ESERCITAZIONE 3

1) $X_n = (X_1 ... X_n)$ t.c. $X_i|\theta$ iid $\forall i = 1 ... n$

$f_X(x,\theta) = \theta(1+x)^{-(1+\theta)}$; $x > 0$, $\theta > 0$

i) Verifica che il modello appartiene alla famiglia esponenziale ed identificare, se esiste, una statistica sufficiente.

$$f_X(x,\theta) = \theta\, (1+x)^{-1}(1+x)^{-\theta} = \frac{1}{1+x}\,\theta(1+x)^{-\theta} = \frac{1}{1+x}\exp\left\{\log\left(\theta(1+x)^{-\theta}\right)\right\}$$

$$= \frac{1}{1+x}\exp\left\{\log\theta - \theta\log(1+x)\right\} = \underbrace{\frac{1}{1+x}}_{h(x)}\exp\left\{\underbrace{-\theta}_{\eta(\theta)}\underbrace{\log(1+x)}_{T(x)} - \underbrace{(-\log\theta)}_{B(\theta)}\right\}$$

Il modello appartiene alla famiglia esponenziale.

La statistica sufficiente è data da $\sum_{i=1}^{n} T(x_i) = \sum_{i=1}^{n}\log(1+x_i)$.

ii) Determina $\hat{\theta}_{MV}$, il nucleo della fdv per un generico campione osservato x_n.

$$L(\theta, x_n) = \prod_{i=1}^{n} f_X(x_i,\theta) = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)}\prod_{i=1}^{n}(1+x_i)^{-\theta} \propto \theta^n\prod_{i=1}^{n}(1+x_i)^{-\theta} = g(\theta, T(x_n))$$

Calcolo $\hat{\theta}_{MV}$: $\ell(\theta, x_n) \propto n\log\theta - \theta\log\prod_{i=1}^{n}(1+x_i) = n\log\theta - \theta\sum_{i=1}^{n}\log(1+x_i)$

$$\frac{d}{d\theta}\ell(\theta, x_n) = \frac{n}{\theta} - \sum_{i=1}^{n}\log(1+x_i) = 0 \Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^{n}\log(1+x_i)}$$

$$\frac{d^2}{d\theta^2}\ell(\theta, x_n) = -\frac{n}{\theta^2} < 0 \;\forall \theta \in \Theta \Rightarrow \hat{\theta} = \hat{\theta}_{MV}$$

iii) Calcolare l'informazione di Fisher osservata

$$I_n^{oss}(x_n, \hat{\theta}_{MV}) = -\ell''(\theta)\Big|_{\theta=\hat{\theta}_{MV}} = \left[\frac{n}{\theta^2}\right]\Big|_{\theta=\hat{\theta}_{MV}} = \frac{n}{\left[\frac{n}{\sum\log(1+x_i)}\right]^2} = \frac{\left[\sum\log(1+x_i)\right]^2}{n}$$

iv) Dato $n = 50$ e $\sum_{i=1}^{n}\log(1+x_i) = 106.6$, calcola $\hat{\theta}_{MV}$, $I_n^{oss}(x_n)$, e l'approssimazione normale per la funzione di verosimiglianza.

$$\hat{\theta}_{MV} = \frac{50}{106.6} \;;\; I_n^{oss}(x_n) = \frac{(106.6)^2}{50}$$

$$\tilde{L}(\theta, x_n) = \exp\left\{-\frac{1}{2}(\theta - \hat{\theta}_{MV})^2 I_n^{oss}(x_n)\right\} = \exp\left\{-\frac{1}{2}\left(\theta - \frac{50}{106.6}\right)^2\frac{(106.6)^2}{50}\right\}$$

2) Dato $\underline{X_n} = (X_1 \dots X_n)$ $\quad X_i | \theta$ iid $\forall i = 1 \dots n$

$$f_X(x, \theta) = \theta^2 (-\log x)\, x^{\theta - 1} \quad , \; x \in (0,1) \quad \theta \in (0, \infty)$$

i) Trova una statistica sufficiente e definire se è minimale

$$L(\theta, \underline{x_n}) = \prod_{i=1}^{n} f_X(x_i; \theta) = \prod_{i=1}^{n} \left(\theta^2 \frac{(-\log x)}{x}\, x^{\theta} \right) \propto \theta^{2n} \prod_{i=1}^{n} x_i^{\theta} = g(\theta, T(\underline{x_n}))$$

la statistica sufficiente è $T(\underline{x_n}) = \prod_{i=1}^{n} x_i$ è anche minimale?

$$f_X(x, \theta) = \theta^2 \frac{(-\log x)}{x}\, x^{\theta} = \frac{-\log x}{x} \exp\left\{ \theta \log x - (-2 \log \theta) \right\}$$

$$= \underbrace{\frac{-\log x}{x}}_{h(x)} \exp\left\{ \underbrace{\theta}_{\eta(\theta)} \underbrace{\log x}_{T(x)} - \underbrace{2 \log\left(\tfrac{1}{\theta}\right)}_{B(\theta)} \right\}$$

Poiché il modello è famiglia esponenziale la statistica sufficiente $T(\underline{x_n}) = \prod_{i=1}^{n} x_i$ è anche minimale.

3) $X \sim \text{Gamma}(\theta_1, \theta_2)$ è famiglia esponenziale? $\quad \underline{\theta} = (\theta_1, \theta_2)$ $\quad \dim(\underline{\theta}) = 2$

$$f_X(x, \theta) = \frac{\theta_2^{\theta_1}}{\Gamma(\theta_1)}\, x^{\theta_1 - 1} e^{-\theta_2 x} = \exp\left\{ \theta_1 \log \theta_2 - \log \Gamma(\theta_1) + (\theta_1 - 1) \log x - \theta_2 x \right\}$$

$h(x) = 1$, $\quad \eta_1(\underline{\theta}) = \theta_1 - 1$ $\quad \eta_2(\theta) = -\theta_2$ $\quad , T_1(x) = \log x$ $\quad , T_2(x) = x$

$B(\theta) = -\theta_1 \log \theta_2 + \log \Gamma(\theta_1)$

ii) Statistica sufficiente per il modello?

$$\underline{T}(\underline{x_n}) = \left(\sum_{i=1}^{n} T_1(x_i), \sum_{i=1}^{n} T_2(x_i) \right) = \left(\sum_{i=1}^{n} \log x_i, \sum_{i=1}^{n} x_i \right) \rightarrow$$

statistica sufficiente minimale per il modello Gamma

4) $X_i|\theta \sim Unif[\theta-1, \theta+1]$ $\forall i=1...n$ iid

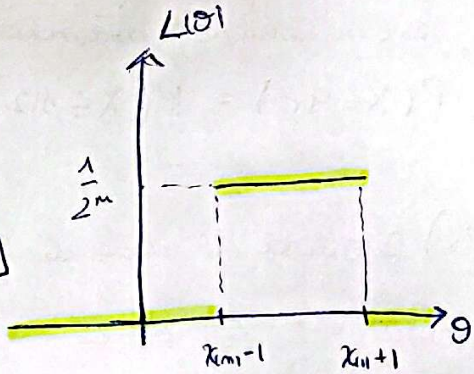
i) Calcolare $\hat{\theta}_{MV}$:

$$f_X(x,\theta) = \frac{1}{(\theta+1)-(\theta-1)} \mathbb{1}_{[\theta-1,\theta+1]}(x) = \frac{1}{2} \mathbb{1}_{[\theta-1,\theta+1]}(x)$$

$$L(\theta, \underline{x}_n) = \prod_{i=1}^{n} \left(\frac{1}{2} \mathbb{1}_{[\theta-1,\theta+1]}(x_i) \right) = \frac{1}{2^n} \prod_{i=1}^{n} \mathbb{1}_{[\theta-1,\theta+1]}(x_i)$$ → mi concentro su questa parte

$$\prod_{i=1}^{n} \mathbb{1}_{[\theta-1,\theta+1]}(x_i) = \begin{cases} 1 & \theta-1 \leq x_i \leq \theta+1 \quad \forall i=1...n \\ 0 & \text{altrimenti} \end{cases} = \begin{cases} 1 & \theta-1 \leq x_{(1)} \leq ... \leq x_{(n)} \leq \theta+1 \\ 0 & \text{altrimenti} \end{cases}$$

$$= \begin{cases} 1 & x_{(n)}-1 \leq \theta \leq x_{(1)}+1 \\ 0 & \text{altrimenti} \end{cases} = \mathbb{1}_{[x_{(n)}-1,\, x_{(1)}+1]}(\theta)$$



$$L(\theta, \underline{x}_n) = \frac{1}{2^n} \mathbb{1}_{[x_{(n)}-1,\, x_{(1)}+1]}(\theta) \propto \mathbb{1}_{[x_{(n)}-1,\, x_{(1)}+1]}(\theta)$$

$$\hat{\theta}_{MV}(\underline{x}_n) = [x_{(n)}-1,\, x_{(1)}+1]$$

ii) Definisci la statistica sufficiente del modello.

$$L(\theta, \underline{x}_n) = \underbrace{\frac{1}{2^n}}_{h(\underline{x}_n)} \underbrace{\mathbb{1}_{[x_{(n)}-1,\, x_{(1)}+1]}(\theta)}_{g(\theta, \underline{T}(\underline{x}_n))} \qquad \underline{T}(\underline{x}_n) = (x_{(1)}, x_{(n)})$$

5) $f_X(x,\theta) = \theta x^{\theta-1}$, $\theta > 0$, $x \in (0,1)$

i) $f_X(x,\theta)$ è funzione di densità ?

$$\int_0^1 f_X(x,\theta)\,dx = \int_0^1 \theta x^{\theta-1}\,dx = \theta \left[\frac{x^\theta}{\theta}\right]_0^1 = \theta \frac{1}{\theta} = 1 \quad \checkmark$$

ii) Determina θ in modo che $\mathbb{E}_\theta[X] = 1/4$

$$\mathbb{E}_\theta[X] = \int_0^1 x\,\theta x^{\theta-1}\,dx = \int_0^1 \theta x^\theta\,dx = \theta \left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1} = \frac{1}{4}$$

$$4\theta = \theta + 1 \rightarrow \theta = 1/3$$

iii) Dato q_r il quantile di livello r di X. Stabiline il valore di r che si ottiene ponendo $q_r = 1/2$

$$P(X \le q_r) = P(X \le 1/2) = \int_0^{1/2} \theta x^{\theta-1}\,dx = \theta \left[\frac{x^\theta}{\theta}\right]_0^{1/2} = \frac{1}{2^\theta} = r(\theta)$$

iv) Determina il valore di θ in modo che $r(\theta) = 1/4$

$$r(\theta) = \frac{1}{2^\theta} = \frac{1}{4} \iff \theta = 2$$