Aritmetica estesa dei limiti: (conclusione)

PROP Se $b_n \to 0^+ \Rightarrow \dfrac{1}{b_n} \to +\infty$

$b_n \to 0^- \Rightarrow \dfrac{1}{b_n} \to -\infty$

COROLLARIO Se $\left.\begin{array}{l} a_n \to \ell \\ b_n \to 0^+ \end{array}\right| \Rightarrow \dfrac{a_n}{b_n} \to \begin{cases} +\infty & \text{se } \ell \in (0,+\infty] \\ -\infty & \text{se } \ell \in [-\infty, 0) \end{cases}$

$\left.\begin{array}{l} a_n \to \ell \\ b_n \to 0^- \end{array}\right| \Rightarrow \dfrac{a_n}{b_n} \to \begin{cases} -\infty & \text{se } \ell \in (0,+\infty] \\ +\infty & \text{se } \ell \in [-\infty, 0) \end{cases}$

DIM

$$\frac{a_n}{b_n} = \underset{\ell}{\underset{\downarrow}{a_n}} \cdot \underset{+\infty}{\underset{\downarrow}{\left(\frac{1}{b_n}\right)}}$$

e quindi si copia lo schema di "$\ell \cdot (+\infty)$"

N.B. "$\dfrac{0}{0}$" è un'altra forma indeterminata, ossia il risultato può variare a seconda della forma effettiva di a_n e b_n che tendono entrambe a zero.

$$\left.\begin{array}{l} a_n = \dfrac{1}{n^2} \\ b_n = \dfrac{1}{n} \end{array}\right. \Rightarrow \frac{a_n}{b_n} = \frac{1/n^2}{1/n} = \frac{n}{n^2} = \frac{1}{n} \to 0$$

Tabella 2.1 Algebra (parziale!) dei limiti in $\mathbb{R}^*$ ($\ell \in \mathbb{R}$)

Somma	Prodotto	Quoziente
$\pm\infty + \ell = \pm\infty$ $+\infty + \infty = +\infty$ $-\infty - \infty = -\infty$	$(\pm\infty)\cdot\ell = \begin{cases} \pm\infty & \text{se } \ell \in (0,+\infty) \\ \mp\infty & \text{se } \ell \in (-\infty,0) \end{cases}$	$\dfrac{\ell}{+\infty} = \begin{cases} 0^+ & \text{se } \ell \in (0,+\infty) \\ 0^- & \text{se } \ell \in (-\infty,0) \end{cases}$
Forme indeterminate $+\infty - \infty$ $\dfrac{0}{0}$ $\quad \pm\infty \cdot 0$ $\dfrac{\pm\infty}{\pm\infty}$	$(+\infty)\cdot(+\infty) = +\infty$ $(+\infty)\cdot(-\infty) = -\infty$ $(-\infty)\cdot(-\infty) = +\infty$	$\dfrac{\ell}{0^+} = \begin{cases} +\infty & \text{se } \ell \in (0,+\infty) \\ -\infty & \text{se } \ell \in (-\infty,0) \end{cases}$ $\dfrac{\ell}{0^-} = \begin{cases} -\infty & \text{se } \ell \in (0,+\infty) \\ +\infty & \text{se } \ell \in (-\infty,0) \end{cases}$ $\dfrac{\pm\infty}{0^+} = \pm\infty, \quad \dfrac{\pm\infty}{0^-} = \mp\infty$

$$\lim_{n\to+\infty} \left(n^{\alpha-1} - n^{-\alpha-1}\right)$$

$\alpha \in \mathbb{R}$. $\forall \alpha$ bisogna calcolare il limite.

$$n^{\alpha-1} \longrightarrow \begin{cases} +\infty & \text{se} \quad \alpha > 1 \\ 1 & \text{se} \quad \alpha = 1 \\ 0^+ & \text{se} \quad \alpha < 1 \end{cases}$$

$$n^{-\alpha-1} \longrightarrow \begin{cases} +\infty & \text{se} \quad \alpha < -1 \\ 1 & \text{se} \quad \alpha = -1 \\ 0^+ & \text{se} \quad \alpha > -1 \end{cases}$$

$$\lim_{n\to+\infty} \left(n^{\alpha-1} - n^{-\alpha-1}\right) = \begin{cases} +\infty - 0 = +\infty & \alpha > 1 \\ 1 - 0 = 1 & \alpha = 1 \\ 0 - 0 = 0 & -1 < \alpha < 1 \\ 0 - 1 = -1 & \alpha = -1 \\ 0 - (+\infty) = -\infty & \alpha < -1 \end{cases}$$

$$\lim_{n\to+\infty} \left(\underbrace{n^5}_{+\infty} + \underbrace{2n^3}_{+\infty} \underbrace{-12}\right) = +\infty.$$

Studio di forme indeterminate: limiti di polinomi

$$\lim_{n\to+\infty} \left(\underbrace{n^5}_{+\infty} - \underbrace{2n^3}_{-\infty} - 12\right) = \left(+\infty - \infty\right)$$

$$= \lim_{n\to+\infty} \underbrace{n^5}_{\to +\infty} \underbrace{\left(1 - \underbrace{\frac{2}{n^2}}_{\to 0} - \underbrace{\frac{12}{n^5}}_{\to 0}\right)}_{\to 1} = (+\infty \cdot 1) = +\infty$$

Questo ragionamento si generalizza a tutti i polinomi.

Se $P_k(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \dots + a_1 n + a_0$

$$= \sum_{j=0}^{k} a_j n^j \qquad \text{dove} \qquad k \in \mathbb{N}$$

$a_j \in \mathbb{R} \quad j = 0, 1, \dots k$

$a_k \neq 0$

$$\lim_{n\to+\infty} P_k(n) = \lim_{n\to+\infty} \underbrace{n^k}_{\to +\infty} \underbrace{\left(a_k + \underbrace{\frac{a_{k-1}}{n}}_{\to 0} + \underbrace{\frac{a_{k-2}}{n^2}}_{\to 0} + \dots + \underbrace{\frac{a_1}{n^{k-1}}}_{\to 0} + \underbrace{\frac{a_0}{n^k}}_{\to 0}\right)}_{\to a_k}$$

$$= \begin{cases} +\infty & \text{se} \quad a_k > 0 \\ -\infty & \text{se} \quad a_k < 0 \end{cases}$$

$$\lim_{n\to+\infty} \left(5n^4 - 3n^2 + 12 - n^6\right) = -\infty$$

Limiti di funzioni razionali fratte (rapporti di polinomi)

$$\lim_{n\to+\infty} \underbrace{\overbrace{3n^2-2n+5}^{\to+\infty}}_{}\,/\,\underbrace{(-n^2+n+1)}_{\to-\infty} = \left(\frac{+\infty}{-\infty}\right) =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^2}\,\overbrace{\left(3-\frac{2}{n}+\frac{5}{n^2}\right)}^{\to 3}}{\cancel{n^2}\,\underbrace{\left(-1+\frac{1}{n}+\frac{1}{n^2}\right)}_{\to -1}} = \frac{3}{-1} = -3$$

$$\lim_{n\to+\infty} \frac{3n^2-2n+5}{-n^3+n+1} = \left(\frac{+\infty}{-\infty}\right) = \lim_{n\to+\infty} \overbrace{\boxed{\frac{\cancel{n^2}\left(3+\text{succ che tende a zero}\right)}{n^{\cancel{3}}\left(-1+\text{succ}^{\text{ne}}\text{ che tende a zero}\right)}}}^{\to -3}$$

$$= 0$$

Notazione importante.

Sia $\{a_n\}$ una successione

Diremo che $\quad a_n = o(1) \quad$ (per $n\to+\infty$)

se $\quad \lim_{n\to+\infty} a_n = 0$

Per es.

$$\frac{1}{n} = o(1)$$

$$\frac{5}{n^2+3} = o(1)$$

OSS Attenzione all'uso di "="
da questo non segue che $\frac{1}{n} = \frac{5}{n^2+3}$

Rifaccio il limite di prima

$$\lim_{n\to+\infty} \frac{3n^2-2n+5}{-n^3+n+1} = \lim_{n\to+\infty} \frac{\cancel{n^2}}{n^{\cancel{3}}}\,\overbrace{\boxed{\frac{3+o(1)}{-1+o(1)}}}^{= -3+o(1)} = 0$$

$$\lim_{n\to+\infty} \frac{3n^3-2n+5}{-n^2+n+1} = \left(\frac{+\infty}{-\infty}\right) = \lim_{n\to+\infty} \frac{n^{\cancel{3}}(3+o(1))}{\cancel{n^2}(-1+o(1))} = +\infty\cdot(-3) =$$

$$= -\infty .$$

Sia $\displaystyle P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0 = \sum_{j=0}^{k} a_j n^j$

$$Q_h(n) = b_h n^h + b_{h-1} n^{h-1} + \dots + b_1 n + b_0 = \sum_{j=0}^{h} b_j n^j$$

$k, h \in \mathbb{N}$ $\quad a_0, \dots a_k;\ b_0, \dots, b_h \in \mathbb{R}$

$\qquad a_k \neq 0, \quad b_h \neq 0$

$$\lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \left(\frac{\pm\infty}{\pm\infty}\right) = \lim_{n\to+\infty} \frac{n^k}{n^h}\boxed{\frac{(a_k + o(1))}{(b_h + o(1))}} \to \frac{a_k}{b_h} =$$

$$= \begin{cases} \pm\infty \ (+\infty \text{ se } a_k, b_h \text{ concordi}, -\infty \text{ se discordi}) & \text{se } k > h \\ \dfrac{a_k}{b_k} & \text{se } k = h \\ 0 \ (0^+ \text{ se } a_k, b_h \text{ concordi}, 0^- \text{ se } a_k, b_h \text{ discordi}) & \text{se } k < h \end{cases}$$

Riassunto: $\displaystyle \lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \lim_{n\to+\infty} \frac{a_k n^k}{b_h n^h}$

Si applica a situazioni simili, anche se non compaiono esattamente polinomi.

$$\lim_{n\to+\infty} \frac{n^\alpha - n}{n^\pi - n^3} = \begin{cases} +\infty & \alpha > \pi \\ 1 & \alpha = \pi \\ 0 \ \begin{cases} 0^+ & \text{se } 1 < \alpha < \pi \\ \equiv 0 & \text{se } \alpha = 1 \\ 0^- & \text{se } \alpha < 1 \end{cases} & \alpha < \pi \end{cases}$$

$$n^\pi - n^3 = n^\pi(1+o(1))$$

$$\longrightarrow n^\alpha - n = \begin{cases} n^\alpha\left(1+o(1)\right) & \alpha>1 \\ 0 & \alpha=1 \\ n\left(-1+o(1)\right) & \alpha<1 \end{cases}$$

$$\lim_{n\to+\infty} \frac{2^n-5^n}{\underbrace{3^n+1}_{\to+\infty}} = \left(\frac{-\infty}{+\infty}\right) = \lim_{n\to+\infty} \frac{5^n}{3^n}\boxed{\frac{\left(-1+o(1)\right)}{\left(1+o(1)\right)}}\to -1 =$$

$$\left[\, 2^n - 5^n = (+\infty-\infty) = 5^n\left(\frac{2^n}{5^n}-1\right) = 5^n\underbrace{\left(-1+o(1)\right)}_{\to -\infty}\,\right]$$

$$\frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \to 0$$

$$= \lim_{n\to+\infty} \underbrace{\left(\frac{5}{3}\right)^n}_{\to+\infty} \underbrace{\left(-1+o(1)\right)}_{\to -1} = -\infty.$$

$$\lim_{n\to+\infty}\left(\underbrace{\sqrt{n^3+n}}_{\to+\infty} \underbrace{-\,n}_{\to-\infty}\right) = (+\infty-\infty)$$

$$= \lim_{n\to+\infty} \underbrace{n^{3/2}}_{\to+\infty}\underbrace{\left[\underbrace{\sqrt{1+\frac{1}{n^2}}}_{\to 1} - \underbrace{\frac{1}{\sqrt{n}}}_{\to 0}\right]}_{=1+o(1)} = +\infty.$$

$$\lim_{n\to+\infty}\left(\sqrt{n^2+5}-2n\right)=(+\infty-\infty)=$$

$$=\lim_{n\to+\infty} n\underbrace{\left[\sqrt{1+\frac{5}{n^2}}-2\right]}_{1-2=-1}=-\infty$$

$$\lim_{n\to+\infty}\left(\sqrt{n^2+5}-n\right)=(+\infty-\infty)=$$

$$=\lim_{n\to+\infty} \underbrace{n}_{+\infty}\underbrace{\left[\sqrt{1+\frac{5}{n^2}}-1\right]}_{1-1=0}=(+\infty\cdot 0)$$

Altro metodo.

$$\sqrt{n^2+5}-n=\frac{\left(\sqrt{n^2+5}-n\right)\left(\sqrt{n^2+5}+n\right)}{\sqrt{n^2+5}+n}=$$

$$\boxed{(A-B)(A+B)=A^2-B^2}$$

$$=\frac{\cancel{n^2}+5-\cancel{n^2}}{\sqrt{n^2+5}+n}=\frac{5}{\sqrt{n^2+5}+n}\longrightarrow\frac{5}{+\infty}=0$$

$$\lim_{n\to+\infty}\left(\sqrt[3]{n^3+5}-n\right)=$$

$$(A-B)\left(A^2+AB+B^2\right)=A^3-B^3$$

$$= \lim_{n\to+\infty} \frac{\left(\sqrt[3]{n^3+5} - n\right)\left((n^3+5)^{2/3} + n\sqrt[3]{n^3+5} + n^2\right)}{\left((n^3+5)^{2/3} + n\sqrt[3]{n^3+5} + n^2\right)} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^3}+5-\cancel{n^3}}{(n^3+5)^{2/3} + n\sqrt[3]{n^3+5} + n^2} = \left(\frac{5}{+\infty}\right) = 0$$

$$\lim_{n\to+\infty} \left(\sqrt[3]{n^3+n^2} - n\right) = (+\infty - \infty) =$$

$$= \lim_{n\to+\infty} \frac{\left(\sqrt[3]{n^3+n^2} - n\right)\left((n^3+n^2)^{2/3} + n\sqrt[3]{(n^3+n^2)} + n^2\right)}{(n^3+n^2)^{2/3} + n\sqrt[3]{n^3+n^2} + n^2} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^3}+n^2-\cancel{n^3}}{(n^3+n^2)^{2/3} + n\sqrt[3]{n^3+n^2} + n^2} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^2}\ 1}{\cancel{n^2}\left[\underbrace{\left(1+\frac{1}{n}\right)^{2/3}}_{\downarrow\ 1} + \underbrace{\sqrt[3]{1+\frac{1}{n}}}_{\downarrow\ 1} + \underset{\downarrow\ 1}{1}\right]} = \frac{1}{3}$$