Trovare sup e inf di $a_n = \frac{n}{n^2+50}$ $\quad n \in \mathbb{N}_+$

Dobbiamo trovare $\sup E$, $\inf E$, dove

$$E = \left\{ x = \frac{n}{n^2+50} \ , \quad n \in \mathbb{N}_+ \right\}$$

<u>oss</u> gli a_n sono >0.

Studiamo la crescenza/decrescenza.

$$a_{n+1} \overset{?}{>} a_n$$

$$\frac{n+1}{(n+1)^2+50} \overset{?}{>} \frac{n}{n^2+50}$$

$$(n+1)(n^2+50) \overset{?}{>} n\left(n^2+2n+51\right)$$

$$\cancel{n^3}+n^2+50n+50 \overset{?}{>} \cancel{n^3}+2n^2+51n$$

$$\left(n^2+n-50 \overset{?}{<} 0 \right)$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+200}}{2} = \frac{-1 \pm \sqrt{201}}{2}$$

(14, 15)

$\cancel{x_1}<0 \qquad 6<x_2<7$.

$$x_1 \underset{\text{sempre}}{<} n < x_2$$

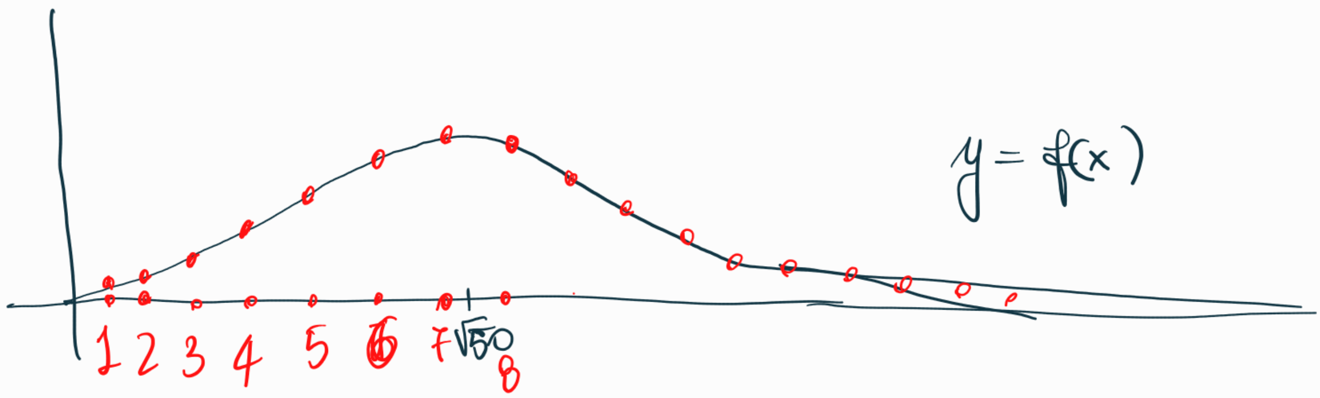
$$a_1 < a_2 < a_3 < a_4 < \dots < a_7 > a_8 > a_9 > a_{10} > \dots$$

$$\sup a_n = \max a_n = a_7 = \frac{7}{49+50} = \frac{7}{99}$$

Alternativa usando la derivata (ancora non fatta a lezione!)

$$f(x) = \frac{x}{x^2+50} \qquad x>0$$

$$f'(x) = \frac{x^2+50-2x^2}{(x^2+50)^2} = \frac{50-x^2}{(x^2+50)^2} > 0 \iff x < \sqrt{50}$$



$$a_7 \overset{?}{>} a_8 \qquad \frac{7}{99} \overset{?}{>} \frac{8}{114} = \frac{4}{57}$$

$$\underset{399}{\underset{\shortparallel}{7 \cdot 57}} \overset{?}{>} 4 \cdot 99 = 396 \qquad \text{sì}$$

Congettura: $\inf a_n = 0$.

Verifica:

1) $0 \overset{?}{\leq} a_n \quad \forall n \in \mathbb{N}_+ \qquad$ <u>sì</u>.

2) Fissato $\varepsilon > 0$, cerco $n \in \mathbb{N}_+$ t.c. $\underline{a_n < 0 + \varepsilon}$

$$a_n < \varepsilon \iff \underline{\frac{n}{n^2+50} < \varepsilon} \iff$$

$$\Leftrightarrow \quad n < \varepsilon(n^2+50)$$

$$\Leftrightarrow \quad \varepsilon n^2 - n + 50\varepsilon > 0 \quad \Leftarrow \quad n > \frac{1+\sqrt{1-200\varepsilon^2}}{2\varepsilon}$$

[Cerco le radici di $\varepsilon x^2 - x + 50\varepsilon$

$$x = \frac{1 \pm \sqrt{1-200\varepsilon^2}}{2\varepsilon}$$

Posso supporre $0 < \varepsilon \leq \frac{1}{\sqrt{200}}$]

<u>OSS</u> In realtà era molto più semplice

Devo trovare $n \in \mathbb{N}_+$ t.c. $a_n < \varepsilon$

$$\frac{n}{n^2+50} < \varepsilon$$

<u>OSS</u>

$$\frac{n}{n^2+50} < \frac{n}{n^2} = \frac{1}{n}$$

e quindi basta scegliere n t.c. $\frac{1}{n} < \varepsilon$,

ossia $n > \frac{1}{\varepsilon}$

<u>OSS 2</u>.

Abbiamo in realtà provato che, fissato $\varepsilon > 0$,

$$\underset{-\varepsilon}{\overset{}{0}} \underset{\vee}{<} a_n < \varepsilon \qquad \forall n \text{ suff}^{\text{te}} \text{ grande}$$

cioè che $\lim_{n\to+\infty} a_n = 0$.

Provare che

$$\lim_{n \to +\infty} \frac{n}{n^2-50} = 0$$

1° modo Fisso $\varepsilon > 0$. cerco k t.c. $\forall n > k$

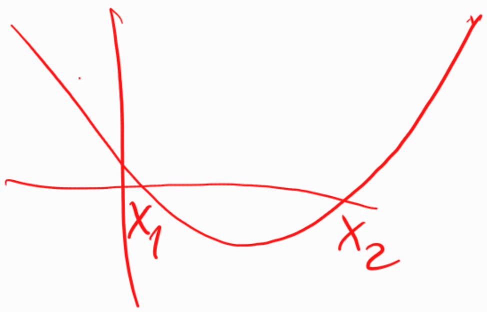
$$\left| \frac{n}{n^2-50} \right| < \varepsilon$$

$\parallel$ $n > 7$

$$\frac{n}{n^2-50}$$

$$\frac{n}{n^2-50} < \varepsilon \iff \varepsilon n^2 - n - 50\varepsilon > 0$$

$$\left(x_{1,2} = \frac{1 \pm \sqrt{1+200\varepsilon^2}}{2\varepsilon} \right)$$



vera se scelgo

$$n > \frac{1+\sqrt{1+200\varepsilon^2}}{2\varepsilon}$$

Prendo $k = \max\left\{ 7, \frac{1+\sqrt{1+200\varepsilon^2}}{2\varepsilon} \right\}$

OSS

$$50 < n \qquad \text{se} \quad n > 50$$

$$-50 > -n$$

$$n^2 - 50 > n^2 - n \qquad n > 50$$

$$\frac{1}{n^2-50} < \frac{1}{n^2-n} \qquad n > 50$$

$$0 < \frac{n}{n^2-50} \underset{n>50}{<} \frac{n}{n^2-n} = \frac{1}{n-1} \to 0 \quad \text{se } n>50$$

$n>7$

Verificare che

$$\lim_{n\to+\infty} \sqrt{\frac{9n}{n+2}} = 3$$

Fissato $\varepsilon>0$ cerco k t.c. $\forall n>k$

$$\underbrace{\left|\sqrt{\frac{9n}{n+2}} - 3\right|}_{\overset{\wedge}{0}} < \varepsilon$$

$$\Updownarrow$$

$$3 - \sqrt{\frac{9n}{n+2}} < \varepsilon$$

$$\Updownarrow$$

$$3-\varepsilon < \sqrt{\frac{9n}{n+2}} \qquad \text{suppongo } \varepsilon \leq 3$$

$$\Updownarrow$$

$$(*) \qquad (3-\varepsilon)^2 < \frac{9n}{n+2} \left(= \frac{9n+18-18}{n+2} = 9 - \frac{18}{n+2} \right)$$

$$\Updownarrow$$

$$9-6\varepsilon+\varepsilon^2 < \frac{9n}{n+2}$$

$$\Updownarrow$$

$$(\cancel{9}-6\varepsilon+\varepsilon^2)n + 2(9-6\varepsilon+\varepsilon^2) < \cancel{9n}$$

$$\Updownarrow$$

$$n\,(6\varepsilon-\varepsilon^2) > 2\,(9-6\varepsilon+\varepsilon^2)$$

$$n\,\underbrace{\varepsilon\,(6-\varepsilon)}_{>0}$$

$$\Updownarrow$$

$$n > \frac{2\,(3-\varepsilon)^2}{\varepsilon\,(6-\varepsilon)} =: K.$$

In alternativa, arrivati a $(*)$ si poteva osservare che $\frac{9n}{n+2} = 2 - \frac{18}{n+2}$, quindi $(*)$ equivale a

$$\frac{18}{n+2} \overset{?}{<} 9 - (3-\varepsilon)^2 = \cancel{9} - \cancel{9} + 6\varepsilon - \varepsilon^2 = \varepsilon\,(6-\varepsilon)$$

Dunque basta prendere $n+2 > \frac{18}{\varepsilon(6-\varepsilon)}$

Mostrare che non è vero che

$$\lim_{n\to+\infty} \sqrt{\frac{9n}{n+2}} = 2$$

Se fosse vero, dovrei, $\forall \varepsilon>0$, trovare K t.c.

$\underline{\forall n > K}$ $\qquad \left| \sqrt{\frac{9n}{n+2}} - 2 \right| < \varepsilon$

$$\sqrt{\frac{9n}{n+2}} > 2 \quad ? \qquad\qquad \frac{9n}{n+2} > 4$$

$9n > 4n + 8 \qquad 5n > 8$

vero per $n \geq 2$.

$$\sqrt{\frac{9n}{n+2}} - 2 < \varepsilon$$

scelgo $\varepsilon = 1/2$

$$\sqrt{\frac{9n}{n+2}} < 2+\varepsilon = \frac{5}{2}$$

$$\Updownarrow$$

$$\frac{9n}{n+2} < \frac{25}{4}$$

$$\Updownarrow$$

$$36n < 25n + 50$$

$$\Updownarrow$$

$$11n < 50 \qquad \text{falso se } n > \frac{50}{11}$$

$$\lim_{n \to +\infty} \frac{1}{2^n + 1} = 0$$

Con i carabinieri:

$$0 < \frac{1}{2^n+1} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \longrightarrow 0$$

visto oggi.

Oppure con la def^{ne}:

Fissato $\varepsilon > 0$, cerco K t.c. $\forall n > K$

$$\left| \frac{1}{2^n+1} \right| < \varepsilon$$

$$\frac{1}{2^n+1} < \varepsilon$$

$$2^n+1 > \frac{1}{\varepsilon}$$

$$2^n > \underbrace{\frac{1}{\varepsilon} - 1}_{>0}$$ suppongo $0 < \varepsilon < 1$

Passo a $\log_2$

$$n > \log_2\left(\frac{1}{\varepsilon} - 1\right) =: k$$

$$\lim_{n\to+\infty} \left(\cos(n^2) - \sqrt{n+1}\right) = -\infty$$

$$\cos(n^2) - \sqrt{n+1} \leq 1 - \sqrt{n+1} < \underbrace{1-\sqrt{n}}_{\downarrow ?\ -\infty}$$

Fissamo $M>0$, cerco k t.c. $\forall n > k$.

$$1 - \sqrt{n} < -M$$

$$\Updownarrow$$

$$\sqrt{n} > M+1$$

$$\Updownarrow$$

$$n > (M+1)^2 =: k$$

Quindi $1-\sqrt{n} \to -\infty$. Quindi, per il thm. dei carabinieri, anche $\cos(n^2) - \sqrt{n+1} \to -\infty$