ESERCITAZIONE 2

1) Data $X \sim N(\vartheta, 1)$ ϑ è un parametro di posizione?
Come verifico se ϑ è parametro di posizione?

1. considero $f_X(x, \vartheta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\vartheta)^2\right\}$

2. Pongo $\vartheta = 0$ e calcolo $f_X(x, \vartheta=0) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} = f_X(x)$

3. Calcolo $g_P(x; \vartheta) = f(x-\vartheta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\vartheta)^2\right\}$

4. Nel punto 3 ho riottenuto la densità del punto 1 $\Rightarrow$ ϑ è parametro di posizione

Data $X \sim Exp(\vartheta)$ ϑ è un parametro di scala?

1. $f_X(x, \vartheta) = \frac{1}{\vartheta} e^{-x/\vartheta}$

2. Pongo $\vartheta = 1 \Rightarrow f_X(x, \vartheta=1) = e^{-x} = f_X(x)$ f. densità standard

3. Calcolo $g_S(x, \vartheta) = \frac{1}{\vartheta} f\left(\frac{x}{\vartheta}\right) = \frac{1}{\vartheta} e^{-x/\vartheta}$

4. Poiché ho riottenuto la densità del punto 1 $\Rightarrow$ ϑ è parametro di scala.

2) Dato $\underline{X}_n = (X_1 \ldots X_n)$ t.c. X_i iid $i=1 \ldots n$ e con $f_X(x, \vartheta) = \frac{\vartheta e^{\vartheta}}{x^{\vartheta+1}}$, $x > e$, $\vartheta > 0$

a) Scrivere $L(\vartheta, \underline{x}_n)$

b) Verifica che $\hat{\vartheta}_{MV} = \frac{n}{\sum_{i=1}^{n} \ln x_i - n}$

a) $L(\vartheta, \underline{x}_n^{oss}) = f_n(\underline{x}_n^{oss}, \vartheta) = \prod_{i=1}^{n} f_X(x_i, \vartheta) = \prod_{i=1}^{n} \left(\frac{\vartheta e^{\vartheta}}{x_i^{\vartheta+1}} \mathbb{1}_{(e,+\infty)}(x) \right)$

$$= \frac{\vartheta^n e^{n\vartheta}}{\left(\prod_{i=1}^{n} x_i\right)^{\vartheta+1}} \prod_{i=1}^{n} \mathbb{1}_{(e,+\infty)}(x_i) = \underbrace{\prod_{i=1}^{n} \left(\frac{\mathbb{1}_{(e,+\infty)}(x_i)}{x_i} \right)}_{h(\underline{x}_n)} \underbrace{\frac{\vartheta^n e^{n\vartheta}}{\left(\prod_{i=1}^{n} x_i\right)^{\vartheta}}}_{g(\vartheta, \underline{x}_n)} \propto \frac{\vartheta^n e^{n\vartheta}}{\left(\prod_{i=1}^{n} x_i\right)^{\vartheta}}$$

b) Osserviamo che $\mathcal{X} = (e, +\infty)$ che non dipende da $\vartheta \Longrightarrow$ problema di stima regolare.

$\ulcorner \ell(\vartheta, \underline{x}_n) = \log L(\vartheta, \underline{x}_n) = \log(h(\underline{x}_n) g(\vartheta, \underline{x}_n)) = \log(h(\underline{x}_n)) + \log(g(\vartheta, \underline{x}_n))$

$\Longrightarrow \frac{d}{d\vartheta} \ell(\vartheta, \underline{x}_n) = \frac{d}{d\vartheta}(\log(h(\underline{x}_n)) + \log(g(\vartheta, \underline{x}_n))) = \frac{d}{d\vartheta}\log(g(\vartheta, \underline{x}_n)) \lrcorner_{\text{nota}}$

$$\ell(\vartheta, \underline{x}_n) = \log\left(\frac{\vartheta^n e^{n\vartheta}}{\left(\prod_{i=1}^{n} x_i\right)^{\vartheta+1}} \prod_{i=1}^{n} \mathbb{1}_{(e,+\infty)}(x_i)\right) = \log(\vartheta^n) + n\vartheta - (\vartheta+1)\log\prod x_i + \log\prod \mathbb{1}(x_i)$$

$$= n\log(\vartheta) + n\vartheta - (\vartheta+1)\sum_{i=1}^{n}\log x_i + \log\prod_{i=1}^{n}\mathbb{1}_{(e,+\infty)}(x_i)$$

$$\frac{d}{d\vartheta}\ell(\vartheta, \underline{x}_n) = \frac{n}{\vartheta} + n - \sum_{i=1}^{n}\log x_i = 0$$

$$\Longrightarrow n + n\vartheta - \vartheta\sum_{i=1}^{n}\log x_i = 0 \Longrightarrow \vartheta\left(\sum_{i=1}^{n}\log x_i - n\right) = n$$

$$\Longrightarrow \hat{\vartheta} = \frac{n}{\sum_{i=1}^{n}\log x_i - n}$$ ↪ candidato

$$\frac{d^2}{d\vartheta^2}\ell(\vartheta, \underline{x}_n) = -\frac{n}{\vartheta^2} < 0 \quad \forall \vartheta \in \Theta \Longrightarrow \left.\frac{d^2}{d\vartheta^2}\ell(\vartheta, \underline{x}_n)\right|_{\vartheta=\hat{\vartheta}} < 0$$

$\Longrightarrow \hat{\vartheta} = \hat{\vartheta}_{MV}$ è stimatore di max verosimiglianza.

3) Dato $\underline{X}_n = (X_1 \dots X_n)$ con X_i iid $i=1 \dots n$ e $f_X(x,\vartheta) = \frac{2x}{\vartheta^2} e^{-\frac{x^2}{\vartheta^2}}$ con $x > 0$ e $\vartheta > 0$.

a) Modello statistico per $\underline{X}_n = (X_1 \dots X_n)$

$(\mathcal{X}^n = \mathbb{R}^+, f_n(\underline{x}_n, \vartheta), \Theta = \mathbb{R}^+)$

$$f_n(\underline{x}_n, \vartheta) = \prod_{i=1}^{n} f_X(x_i, \vartheta) = \prod_{i=1}^{n}\left(\frac{2x_i}{\vartheta^2} e^{-\frac{x_i^2}{\vartheta^2}}\right) = \left(\prod_{i=1}^{n} x_i\right)\left(\frac{2}{\vartheta^2}\right)^n e^{-\frac{\sum_{i=1}^{n} x_i^2}{\vartheta^2}}$$

b) Determina $L(\theta, \underline{x}_n)$, il suo nucleo e $\hat{\vartheta}_{MV}$ per un generico campione osservato $\underline{x}_n$.

$$L(\theta, \underline{x}_n^{oss}) = \prod_{i=1}^{n} f_n(x_i, \theta) = \left(\prod_{i=1}^{n} x_i\right)\left(\frac{2}{\theta^2}\right)^n e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta^2}} = \underbrace{\left(\prod_{i=1}^{n} 2x_i\right)}_{h(\underline{x}_n)} \underbrace{\theta^{-2n} e^{-\sum_{i=1}^{n} x_i^2/\theta^2}}_{\substack{g(\theta, \underline{x}_n) \\ \shortparallel \\ \text{nucleo.}}}$$

Per calcolare $\hat{\vartheta}_{MV}$ posso concentrarmi solo sul nucleo della fdv $\Longrightarrow$

$$\ell(\theta, \underline{x}_n) = \log L(\theta, \underline{x}_n) \propto \log\left(\theta^{-2n} e^{-\sum_{i=1}^{n} x_i^2/\theta^2}\right) =$$

$$= -2n \log\theta - \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

$$\frac{d}{d\theta} \ell(\theta, \underline{x}_n) = -\frac{2n}{\theta} + 2\frac{\sum_{i=1}^{n} x_i^2}{\theta^3} = 0 \implies \hat{\theta} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$$

$$\left.\frac{d^2}{d\theta^2} \ell(\theta, \underline{x}_n)\right|_{\theta=\hat{\theta}} = \left.\left(\frac{2n}{\theta^2} - \frac{6\sum_{i=1}^{n} x_i^2}{\theta^4}\right)\right|_{\theta=\hat{\theta}} = \left(\frac{2n^2}{\sum_{i=1}^{n} x_i^2} - \frac{6n^2}{\sum_{i=1}^{n} x_i^2}\right) < 0$$

quindi $\hat{\theta} = \hat{\vartheta}_{MV} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$

c) Calcolare l'informazione di Fisher $I_n^{oss}(\hat{\vartheta}_{MV})$

$$I_n^{oss}(\hat{\vartheta}_{MV}) = -\frac{d^2}{d\theta^2} \ell(\hat{\vartheta}_{MV}, \underline{x}_n) = \frac{4n^2}{\sum_{i=1}^{n} x_i^2}$$

d) Scrivere l'approssimazione normale della fdv.

$$\bar{L}(\theta, \underline{x}_n) = \exp\left\{-\frac{1}{2}(\theta - \hat{\vartheta}_{MV})^2 I_n(\hat{\vartheta}_{MV})\right\} = \exp\left\{-\frac{1}{2}\left(\theta - \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}\right)^2 \frac{4n^2}{\sum_{i=1}^{n} x_i^2}\right\}$$

4) Dato $\underline{X}_n = (X_1 \dots X_n)$ con X_i $i=1 \dots n$ iid

$$f_X(x,\vartheta) = 2\vartheta x e^{-\vartheta x^2} \mathbb{1}_{(0,+\infty)}(x) \quad \text{con } \vartheta > 0$$

a) Scrivere la fdv $L(\vartheta, \underline{x}_n)$

$$L(\vartheta, \underline{x}_n) = \prod_{i=1}^{n} \left(2\vartheta x_i e^{-\vartheta x_i^2} \mathbb{1}_{(0,+\infty)}(x_i) \right) = \left(\prod_{i=1}^{n} x_i \mathbb{1}_{(0,+\infty)}(x_i) \right) (2\vartheta)^n e^{-\vartheta \sum_{i=1}^{n} x_i^2}$$

b) Individua il nucleo di $L(\vartheta, \underline{x}_n)$ e una statistica sufficiente per il modello

$$L(\vartheta, \underline{x}_n) = \underbrace{\left(\prod_{i=1}^{n} 2x_i \mathbb{1}_{(0,+\infty)}(x_i) \right)}_{h(\underline{x}_n)} \underbrace{\vartheta^n e^{-\vartheta \sum_{i=1}^{n} x_i^2}}_{g(\vartheta, \underline{x}_n)}$$

Per criterio di fattorizzazione di Fisher $T(\underline{x}_n) = \sum_{i=1}^{n} x_i^2$ è statistica sufficiente

c) Calcola $\hat{\vartheta}_{MV}$

$$\ell(\vartheta, \underline{x}_n) \propto \log(g(\vartheta, \underline{x}_n)) = n \log \vartheta - \vartheta \sum_{i=1}^{n} x_i^2$$

$$\frac{d}{d\vartheta} \ell(\vartheta, \underline{x}_n) = \frac{n}{\vartheta} - \sum_{i=1}^{n} x_i^2 = 0 \implies \hat{\vartheta} = \frac{n}{\sum_{i=1}^{n} x_i^2}$$

$$\frac{d^2}{d\vartheta^2} \ell(\vartheta, \underline{x}_n) = -\frac{n}{\vartheta^2} < 0 \quad \forall \vartheta \in \Theta \implies \hat{\vartheta}_{MV} = \hat{\vartheta} = \frac{n}{\sum_{i=1}^{n} x_i^2}$$

5) $\underline{X_n} = (X_1 - - - X_n)$ con X_i iid $i = 1 \dots n$

$$f_X(x, \vartheta) = \frac{2x}{\vartheta^2} \mathbb{1}_{(0,\vartheta)}(x) \quad \text{con } \vartheta > 0$$

a) Calcolare $\mathbb{E}_\vartheta[\bar{X}_n]$ dove $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\mathbb{E}_\vartheta[\bar{X}_n] = \mathbb{E}_\vartheta\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] = \mathbb{E}[X_i]$$

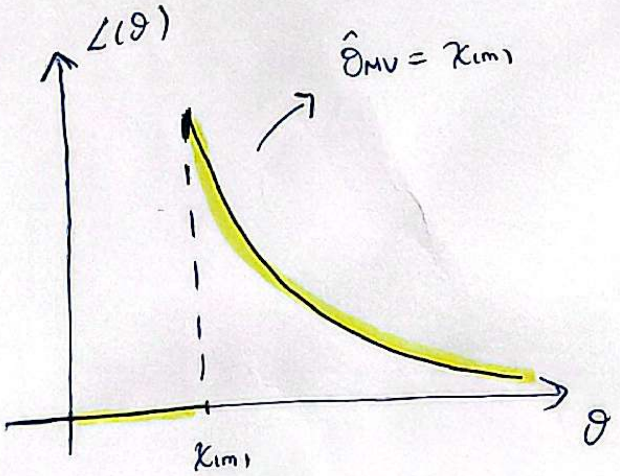
$$= \int_0^\vartheta \frac{2x^2}{\vartheta^2}\,dx = \frac{2}{\vartheta^2}\left[\frac{x^3}{3}\right]_0^\vartheta = \frac{2}{\vartheta^2}\frac{\vartheta^3}{3} = \frac{2}{3}\vartheta$$

b) Calcolare $L(\vartheta, \underline{x_n})$ e $\hat{\vartheta}_{MV}$

$$L(\vartheta, \underline{x_n}) = \prod_{i=1}^{n}\left(\frac{2x_i}{\vartheta^2}\mathbb{1}_{(0,\vartheta)}(x_i)\right) = \left(\prod_{i=1}^{n} 2x_i\right)\vartheta^{-2n}\prod_{i=1}^{n}\mathbb{1}_{(0,\vartheta)}(x_i) \propto \vartheta^{-2n}\mathbb{1}_{(x_{(n)}, +\infty)}(\vartheta)$$

$$\prod_{i=1}^{n}\mathbb{1}_{(0,\vartheta)}(x_i) = \begin{cases} 1 & \text{se } 0 < x_i < \vartheta \quad \forall i = 1 \dots n \\ 0 & \text{altrimenti} \end{cases} = \begin{cases} 1 & \text{se } 0 < x_{(1)} < \dots < x_{(n)} < \vartheta \\ 0 & \text{altrimenti} \end{cases}$$

$$= \mathbb{1}_{(x_{(n)}, +\infty)}(\vartheta)$$



6) Data $f_X(x,\vartheta) = a \dfrac{x^2}{\vartheta^3} \mathbb{1}_{[0,2\vartheta]}(x)$ con $\vartheta > 0$

i) Determina a in modo che f_X sia una densità

Deve valere che $\displaystyle\int_{\mathcal{X}} f_X(x,\vartheta)\,dx = 1 \implies \int_{\mathcal{X}} a \frac{x^2}{\vartheta^3} \mathbb{1}_{[0,2\vartheta]}(x)$

impongo l'uguaglianza

$$= \frac{a}{\vartheta^3}\int_0^{2\vartheta} x^2\,dx = \frac{a}{\vartheta^3}\left[\frac{x^3}{3}\right]_0^{2\vartheta} = \frac{a}{\vartheta^3}\left[\frac{8\vartheta^3}{3}\right] = \frac{8}{3}a = 1$$

$\implies a = 3/8$

ii) Calcola $\hat{\vartheta}_{MV}$ data $f_X(x,\vartheta) = \dfrac{3}{8}\dfrac{x^2}{\vartheta^3}\mathbb{1}_{[0,2\vartheta]}(x)$

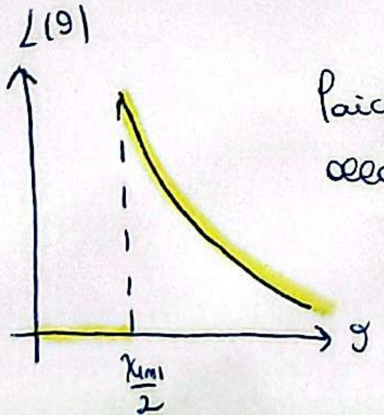
$$L(\vartheta, \underline{x}_n) = \prod_{i=1}^{n}\left(\frac{3}{8}\frac{x_i^2}{\vartheta^3}\mathbb{1}_{[0,2\vartheta]}(x_i)\right) = \left(\frac{3}{8}\right)^n \frac{\prod_{i=1}^{n} x_i^2}{\vartheta^{3n}} \prod_{i=1}^{n}\mathbb{1}_{[0,2\vartheta]}(x_i)$$

Si tratta di un problema di stima non regolare (supporto $[0,2\vartheta]$ dipende da ϑ)

$$\prod_{i=1}^{n}\mathbb{1}_{[0,2\vartheta]}(x_i) = \begin{cases} 1 & 0 \le x_i \le 2\vartheta \quad \forall i = 1..n \\ 0 & \text{altrimenti} \end{cases} = \begin{cases} 1 & \vartheta \ge x_i/2 \quad \forall i \\ 0 & \text{altrimenti} \end{cases}$$

$$= \begin{cases} 1 & \frac{x_{(n)}}{2} \le \vartheta \\ 0 & \text{altrimenti} \end{cases} = \mathbb{1}_{\left[\frac{x_{(n)}}{2}, +\infty\right)}(\vartheta)$$

$$L(\vartheta, \underline{x}_n) = \left(\frac{3}{8}\right)^2 \frac{\prod_{i=1}^{n} x_i^2}{\vartheta^{3n}} \mathbb{1}_{\left[\frac{x_{(n)}}{2}, \infty\right)}(\vartheta) \propto \vartheta^{-3n}\mathbb{1}_{\left[\frac{x_{(n)}}{2}, \infty\right)}(\vartheta)$$



Poiché la $L(\vartheta)$ è decrescente di ϑ nell'insieme $\left[\frac{x_{(n)}}{2}, \infty\right)$ allora $\hat{\vartheta}_{MV}(\underline{x}_n) = \dfrac{x_{(n)}}{2}$