ESERCITAZIONE 4

1) $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$ indipendenti tra loro. Determina le distribuzioni delle seguenti v.a.

i) $U = (X_1 - X_2) \sim ?$

$$\mathbb{E}[U] = \mathbb{E}[X_1 - X_2] = \mathbb{E}[X_1] - \mathbb{E}[X_2] = 0 - 0 = 0$$
$$V[U] = V(X_1 - X_2) = V(X_1) + V(X_2) = 1 + 1 = 2$$
$$\Rightarrow U \sim N(0,2)$$

ii) $Z = \frac{U}{\sqrt{2}} \sim ?$

$$\mathbb{E}[Z] = \mathbb{E}[U/\sqrt{2}] = \frac{1}{\sqrt{2}} \mathbb{E}[U] = \frac{1}{\sqrt{2}} 0 = 0$$
$$V[Z] = V[U/\sqrt{2}] = \frac{1}{2} V[U] = \frac{1}{2} 2 = 1$$
$$\Rightarrow Z = \frac{U}{\sqrt{2}} \sim N(0,1)$$

iii) $Z^2 = \frac{U^2}{2} = ?$ $\rightarrow$ $Z^2 = N(0,1)^2 = \chi^2_1$
$$= \frac{U^2}{2}$$

2) $\underline{X}_n = (X_1, \dots, X_n)$ t.c. $X_i | \mu, \sigma^2 \sim N(\mu, \sigma^2)$ $\forall i = 1 \dots n$

Determinare $\mathbb{E}[H_n]$ e $V[H_n]$ essendo $H_n = a S_n^2 + b \bar{X}_n^2$ con $a, b \in \mathbb{R}$

$$\mathbb{E}[H_n] = \mathbb{E}[a S_n^2 + b \bar{X}_n^2] = a\mathbb{E}[S_n^2] + b\,\mathbb{E}[\bar{X}_n^2] = a V[X] + b\,\mathbb{E}[\bar{X}_n^2]$$
$$= a\sigma^2 + b\left(V[\bar{X}_n] + \mathbb{E}[\bar{X}_n]^2\right)$$

$\rightarrow$ poiché $V[X] = \mathbb{E}[X^2] - \mathbb{E}(X)^2$
$\mathbb{E}[X^2] = V[X] + \mathbb{E}(X)^2$

$$= a\sigma^2 + b\left(\frac{V(X)}{n} + \mathbb{E}[X]^2\right)$$
$$= a\sigma^2 + b\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$\rightarrow$ questo poiché per il caso normale S_n^2 e $\bar{X}_n$ sono indipendenti.

$$V[H_n] = V[a S_n^2 + b \bar{X}_n^2] = a^2 V[S_n^2] + b^2 V[\bar{X}_n^2]$$

$$\frac{(n-1)}{\sigma^2} S_n^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sim \chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{1}{2}\right)$$

$$\Rightarrow V\left[\frac{(n-1) S_n^2}{\sigma^2}\right] = \underbrace{\frac{n-1}{2}\, 4}_{\text{Varianza della } G^a\left(\frac{n-1}{2}, \frac{1}{2}\right)} = 2(n-1) \Rightarrow \frac{(n-1)^2}{\sigma^4} V[S_n^2] = 2(n-1)$$

$$\rightarrow V[S_n^2] = \frac{2\sigma^4}{n-1}$$

Si suppone che $\mu = 0 \Rightarrow \bar{X}_n \sim N(\mu=0, \sigma^2/m) = N(0, \sigma^2/m)$

$$\bar{X}_n^2 = \left(\frac{\bar{X}_n \sqrt{m}}{\sigma} \frac{\sigma}{\sqrt{n}}\right)^2 = \frac{\sigma^2}{m}\left(\frac{\bar{X}_n}{\sigma}\sqrt{m}\right)^2 = \frac{\sigma^2}{m} Z^2 \quad \text{con } Z = \frac{\bar{X}_n \sqrt{m}}{\sigma} \sim N(0,1)$$

quindi $Z^2 \sim \chi^2_1 = G^R\left(\frac{1}{2}, \frac{1}{2}\right)$

quindi $\bar{X}_n^2 = \frac{\sigma^2}{m} Z^2 = \frac{\sigma^2}{m}\chi^2_1 \sim \mathrm{Gamma}\left(\frac{1}{2}, \frac{m}{2\sigma^2}\right)$

$$V(\bar{X}_n^2) = \frac{1}{2}\left(\frac{2\sigma^2}{m}\right)^2 = \frac{2\sigma^4}{m^2}$$

otteniamo quindi $V(H_n) = a^2 \frac{2\sigma^4}{m-1} + b^2 \frac{2\sigma^4}{m^2}$

3) $X_i | \theta$ iid $i=1 \dots n$ $\quad f_X(x, \theta) = 2\theta^2 x \, \mathbb{1}_{[0, 1/\theta]}(x)$ con $\theta > 0$. Determina $\mathbb{E}_\theta[X]$ e lo stimatore dei momenti di θ.

$$\mathbb{E}_\theta[X] = \int_0^{1/\theta} 2\theta^2 x^2 \, dx = 2\theta^2 \left[\frac{x^3}{3}\right]_0^{1/\theta} = 2\theta^2 \frac{1}{3\theta^3} = \frac{2}{3}\frac{1}{\theta}$$

$$\mu_1(\theta) = m_1(\underline{X}_n) \Rightarrow \mathbb{E}_\theta[X] = \bar{X}_n \Rightarrow \frac{2}{3}\frac{1}{\theta} = \bar{X}_n \Rightarrow \hat{\theta}_M = \frac{2}{3\bar{X}_n}$$

4) $X_i | \theta \sim \mathrm{Unif}[-\theta, \theta]$ con $\theta > 0$. Determina stimatore dei momenti di θ.

$$f(x, \theta) = \frac{1}{2\theta} \mathbb{1}_{[-\theta, \theta]}(x) \Rightarrow \mathbb{E}[X] = \frac{(\theta - \theta)^2}{2} = 0 \Rightarrow \mathbb{E}[X] = \bar{X}_n \Rightarrow 0 = \bar{X}_n$$

Utilizziamo il momento secondo $\mathbb{E}[X^2] = \frac{1}{m}\sum X_i^2$

$$\mathbb{E}[X^2] = V[X] + \mathbb{E}[X]^2 = \frac{4\theta^2}{12} + 0 = \frac{4\theta^2}{12} = \frac{1}{m}\sum X_i^2 \Rightarrow \hat{\theta}_{M_m} = \sqrt{\frac{3}{m}\sum_{i=1}^m X_i^2}$$

5) $X_i | \theta \sim Ga^R(\theta, 2)$ $\theta > 0$ calcola $\hat{\theta}_{mom}$ e il suo valore atteso e varianza e MSE.

i)
$$\mu_1(\theta) = \mathbb{E}_\theta[X] = \frac{\theta}{2} \; ; \; m_1(\underline{X}_n) = \frac{1}{m}\sum_{i=1}^m X_i = \bar{X}_n \Rightarrow \frac{\theta}{2} = \bar{X}_n \Rightarrow \hat{\theta}_{mom} = 2\bar{X}_n$$

ii)
$$\mathbb{E}[\hat{\theta}_{mom}] = \mathbb{E}[2\bar{X}_n] = 2\mathbb{E}[\bar{X}_n] = 2\mathbb{E}[X] = 2\frac{\theta}{2} = \theta \quad \forall \theta$$

$$V[\hat{\theta}_{mom}] = V[2\bar{X}_n] = 4V[\bar{X}_n] = 4\frac{V[X]}{m} = 4\frac{\theta}{4m} = \frac{\theta}{m}$$

$$MSE(\hat{\theta}_{mom}) = V_\theta[\hat{\theta}_{mom}] + B_\theta^2(\hat{\theta}_{mom}) = V_\theta[\hat{\theta}_{mom}] = \theta / m \quad \forall \theta$$

6) $X_1 \dots X_n \sim Pois(\vartheta)$ iid

Calcolare lo STV di $g(\vartheta) = \mathbb{P}(X=0;\vartheta)$ quando $\sum x_i = 6$ e $n=2$

$$f_X(x,\vartheta) = \frac{e^{-\vartheta}\vartheta^x}{x!}$$

$$L(\vartheta, \underline{x}_1) \propto e^{-n\vartheta}\vartheta^{\sum x_i}$$

$$\ell(\vartheta, \underline{x}_1) \propto -n\vartheta + \sum x_i \log\vartheta$$

$$\frac{d}{d\vartheta}\ell(\vartheta) = -n + \frac{\sum x_i}{\vartheta} = 0 \Rightarrow \hat{\vartheta}_{MV} = \frac{\sum x_i}{n} = \bar{x}_n$$

Per la proprietà di equivarianza dello STV $\lambda = g(\vartheta) \Rightarrow \hat{\lambda}_{MV} = g(\hat{\vartheta}_{MV})$

quindi $\hat{\lambda}_{MV} = g(\hat{\vartheta}_{MV}) = g(\bar{x}_n) = \mathbb{P}(X=0;\bar{x}_n)$

$$= \frac{e^{-\hat{\vartheta}_{MV}}\hat{\vartheta}_{MV}^{0}}{0!} = e^{-\hat{\vartheta}_{MV}} = e^{-\bar{x}_n} = e^{-\frac{6}{2}} = e^{-3}$$

7) Sia X v.a. che assume valori $(0,1,2)$ con probabilità $(1-\frac{\vartheta}{2}, \frac{\vartheta}{4}, \frac{\vartheta}{4})$

determina $\mathbb{E}[X]$ e lo stimatore dei momenti di ϑ.

$$\mathbb{E}[X] = 0\cdot\left(1-\frac{\vartheta}{2}\right) + 1\cdot\frac{\vartheta}{4} + 2\cdot\frac{\vartheta}{4} = \frac{\vartheta}{4} + 2\frac{\vartheta}{4} = \frac{3}{4}\vartheta$$

$$\mathbb{E}[X] = \frac{3}{4}\vartheta = \bar{X}_n \Rightarrow \hat{\vartheta}_{MoM} = \frac{4}{3}\bar{X}_n$$