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$\mathbb{R}^n = \{ x = (x_1, \dots, x_n) ; x_i \in \mathbb{R}, i = 1, \dots, n \}$

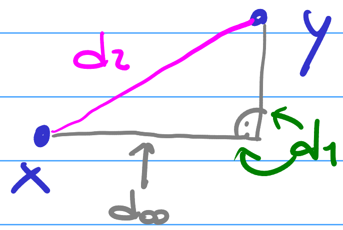
$\mathbb{R}^n$ spazio vettoriale e spazio metrico

$d_2(x,y) := \left[(x_1 - y_1)^2 + \dots + (x_n - y_n)^2 \right]^{1/2}, \quad x, y \in \mathbb{R}^n$

distanza euclidea

$d_\infty(x,y) := \max_{i=1,\dots,n} |x_i - y_i|$

$d_1(x,y) := |x_1 - y_1| + \dots + |x_n - y_n|$ distanza di Manhattan



$d_D(x,y) = \begin{cases} 1 & \text{se } x \neq y \\ 0 & \text{se } x = y \end{cases}$ distanza discreta

DEF. $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ si dice distanza se:

i. $d(x,y) \geq 0 \quad \forall x, y \in \mathbb{R}^n$ e $d(x,y) = 0$ sse $x = y$

ii. $d(x,y) = d(y,x) \quad \forall x, y \in \mathbb{R}^n$

iii. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x, y, z \in \mathbb{R}^n$

DEF. $N: \mathbb{R}^n \to \mathbb{R}$ si dice norma se:

i. $N(u) \geq 0 \quad \forall u \in \mathbb{R}^n$ e $N(u) = 0$ se $u = 0$

ii. $N(\lambda u) = |\lambda| N(u) \quad \forall \lambda \in \mathbb{R}, \forall u \in \mathbb{R}^n$

iii. $N(u+w) \leq N(u) + N(w) \quad \forall u, w \in \mathbb{R}^n$

$$\|u\|_2 = [u_1^2 + \dots + u_n^2]^{1/2}$$

$$\|u\|_\infty = \max_i |u_i|$$

$$\|u\|_1 = |u_1| + \dots + |u_n|$$

OSS. se N è una norma allora $d_N(u,w) = N(u-w)$ è una distanza

DEF. $S: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ è un prodotto scalare se:

i. $S(w,w) \geq 0 \quad \forall w \in \mathbb{R}^n$ e $S(w,w) = 0$ se $w = 0$

ii. $S(u,w) = S(w,u) \quad \forall u, w \in \mathbb{R}^n$

iii. $S(\alpha u_1 + \beta u_2, w) = \alpha S(u_1, w) + \beta S(u_2, w) \quad \forall \alpha, \beta \in \mathbb{R} \; \forall u_1, u_2, w \in \mathbb{R}^n$

OSS. $\|u\|_S = [S(u,u)]^{1/2}$ è una norma se S è un prodotto scalare

$$u \cdot w = \sum_{i=1}^{n} u_i w_i$$ prodotto scalare euclideo

$$\|u\|_2 = [u \cdot u]^{1/2}$$

CAUCHY-SCHWARTZ: $\forall u, w \in \mathbb{R}^n$ vale $|S(u,w)| \leq \|u\|_S \|w\|_S$

IDENTITÀ DEL PARALLELOGRAMMA: $\|u+w\|_S^2 + \|u-w\|_S^2 = 2(\|u\|_S^2 + \|w\|_S^2)$

DEF. dati $u, w \in \mathbb{R}^3$ definiamo il prodotto vettoriale:

$$u \wedge w = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (u_2 w_3 - u_3 w_2, u_3 w_1 - u_1 w_3, u_1 w_2 - u_2 w_1)$$

$$u \wedge w = - w \wedge u$$

DEF. curva è una coppia di oggetti $(x, [a,b])$ dove
$[a,b] \subseteq \mathbb{R}$ e $x : [a,b] \to \mathbb{R}^n$ continua
$\mathrm{Im}(x) = x([a,b])$ si chiama sostegno o immagine della curva.
$x \in C^1([a,b], \mathbb{R}^n)$ è una curva regolare se $\|x'(t)\|_2 \neq 0$ $\forall t \in (a,b)$

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

$x \in C^1$ se $x_i \in C^1([a,b], \mathbb{R})$ $\forall i = 1, \dots, n$

$$\frac{x(t+h) - x(t)}{h} = \left(\frac{1}{h}[x_1(t+h) - x_1(t)]; \dots, \frac{1}{h}[x_n(t+h) - x_n(t)] \right)$$

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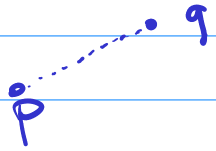
curva semplice se $x(t) \neq x(s)$ se $t \neq s$ $t, s \in (a,b)$
curva chiusa se $x(a) = x(b)$
curva piana se $\exists \pi$ iperpiano di $\mathbb{R}^n$ t.c.
$\pi \supseteq \gamma = x([a,b]) = \mathrm{Im}(x)$

ESEMPI

$p, q \in \mathbb{R}^3$ $\qquad t \in [0,1]$ $\qquad$ **SEGMENTO**



$$x(t) = p + t(q-p) = (1-t)p + tq$$

$$x(t) = ((1-t)p_1 + tq_1, (1-t)p_2 + tq_2, (1-t)p_3 + tq_3)$$

$$x'(t) = q - p \neq 0 \quad (\text{se } p \neq q!)$$

CIRCONFERENZA/ELLISSE

$$x(t) = (a\cos(t), b\sin(t), 0) \qquad a,b>0$$
$$t \in [0, 2\pi]$$

$$x'(t) = (-a\sin(t), b\cos(t), 0)$$

$$\|x'(t)\|_2^2 = a^2\sin^2(t) + b^2\cos^2(t) \cancel{\stackrel{?}{=}} 0$$

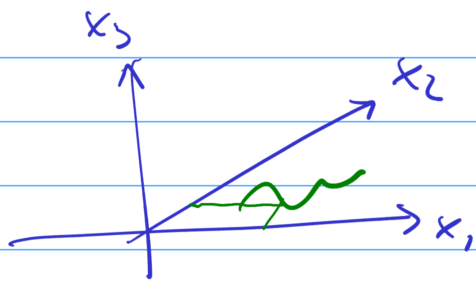
$\Rightarrow$ $(x, [0,2\pi])$ è una curva regolare il cui sostegno è un'ellisse

GRAFICO DI UNA FUNZIONE

$(f, [a,b])$ funzione $f : [a,b] \to \mathbb{R}$, $f \in C^1$
$\qquad s \mapsto f(s)$

$$x(t) = (t, f(t), 0)$$

$$x'(t) = (1, f'(t), 0) \neq 0$$
sempre!



ELICA

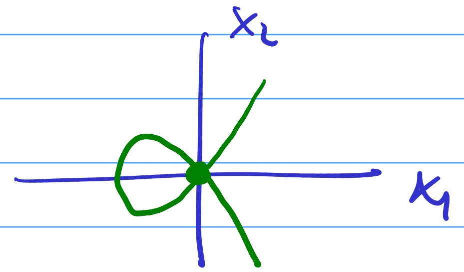
$$x(t) = (R\cos(t), R\sin(t), pt) \qquad R>0, p\neq 0$$
$$t \in [0,T], T>0$$

$$\|x'(t)\|_2^2 = R^2\sin^2(t) + R^2\cos^2(t) + p^2 = R^2 + p^2$$

$$x'(t) = (-R\sin(t), R\cos(t), p)$$

"FOLIUM" DI CARTESIO

$$x(t) = (t(t-1), t(t-1)(2t-1), 0) \qquad t \in [-1,2]$$



DEF. $x \in C^1([a,b], \mathbb{R}^n)$ curva regolare

$T(t_0) := \dfrac{x'(t_0)}{\|x'(t_0)\|_2}$ e' il versore tangente a $\gamma = Im(x)$ nel punto $x(t_0)$, $t_0 \in [a,b]$

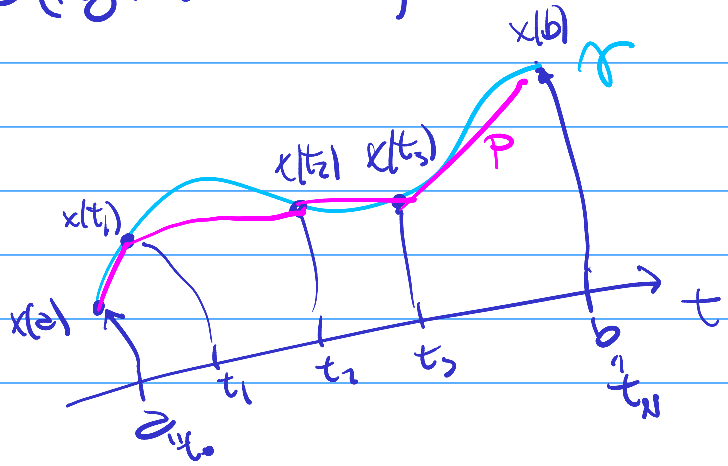
$r(t) = x(t_0) + T(t_0)(t - t_0)$, $t \in \mathbb{R}$

retta tangente a γ nel punto $x(t_0)$

DEF. $x \in C^1([a,b], \mathbb{R}^n)$ curva (regolare a tratti)

$$L(\gamma) = \sup_{p \in \mathcal{P}} L(p)$$

$\gamma = Im(x)$



TEOR. $x \in C^1([a,b],\mathbb{R}^n)$ curva regolare a tratti, allora

$$L(\gamma) = \int_a^b \|x'(t)\|_2 \, dt$$

ESEMPIO

$$x(t) = (R\cos(t), R\,\text{sen}(t), 0) \quad , R>0, \; t\in[0,2\pi]$$

$$L(\text{Im}(x)) = \int_0^{2\pi} \sqrt{R^2\cos^2(t) + R^2\,\text{sen}^2(t) + 0^2}\, dt = \int_0^{2\pi} R\, dt$$

$$= 2\pi R$$

$$y(t) = (R\,\text{sen}(2\pi t), -R\cos(2\pi t), 0) \quad t\in[0,1]$$

$$y'(t) = (2\pi R\cos(2\pi t), 2\pi R\,\text{sen}(2\pi t), 0)$$

$$L(\text{Im}(y)) = \int_0^1 \sqrt{4\pi^2 R^2}\, dt = 2\pi R$$

$$d_D(x,y) = \begin{cases} 1 & x\neq y \\ 0 & x=y \end{cases} \quad \nexists \; \|\cdot\|_D \text{ tale che } d_D(x,y) = \|x-y\|_D$$

$$y = 0 \in \mathbb{R}^n \quad x = \lambda e_1 \Rightarrow d_D(x,y) = 1 \quad \forall \lambda \neq 0$$

$$\text{se } \exists \; \|\cdot\|_D \quad \|x-y\|_D = |\lambda| \underset{\neq 0}{\|e_1\|_D} \longrightarrow +\infty \quad \text{se } \lambda \to +\infty$$

$\boxed{02/10/2025}$

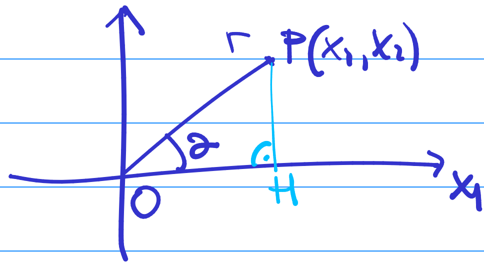
ESEMPIO 1 $\qquad f\in C^1([a,b],\mathbb{R}) \Rightarrow x(t)=(t,f(t),0)$

$$x'(t)=(1,f'(t),0)$$

$$L(\gamma)=L(Im(x))=\int_a^b \|x'(t)\|_2\,dt=\int_a^b\sqrt{1+|f'(t)|^2}\,dt$$

ESEMPIO 2 $\qquad r=r(\theta)\quad \theta\in[a,b]\quad r\in C^1([a,b],\mathbb{R})$

$$\begin{cases} x_1(\theta)=r(\theta)\cos(\theta)\\ x_2(\theta)=r(\theta)\operatorname{sen}(\theta)\\ x_3(\theta)=0\end{cases}$$



$$x_1'(\theta)=r'(\theta)\cos(\theta)-r(\theta)\operatorname{sen}(\theta)$$
$$x_2'(\theta)=r'(\theta)\operatorname{sen}(\theta)+r(\theta)\cos(\theta)$$

$$\begin{aligned}\|x'(\theta)\|_2^2 &= |r'(\theta)|^2\cos^2(\theta)+|r(\theta)|^2\operatorname{sen}^2(\theta)-\cancel{2r'(\theta)r(\theta)\operatorname{sen}(\theta)\cos(\theta)}\\ &\quad+|r'(\theta)|^2\operatorname{sen}^2(\theta)+|r(\theta)|^2\cos^2(\theta)+\cancel{2r'(\theta)r(\theta)\operatorname{sen}(\theta)\cos(\theta)}\\ &= |r'(\theta)|^2+|r(\theta)|^2\end{aligned}$$

$$L(\gamma)=\int_a^b\left[|r'(\theta)|^2+|r(\theta)|^2\right]^{1/2}d\theta$$

ESEMPIO 3 $\qquad x(t)=(R(t-\operatorname{sen}(t)),R(1-\cos(t)),0)\quad t\in[0,\pi],\ R>0$

$$x'(t)=R(1-\cos(t),\operatorname{sen}(t),0)$$

$$L(\text{cicloide})=\int_0^\pi R\sqrt{(1-\cos(t))^2+\operatorname{sen}^2(t)}\,dt$$
$$=R\int_0^\pi\sqrt{2-2\cos(t)}\,dt=2R\int_0^\pi\sqrt{\frac{1-\cos(t)}{2}}\,dt$$

$$= 2R\int_0^{\pi}\sqrt{\operatorname{sen}^2(t/2)}\,dt = 2R\int_0^{\pi}\operatorname{sen}\left(\frac{t}{2}\right)dt$$

$$= 4R\left[-\cos\left(\frac{t}{2}\right)\right]_0^{\pi} = 4R$$

ESEMPIO $\quad x(t) = (\operatorname{sen}(t), \operatorname{sen}(2t), \cos(t)) \quad t\in[0,2\pi]$

$$x'(t) = (\cos(t), 2\cos(2t), -\operatorname{sen}(t))$$

$$\|x'(t)\|_2^2 = 1+4\cos^2(2t) \geq 1 \quad \forall t$$

$$L(\gamma) = \int_0^{2\pi}\sqrt{1+4\cos^2(2t)}\,dt = \int_0^{2\pi}\sqrt{5-4\operatorname{sen}^2(2t)}\,dt$$

$$= \sqrt{5}\int_0^{2\pi}\sqrt{1-\frac{4}{5}\operatorname{sen}^2(2t)}\,dt$$

ESEMPIO $\quad x(t) = (R\cos(t), R\operatorname{sen}(t), pt) \quad t\in[0,T_0],\ R>0,\ p\neq 0$

$$x'(t) = (-R\operatorname{sen}(t), R\cos(t), p)$$

$$L(\text{elica}) = \int_0^{T_0}\|x'(t)\|_2\,dt = \int_0^{T_0}\sqrt{R^2+p^2}\,dt = \sqrt{R^2+p^2}\,T_0$$

DEF.

$$\int_\gamma F(x)\,ds := \int_a^b F(x(t))\,\|x'(t)\|\,dt$$

$\gamma = \operatorname{Im}(x) \quad x\in C^1([a,b],\mathbb{R}^3)$ regolare a tratti

$F: A\to\mathbb{R} \quad A\supseteq\gamma$

ESEMPIO $\quad \gamma=\operatorname{Im}(x)$ è una "distribuzione" di massa di densità (lineare) $\delta(x)$

massa totale $\quad M = \int_\gamma \delta(x)\,ds$

G centro di massa della distribuzione

$$G_i = \frac{1}{M} \int_\gamma x_i \, \delta(x) \, ds$$

$\delta(x) = \delta_0 \in \mathbb{R}^+ = (0, +\infty)$ densità di massa costante, γ elica

$$M = \int_\gamma \delta_0 \, ds = \int_a^b \delta_0 \, \|x'(t)\|_2 \, dt = \delta_0 \sqrt{R^2+p^2}\,(b-a)$$

per motivi di simmetria $G_1 = G_2 = 0$ (se $b = a + 2k\pi$)

$$G_3 = \frac{1}{M} \int_\gamma x_3 \, \delta(x) \, ds = \frac{1}{M} \int_a^b \delta_0 \, p t \sqrt{R^2+p^2} \, dt$$

$$= \frac{1}{\delta_0 \sqrt{R^2+p^2}\,(b-a)} \, \delta_0 \, p \sqrt{R^2+p^2} \int_a^b t \, dt$$

$$= \frac{p}{(b-a)} \left[\frac{1}{2} t^2 \right]_a^b = \frac{1}{2} p (b+a)$$

ESEMPIO

$$I_a = \int_\gamma \mathrm{dist}^2(x, a) \, \delta(x) \, ds$$

$\{x_1^2 + x_3^2 = R^2, \ x_2 = 0, \ x_3 \geq 0\} = \gamma$ $\quad x(t) = R(\cos(t), 0, \mathrm{sen}(t))$

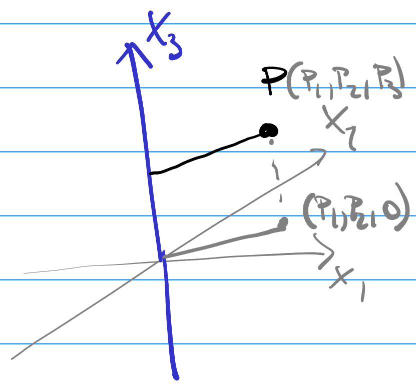
$t \in [0, \pi]$

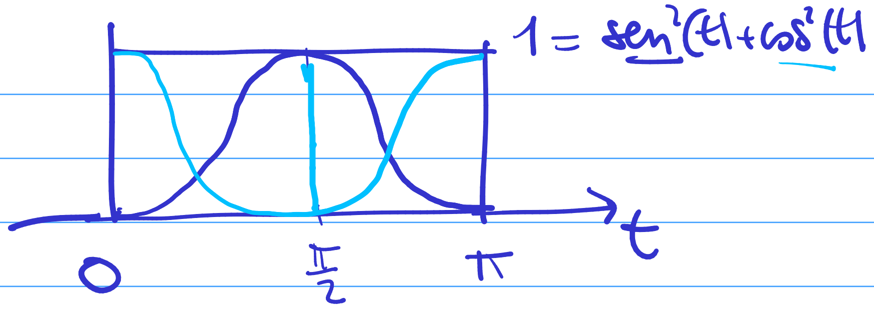
$a = \{x_1 = x_2 = 0\}$ $\quad \delta(x) = 1$

$$I_a = \int_\gamma \mathrm{dist}^2(x, a) \, ds$$

$$= \int_0^\pi (x_1^2 + x_2^2) \, R \, dt = \int_0^\pi R^2 \cos^2(t) \cdot R \, dt$$

$$= R^3 \int_0^\pi \cos^2(t) \, dt = \frac{\pi}{2} R^3$$





$$\int_0^{\pi} \cos^2(t)\,dt = \frac{1}{2}\int_0^{\pi} (1+\cos(2t))\,dt = \frac{\pi}{2}$$