigonometric limit (or from L'Hôpital's rule). ■

Part 2 of Proposition 2.3 provides a precise way of measuring the bending of a path.

DEFINITION 2.4 The **curvature** κ of a path $\mathbf{x}$ in $\mathbf{R}^3$ is the angular rate of change of the direction of $\mathbf{T}$ per unit change in distance along the path.

The reason for taking the rate of change of $\mathbf{T}$ *per unit change in distance* in the definition of κ is so that the curvature is an intrinsic quantity (which we certainly want it to be). Figure 3.22 should help you develop some intuition about κ.

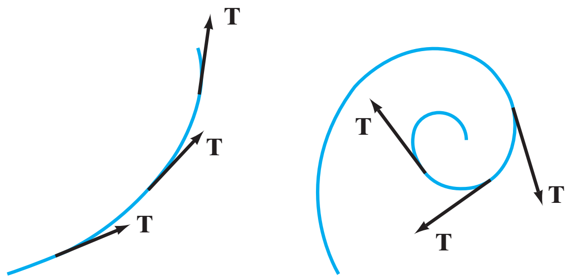

Figure 3.22 In the left figure, κ is not large, since the path's unit tangent vector turns only a small amount per unit change in distance along the path. In the right figure, κ is much larger, because $\mathbf{T}$ turns a great deal relative to distance traveled.

Because $\|d\mathbf{T}/dt\|$ measures the angular rate of change of the direction of $\mathbf{T}$ per unit change in parameter (by part 2 of Proposition 2.3) and ds/dt is the rate of change of distance per unit change in parameter, we see that

$$\kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\| \frac{d\mathbf{T}}{ds} \right\|, \tag{8}$$

where the last equality holds by the chain rule. It is formula (8) that we will use when making calculations.

EXAMPLE 6 For the circle $\mathbf{x}(t) = (a\cos t, a\sin t)$, $0 \le t < 2\pi$,

$$\mathbf{x}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}, \qquad \|\mathbf{x}'(t)\| = \frac{ds}{dt} = a,$$

so that

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.$$

Hence,

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{a}\|-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}\| = \frac{1}{a}.$$

Thus, we see that the curvature of a circle is always constant with value equal to the reciprocal of the radius. Therefore, the smaller the circle, the greater the curvature. (Draw a sketch to convince yourself.) ◆

EXAMPLE 7 If **a** and **b** are constant vectors in $\mathbf{R}^3$ and $\mathbf{a} \neq \mathbf{0}$, the path

$$\mathbf{x}(t) = \mathbf{a}\,t + \mathbf{b}$$

traces a line. We have

$$\mathbf{x}'(t) = \mathbf{a},$$

so

$$\frac{ds}{dt} = \|\mathbf{a}\|.$$

Hence,

$$\mathbf{T}(t) = \frac{\mathbf{a}}{\|\mathbf{a}\|},$$

which is a constant vector. Thus, $\mathbf{T}'(t) \equiv \mathbf{0}$ and formula (8) implies immediately that κ is zero, which agrees with the intuitive fact that a line doesn't curve. ◆

EXAMPLE 8 Returning to our friend the helix

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt),$$

we have already seen that

$$\frac{ds}{dt} = \sqrt{a^2 + b^2} \quad \text{and} \quad \mathbf{T}(t) = \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2 + b^2}}.$$

Thus, formula (8) gives

$$\kappa = \frac{1}{\sqrt{a^2+b^2}} \left\| \frac{-a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j}}{\sqrt{a^2+b^2}} \right\| = \frac{a}{a^2+b^2}.$$

We see that the curvature of the helix is constant, just like the circle. In fact, as b approaches zero, the helix degenerates to a circle, and the resulting curvature is consistent with that of Example 6.

We can also compute the curvature from the parametrization given by arc-length. The same helix is also described by

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right),$$

and we have

$$\mathbf{T}(s) = \frac{d\mathbf{x}}{ds} = -\frac{a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}.$$

We can, therefore, compute

$$\frac{d\mathbf{T}}{ds} = -\frac{a}{a^2+b^2}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} - \frac{a}{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j},$$

and hence, from formula (8), that

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{a}{a^2+b^2},$$

which checks. ◆

The Moving Frame and Torsion

We now introduce a triple of mutually orthogonal unit vectors that "travel" with a given path $\mathbf{x}: I \to \mathbf{R}^3$, known as the **moving frame** of the path. (Note: In general, the term "frame" means an ordered collection of mutually orthogonal unit vectors in $\mathbf{R}^n$.) These vectors should be thought of as a set of special vector "coordinate axes" that move from point to point along the path.

To begin, assume that (i) $\mathbf{x}'(t) \neq \mathbf{0}$ and (ii) $\mathbf{x}'(t) \times \mathbf{x}''(t) \neq \mathbf{0}$ for all t in I. (The first condition assures us that $\mathbf{x}$ never has zero speed and the second that $\mathbf{x}$ is not a straight-line path.) Then the first vector of the moving frame is just the unit tangent vector:

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

(Now you see why condition (i) is needed.) For a second vector orthogonal to $\mathbf{T}$, recall that part 1 of Proposition 2.3 says that $d\mathbf{T}/dt$ must be perpendicular to $\mathbf{T}$. Hence, we define

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}. \tag{9}$$

(That $d\mathbf{T}/dt$ is not zero follows from assumptions (i) and (ii).) The vector $\mathbf{N}$ is called the **principal normal vector** of $\mathbf{x}$. By the chain rule, $\mathbf{N}$ is also given by

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|}. \tag{10}$$

Since $\kappa = \|d\mathbf{T}/ds\|$ by formula (8), we also see that

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}. \tag{11}$$

At a given point P along the path, the vectors $\mathbf{T}$ and $\mathbf{N}$ (and also the vectors $\mathbf{x}'$ and $\mathbf{x}''$) determine what is called the **osculating plane** of the path at P. (See Figure 3.23.) This is the plane that "instantaneously" contains the path at P. (More

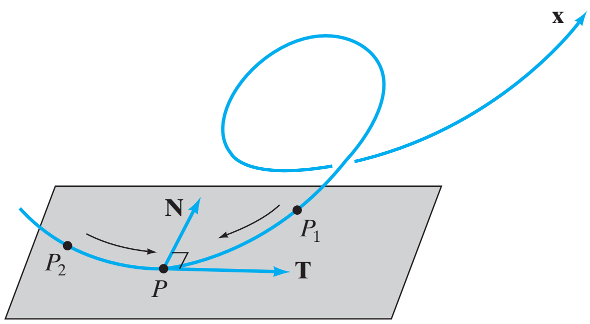

Figure 3.23 The osculating plane of the path $\mathbf{x}$ at the point P.

precisely, it is the plane obtained by taking points P_1 and P_2 on the path near P and finding the limiting position of the plane through P, P_1, and P_2 as P_1 and P_2 approach P along $\mathbf{x}$. The word "osculating" derives from the Latin *osculare,* meaning "to kiss.")

Now that we have defined two orthogonal unit vectors $\mathbf{T}$ and $\mathbf{N}$, we can produce a third unit vector perpendicular to both:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}. \tag{12}$$

The vector $\mathbf{B}$, called the **binormal vector**, is defined so that the ordered triple $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed system. Thus, $\mathbf{B}$ is a unit vector since

$$\|\mathbf{B}\| = \|\mathbf{T}\|\,\|\mathbf{N}\| \sin\frac{\pi}{2} = 1 \cdot 1 \cdot 1 = 1.$$

EXAMPLE 9 For the helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, the moving frame vectors are

$$\mathbf{T}(t) = \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\,\mathbf{k}}{\sqrt{a^2+b^2}}$$

(as we have already seen),

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{(-a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j})/\sqrt{a^2+b^2}}{a/\sqrt{a^2+b^2}} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j},$$

and

$$\mathbf{B}(t) = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t/\sqrt{a^2+b^2} & a\cos t/\sqrt{a^2+b^2} & b/\sqrt{a^2+b^2} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \left(\frac{b}{\sqrt{a^2+b^2}}\sin t\right)\mathbf{i} - \left(\frac{b}{\sqrt{a^2+b^2}}\cos t\right)\mathbf{j} + \left(\frac{a}{\sqrt{a^2+b^2}}\right)\mathbf{k}.$$ ◆

Equation (11) says that the derivative of $\mathbf{T}$ (with respect to arclength) is a scalar function (namely, the curvature) multiple of the principal normal $\mathbf{N}$. This is not surprising, since $\mathbf{N}$ is defined to be parallel to the derivative of $\mathbf{T}$. A more remarkable result (see the addendum at the end of this section) is that the derivative of the binormal vector is also always parallel to the principal normal; that is,

$$\frac{d\mathbf{B}}{ds} = (\text{scalar function})\,\mathbf{N}.$$

The standard convention is to write this scalar function with a negative sign, so we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \tag{13}$$

The scalar function τ thus defined is called the **torsion** of the path $\mathbf{x}$. Roughly speaking, the torsion measures how much the path twists out of the plane, how "three-dimensional" $\mathbf{x}$ is. Note that, according to our conventions, the curvature κ is always nonnegative (why?), while τ can be positive, negative, or zero.

EXAMPLE 10 Consider again the case of circular motion. Thus, let $\mathbf{x}(t) = (a\cos t, a\sin t)$. Then, as shown in Example 6,

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}, \quad \text{and} \quad \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{1}{a}.$$

Now we calculate

$$\mathbf{N} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j},$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \mathbf{k}, \quad \text{a constant vector.}$$

Hence, $d\mathbf{B}/ds \equiv \mathbf{0}$, so there is no torsion. This makes sense, since a circle does not twist out of the plane. ◆

EXAMPLE 11 Let $\mathbf{x}(t) = (e^t\cos t, e^t\sin t, e^t)$. We calculate $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ and identify the curvature and torsion of $\mathbf{x}$.

To begin, we have

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{e^t(\cos t - \sin t)\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t\,\mathbf{k}}{\sqrt{3}\,e^t}$$
$$= \frac{1}{\sqrt{3}}\left((\cos t - \sin t)\,\mathbf{i} + (\cos t + \sin t)\,\mathbf{j} + \mathbf{k}\right).$$

From this, we may compute

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\frac{1}{\sqrt{3}}(-(\sin t + \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j})}{\sqrt{3}\,e^t}$$
$$= \frac{e^{-t}}{3}(-(\sin t + \cos t)\,\mathbf{i} + (\cos t - \sin t)\mathbf{j}),$$

so that the curvature is

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\sqrt{2}\,e^{-t}}{3}.$$

Now we determine the remainder of the moving frame:

$$\mathbf{N} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2}}(-(\sin t + \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}),$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{\sqrt{6}}((\sin t - \cos t)\,\mathbf{i} - (\sin t + \cos t)\,\mathbf{j} + 2\mathbf{k}).$$

Finally, to find the torsion, we calculate

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\frac{1}{\sqrt{6}}((\cos t + \sin t)\,\mathbf{i} + (\sin t - \cos t)\,\mathbf{j})}{\sqrt{3}\,e^t}$$
$$= \frac{e^{-t}}{3\sqrt{2}}((\cos t + \sin t)\,\mathbf{i} + (\sin t - \cos t)\,\mathbf{j})$$
$$= -\frac{e^{-t}}{3}\mathbf{N},$$

so

$$\tau = \frac{e^{-t}}{3}.$$ ◆

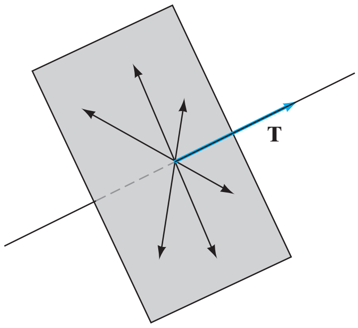

Figure 3.24 Any vector in the plane perpendicular to **T** can be used for **N**.

EXAMPLE 12 If **a** and **b** are vectors in $\mathbf{R}^3$, then the straight-line path $\mathbf{x}(t) = \mathbf{a}\,t + \mathbf{b}$ has, as we saw in Example 7, $\mathbf{T} = \mathbf{a}/\|\mathbf{a}\|$. Thus, both $d\mathbf{T}/dt$ and $d\mathbf{T}/ds$ are identically zero. Hence, $\kappa \equiv 0$ (as shown in Example 7) and **N** cannot be defined using formula (9). From geometric considerations, any unit vector perpendicular to **T** can, in principle, be used for **N**. (See Figure 3.24.) If we choose one such vector, then **B** can be calculated from formula (12). Since **T**, **N**, and **B** are all constant, τ must be zero. This is an example of a moving frame that is *not* uniquely determined by the path **x** and serves to illustrate why the assumption $\mathbf{x}' \times \mathbf{x}'' \neq \mathbf{0}$ was made. ◆

It is important to realize that the moving frame, curvature, and torsion are quantities that are *intrinsic* to the curve traced by the path. That is, any parametrized path that traces the same curve (in the same direction) must necessarily have the same **T**, **N**, **B** vector functions and the same curvature and torsion. This is because all of these quantities can be defined entirely in terms of the intrinsic arclength parameter s. (See Definition 2.2 and formulas (6), (8), (10), (11), (12), and (13).)

Another important fact is that the curvature function κ and the torsion function τ together determine all the geometric information regarding the shape of the curve, except for the curve's particular position in space. To be more precise, we have the following theorem, whose proof we omit:

THEOREM 2.5 Let s be the arclength parameter and suppose C_1 and C_2 are two curves of class C^3 in $\mathbf{R}^3$. Assume that the corresponding curvature functions κ_1 and κ_2 are strictly positive. Then if $\kappa_1(s) \equiv \kappa_2(s)$ and $\tau_1(s) \equiv \tau_2(s)$, the two curves must be congruent (in the sense of high school geometry). In fact, given any two continuous functions κ and τ, where $\kappa(s) > 0$ for all s in the closed interval $[0, L]$, there is a unique curve parametrized by arclength on $[0, L]$ (up to position in space) whose curvature and torsion are κ and τ, respectively.

Tangential and Normal Components of Velocity and Acceleration; Other Curvature Formulas

As we have seen, the moving frame provides us with an intrinsic set of vectors, like coordinate axes, that are special to the particular curve traced by a path. In contrast, the velocity and acceleration vectors of a path are definitely *not* intrinsic quantities but depend on the particular parametrization chosen as well as on the shape of the path. (The speed of a path is entirely independent of the geometry of the curve traced.) We can get some feeling for the relationship between the intrinsic notion of the moving frame and the extrinsic quantities of velocity and acceleration by expressing the latter two vector functions in terms of the moving frame vectors.

Thus, we begin with a C^2 path $\mathbf{x}\colon I \to \mathbf{R}^3$ having $\mathbf{x}' \neq \mathbf{0}$ and $\mathbf{x}' \times \mathbf{x}'' \neq \mathbf{0}$. For notational convenience, let $\dot{s}$ denote ds/dt and $\ddot{s}$ denote d^2s/dt^2. From Definition 2.2, we know that $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$ and so, since the speed $\dot{s} = ds/dt = \|\mathbf{v}\|$, we have

$$\mathbf{v}(t) = \dot{s}\mathbf{T}. \tag{14}$$

This formula says that the velocity is always parallel to the unit tangent vector, something we know well. To obtain a similar result for acceleration, we can differentiate (14) and apply the product rule:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d}{dt}(\dot{s}\mathbf{T}) = \ddot{s}\mathbf{T} + \dot{s}\frac{d\mathbf{T}}{dt}. \tag{15}$$

Next, we express $d\mathbf{T}/dt$ in terms of the $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ frame. Formula (11) gives the derivative of $d\mathbf{T}/ds$ in terms of $\mathbf{N}$. The chain rule says that $d\mathbf{T}/ds = (d\mathbf{T}/dt)/(ds/dt)$. Thus, from formula (11), we have

$$\frac{d\mathbf{T}}{dt} = \dot{s}\frac{d\mathbf{T}}{ds} = \dot{s}\kappa\mathbf{N}.$$

Hence, we may rewrite equation (15) as

$$\mathbf{a}(t) = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}. \tag{16}$$

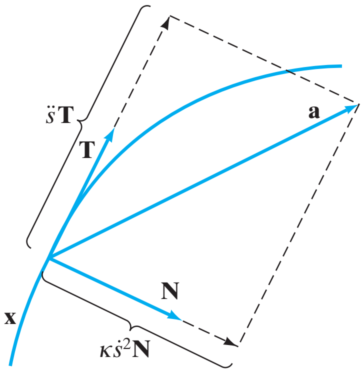

Figure 3.25 Decomposition of acceleration **a** into tangential and normal components.

WARNING $\ddot{s} = d^2s/dt^2$ is the derivative of the speed, which is a scalar function. The acceleration **a** is the derivative of *velocity* and so is a vector function.

Note that formula (16) shows that the acceleration has no component in the direction of the binormal vector **B**. Therefore, both velocity and acceleration are vectors that lie in the osculating plane of the path. (See Figure 3.25.)

At first glance, it may not appear to be especially easy to use formula (16) to resolve acceleration into its tangential and normal components because of the curvature term. However,

$$\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} = (\ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N})\cdot(\ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}) = \ddot{s}^2 + (\kappa\dot{s}^2)^2,$$

since $\mathbf{T}$ and $\mathbf{N}$ are perpendicular vectors. Consequently, we may calculate the components as follows:

Tangential component of acceleration $= a_{\text{tang}} = \ddot{s}$.

Normal component of acceleration $= a_{\text{norm}} = \kappa\dot{s}^2 = \sqrt{\|\mathbf{a}\|^2 - a_{\text{tang}}^2}$.

EXAMPLE 13 Let $\mathbf{x}(t) = (t, 2t, t^2)$. Then $\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{a}(t) = 2\mathbf{k}$. We have $\dot{s} = \|\mathbf{v}(t)\| = \sqrt{5 + 4t^2}$. Therefore,

$$a_{\text{tang}} = \ddot{s} = \frac{4t}{\sqrt{5 + 4t^2}}.$$

Since $\|\mathbf{a}\| = 2$, we see that

$$a_{\text{norm}} = \sqrt{\|\mathbf{a}\|^2 - a_{\text{tang}}^2} = \sqrt{4 - \frac{16t^2}{5 + 4t^2}} = \frac{2\sqrt{5}}{\sqrt{5 + 4t^2}}.$$ ◆

Formulas (14) and (16) enable us to find an alternative equation for the curvature of the path. We simply calculate that

$$\mathbf{v}\times\mathbf{a} = (\dot{s}\mathbf{T})\times(\ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}) = \dot{s}\ddot{s}(\mathbf{T}\times\mathbf{T}) + \kappa\dot{s}^3(\mathbf{T}\times\mathbf{N}) = \kappa\dot{s}^3\mathbf{B}.$$

Recalling that $\dot{s} = \|\mathbf{v}\|$, we have, by taking magnitudes,

$$\|\mathbf{v} \times \mathbf{a}\| = \kappa \|\mathbf{v}\|^3 \, \|\mathbf{B}\| = \kappa \|\mathbf{v}\|^3,$$

since **B** is a unit vector. Thus,

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}. \tag{17}$$

This relatively simple formula expresses the curvature (an intrinsic quantity) in terms of the nonintrinsic quantities of velocity and acceleration.

EXAMPLE 14 For the path $\mathbf{x}(t) = (2t^3 + 1, t^4, t^5)$, we have

$$\mathbf{v}(t) = 6t^2\mathbf{i} + 4t^3\mathbf{j} + 5t^4\mathbf{k}$$

and

$$\mathbf{a}(t) = 12t\mathbf{i} + 12t^2\mathbf{j} + 20t^3\mathbf{k}.$$

You can check that

$$\|\mathbf{v}\| = t^2\sqrt{25t^4 + 16t^2 + 36}$$

and

$$\|\mathbf{v} \times \mathbf{a}\| = \left\|4t^4(5t^2\mathbf{i} - 15t\mathbf{j} + 6\mathbf{k})\right\| = 4t^4\sqrt{25t^4 + 225t^2 + 36}.$$

Therefore, formula (17) yields

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{4(25t^4 + 225t^2 + 36)^{1/2}}{t^2(25t^4 + 16t^2 + 36)^{3/2}},$$

which is certainly a more convenient way to determine curvature in this case. ◆

Summary

You have seen many formulas in this section, and, at first, it may seem difficult to sort out the primary statements from the secondary results. We list the more fundamental facts here:

For a path $\mathbf{x}\colon I \to \mathbf{R}^3$:

Nonintrinsic quantities:

Velocity $\mathbf{v}(t) = \mathbf{x}'(t)$.

Speed $\dfrac{ds}{dt} = \|\mathbf{v}(t)\|$.

Acceleration $\mathbf{a}(t) = \mathbf{x}''(t)$.

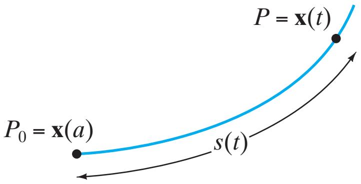

Figure 3.26 The arclength function.

Arclength function: (See Figure 3.26.)

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\,d\tau \quad (\text{basepoint is } P_0 = \mathbf{x}(a))$$

Intrinsic quantities:

The moving frame:

$$\text{Unit tangent vector } \mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

$$\text{Principal normal vector } \mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

$$\text{Binormal vector } \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

$$\text{Curvature } \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt}.$$

$$\text{Torsion } \tau \text{ is defined so that } \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Additional formulas:

$$\mathbf{v}(t) = \dot{s}\,\mathbf{T} \quad (\dot{s} \text{ is speed}).$$

$$\mathbf{a}(t) = \ddot{s}\,\mathbf{T} + \kappa\dot{s}^2\,\mathbf{N} \quad (\ddot{s} \text{ is derivative of speed}).$$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

Addendum: More About Torsion and the Frenet–Serret Formulas

We now derive formula (13), the basis for the definition of the torsion of a curve. That is, we show that the derivative of the binormal vector $\mathbf{B}$ (with respect to arclength) is always parallel to the principal normal $\mathbf{N}$ (i.e., that $d\mathbf{B}/ds$ is a scalar function times $\mathbf{N}$). The two main ingredients in our derivation are part 1 of Proposition 2.3 and the product rule.

We begin by noting that, since the ordered triple of vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ forms a frame for $\mathbf{R}^3$, any moving vector, including $d\mathbf{B}/ds$, can be expressed as a **linear combination** of these vectors; that is, we must have

$$\frac{d\mathbf{B}}{ds} = a(s)\mathbf{T} + b(s)\mathbf{N} + c(s)\mathbf{B}, \tag{18}$$

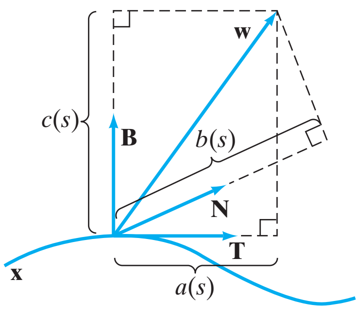

Figure 3.27 $\mathbf{w}(s) = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$.

where a, b, and c are appropriate scalar-valued functions. (Because $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are mutually perpendicular unit vectors, any (moving) vector $\mathbf{w}$ in $\mathbf{R}^3$ can be decomposed into its components with respect to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ in much the same way that it can be decomposed into $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components—see Figure 3.27.) To find the particular values of the component functions a, b, and c, it turns out that we can solve for each function by applying appropriate dot products to equation (18). Specifically,

$$\begin{aligned}\frac{d\mathbf{B}}{ds}\cdot\mathbf{T} &= a(s)\mathbf{T}\cdot\mathbf{T} + b(s)\mathbf{N}\cdot\mathbf{T} + c(s)\mathbf{B}\cdot\mathbf{T}\\ &= a(s)\cdot 1 + b(s)\cdot 0 + c(s)\cdot 0\\ &= a(s),\end{aligned}$$

and, similarly,

$$\frac{d\mathbf{B}}{ds}\cdot\mathbf{N} = b(s), \qquad \frac{d\mathbf{B}}{ds}\cdot\mathbf{B} = c(s).$$

From Proposition 1.7, $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$ and, hence, c must be zero. To find a, we use an ingenious trick with the product rule: Because $\mathbf{T}\cdot\mathbf{B} = 0$, it follows that $d/ds(\mathbf{T}\cdot\mathbf{B}) = 0$. Now, by the product rule,

$$\frac{d}{ds}(\mathbf{T}\cdot\mathbf{B}) = \mathbf{T}\cdot\frac{d\mathbf{B}}{ds} + \frac{d\mathbf{T}}{ds}\cdot\mathbf{B}.$$

Consequently, $(d\mathbf{B}/ds)\cdot\mathbf{T} = -(d\mathbf{T}/ds)\cdot\mathbf{B}$. Thus,

$$\begin{aligned}a(s) = \frac{d\mathbf{B}}{ds}\cdot\mathbf{T} &= -\frac{d\mathbf{T}}{ds}\cdot\mathbf{B}\\ &= -\kappa\mathbf{N}\cdot\mathbf{B} \qquad \text{by formula (11),}\\ &= 0,\end{aligned}$$

and equation (18) reduces to

$$\frac{d\mathbf{B}}{ds} = b(s)\mathbf{N}.$$

No further reductions are possible, and we have proved that the derivative of $\mathbf{B}$ is parallel to $\mathbf{N}$. The torsion τ can, therefore, be defined by $\tau(s) = -b(s)$.

Formulas (11) and (13) gave us intrinsic expressions for $d\mathbf{T}/ds$ and $d\mathbf{B}/ds$, respectively. We can complete the set by finding an expression for $d\mathbf{N}/ds$. The method is the same as the one just used. Begin by writing

$$\frac{d\mathbf{N}}{ds} = a(s)\mathbf{T} + b(s)\mathbf{N} + c(s)\mathbf{B}, \tag{19}$$

where a, b, and c are suitable scalar functions. Taking the dot product of equation (19) with, in turn, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, yields the following:

$$a(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{T}, \qquad b(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{N}, \qquad c(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{B}.$$

The "product rule trick" used here then reveals that

$$\begin{aligned}a(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{T} &= -\mathbf{N}\cdot\frac{d\mathbf{T}}{ds}\\ &= -\mathbf{N}\cdot\kappa\mathbf{N} \qquad \text{by formula (11)}\\ &= -\kappa,\end{aligned}$$

and

$$\begin{aligned}c(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{B} &= -\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}\\ &= -\mathbf{N}\cdot(-\tau\mathbf{N}) \qquad \text{by formula (13)}\\ &= \tau.\end{aligned}$$

Moreover, we may differentiate the equation $\mathbf{N}\cdot\mathbf{N} = 1$ to find

$$b(s) = \frac{d\mathbf{N}}{ds}\cdot\mathbf{N} = -\mathbf{N}\cdot\frac{d\mathbf{N}}{ds},$$

which implies that $b(s)$ is zero. Hence, equation (19) becomes

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

The formulas for $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$ are usually taken together as

$$\begin{cases} \mathbf{T}'(s) = \kappa\mathbf{N} \\ \mathbf{N}'(s) = -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}'(s) = -\tau\mathbf{N} \end{cases}$$

and are known as the **Frenet–Serret formulas** for a curve in space. They are so named for Frédéric-Jean Frenet and Joseph Alfred Serret, who published them separately in 1852 and 1851, respectively. The Frenet–Serret formulas give a system of differential equations for a curve and are key to proving a result like Theorem 2.5. They are often written in matrix form, in which case, they have an especially appealing appearance, namely,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

3.2 Exercises

Calculate the length of each of the paths given in Exercises 1–6.

1. $\mathbf{x}(t) = (2t + 1, 7 - 3t)$, $-1 \le t \le 2$

2. $\mathbf{x}(t) = t^2\,\mathbf{i} + \frac{2}{3}(2t + 1)^{3/2}\,\mathbf{j}$, $0 \le t \le 4$

3. $\mathbf{x}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$, $0 \le t \le 2$

4. $\mathbf{x}(t) = 7\mathbf{i} + t\,\mathbf{j} + t^2\mathbf{k}$, $1 \le t \le 3$

5. $\mathbf{x}(t) = (t^3, 3t^2, 6t)$, $-1 \le t \le 2$

6. $\mathbf{x}(t) = (\ln(\cos t), \cos t, \sin t)$, $\frac{\pi}{6} \le t \le \frac{\pi}{3}$

7. $\mathbf{x}(t) = (\ln t, t^2/2, \sqrt{2}t)$, $1 \le t \le 4$

8. $\mathbf{x}(t) = (2t\cos t, 2t\sin t, 2\sqrt{2}t^2)$, $0 \le t \le 3$

9. The path $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$, where a is a positive constant, traces a curve known as an **astroid** or a **hypocycloid of four cusps.** Sketch this curve and find its total length. (Be careful when you do this.)

10. If f is a continuously differentiable function, show how Definition 2.1 may be used to establish the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

for the length of the curve $y = f(x)$ between $(a, f(a))$ and $(b, f(b))$.

11. Use Exercise 10 or Definition 2.1 (or both) to calculate the length of the line segment $y = mx + b$ between (x_0, y_0) and (x_1, y_1). Explain your result with an appropriate sketch.

12. (a) Calculate the length of the line segment determined by the path

$$\mathbf{x}(t) = (a_1 t + b_1, a_2 t + b_2)$$

as t varies from t_0 to t_1.

(b) Compare your result with that of Exercise 11.

(c) Now calculate the length of the line segment determined by the path $\mathbf{x}(t) = \mathbf{a}\,t + \mathbf{b}$ as t varies from t_0 to t_1.

13. This problem concerns the path $\mathbf{x} = |t - 1|\,\mathbf{i} + |t|\,\mathbf{j}$, $-2 \le t \le 2$.

(a) Sketch this path.

(b) The path fails to be of class C^1 but is piecewise C^1. Explain.

(c) Calculate the length of the path.

14. Consider the path $\mathbf{x}(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

(a) Argue that the path spirals toward the origin as $t \to +\infty$.

(b) Show that, for any a, the improper integral

$$\int_a^\infty \|\mathbf{x}'(t)\|\, dt$$

converges.

(c) Interpret what the result in part (b) says about the path $\mathbf{x}$.

15. Suppose that a curve is given in polar coordinates by an equation of the form $r = f(\theta)$, where f is of class C^1. Use Definition 2.1 to derive the formula

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

for the length of the curve between the points $(f(\alpha), \alpha)$ and $(f(\beta), \beta)$ (given in polar coordinates).

16. (a) Find the arclength parameter $s = s(t)$ for the path

$$\mathbf{x}(t) = e^{at}\cos bt\,\mathbf{i} + e^{at}\sin bt\,\mathbf{j} + e^{at}\,\mathbf{k}.$$

(b) Express the original parameter t in terms of s and, thereby, reparametrize $\mathbf{x}$ in terms of s.

Determine the moving frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, and compute the curvature and torsion for the paths given in Exercises 17–20.

17. $\mathbf{x}(t) = 5\cos 3t\,\mathbf{i} + 6t\,\mathbf{j} + 5\sin 3t\,\mathbf{k}$

18. $\mathbf{x}(t) = (\sin t - t\cos t)\,\mathbf{i} + (\cos t + t\sin t)\,\mathbf{j} + 2\mathbf{k}$, $t \ge 0$

19. $\mathbf{x}(t) = \left(t, \frac{1}{3}(t+1)^{3/2}, \frac{1}{3}(1-t)^{3/2}\right)$, $-1 < t < 1$

20. $\mathbf{x}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1)$

21. (a) Use formula (17) in this section to establish the following well-known formula for the curvature of a plane curve $y = f(x)$:

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

(Assume that f is of class C^2.)

(b) Use your result in (a) to find the curvature of $y = \ln(\sin x)$.

22. (a) Let $\mathbf{x}(s) = (x(s), y(s))$ be a plane curve parametrized by arclength. Show that the curvature is given by the formula

$$\kappa = |x'y'' - x''y'|.$$

(b) Show that $\mathbf{x}(s) = \left(\frac{1}{2}(1-s^2), \frac{1}{2}(\cos^{-1} s - s\sqrt{1-s^2})\right)$ is parametrized by arclength, and compute its curvature.

In Exercises 23–26, (a) use a computer algebra system to calculate the curvature κ of the indicated path $\mathbf{x}$ and (b) plot the path $\mathbf{x}$ and, separately, plot the curvature κ as a function of t over the indicated interval for t and value(s) of the constants.

23. $\mathbf{x}(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$; $a = 2, b = 1$

24. $\mathbf{x}(t) = (2a(1+\cos t)\cos t, 2a(1+\cos t)\sin t)$, $0 \le t \le 2\pi$; $a = 1$

25. $\mathbf{x}(t) = (2a\cos t(1+\cos t) - a, 2a\sin t(1+\cos t))$, $0 \le t \le 2\pi$; $a = 1$

26. $\mathbf{x}(t) = (a\sin nt, b\sin mt)$, $0 \le t \le 2\pi$; $a = 3$, $b = 2, n = 4, m = 3$

Find the tangential and normal components of acceleration for the paths given in Exercises 27–32.

27. $\mathbf{x}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$

28. $\mathbf{x}(t) = (2t, e^{2t})$

29. $\mathbf{x}(t) = (e^t\cos 2t, e^t\sin 2t)$

30. $\mathbf{x}(t) = (4\cos 5t, 5\sin 4t, 3t)$

31. $\mathbf{x}(t) = (t, t, t^2)$

32. $\mathbf{x}(t) = \frac{3}{5}(1-\cos t)\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{4}{5}\cos t\,\mathbf{k}$

33. (a) Show that the tangential and normal components of acceleration a_{tang} and a_{norm} satisfy the equations

$$a_{\text{tang}} = \frac{\mathbf{x}' \cdot \mathbf{x}''}{\|\mathbf{x}'\|}, \qquad a_{\text{norm}} = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|}.$$

(b) Use these formulas to find the tangential and normal components of acceleration for the path $\mathbf{x}(t) = (t+2)\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}$.

34. Use Exercise 33 to show that, for the plane curve $y = f(x)$,

$$a_{\text{tang}} = \frac{f'(x)f''(x)}{\sqrt{1 + (f'(x))^2}},$$

$$a_{\text{norm}} = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}}.$$

35. Establish the following formula for the torsion:

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2}.$$

36. Show that $\kappa\tau = -\mathbf{T}' \cdot \mathbf{B}'$, where differentiation is with respect to the arclength parameter s.

37. Show that if $\mathbf{x}$ is a path parametrized by arclength and $\mathbf{x}' \times \mathbf{x}'' \ne \mathbf{0}$, then

$$\kappa^2\tau = (\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''.$$

38. Suppose $\mathbf{x}\colon I \to \mathbf{R}^3$ is a path with $\mathbf{x}'(t) \times \mathbf{x}''(t) \ne \mathbf{0}$ for all $t \in I$. The **osculating plane** to the path at $t = t_0$ is the plane containing $\mathbf{x}(t_0)$ and determined by (i.e., parallel to) the tangent and normal vectors $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$.