Risk Analysis, Vol. 1, No. 1, 1981

On The Quantitative Definition of Risk

Stanley Kaplan[1] and B. John Garrick[2]

Received July 14, 1980

A quantitative definition of risk is suggested in terms of the idea of a "set of triplets." The definition is extended to include uncertainty and completeness, and the use of Bayes' theorem is described in this connection. The definition is used to discuss the notions of "relative risk," "relativity of risk," and "acceptability of risk."

KEY WORDS: risk; uncertainty; probability; Baye's theorem; decision.

1. INTRODUCTION

As readers of this journal are well aware, we are not able in life to avoid risk but only to choose between risks. Rational decision-making requires, therefore, a clear and quantitative way of expressing risk so that it can be properly weighed, along with all other costs and benefits, in the decision process.

The purpose of this paper is to provide some suggestions and contributions toward a uniform conceptual/linguistic framework for quantifying and making precise the notion of risk. The concepts and definitions we shall present in this connection have shown themselves to be sturdy and serviceable in practical application to a wide variety of risk situations. They have demonstrated in the courtroom and elsewhere the ability to improve communication and greatly diminish the confusion and controversy that often swirls around public decision making involving risk. We hope therefore with this paper to widen the understanding and adoption of this framework, and to that end adopt a leisurely and tutorial place.

We begin in the next section with a short discussion of several qualitative aspects of the notion of risk. We then proceed to a first-pass or first-level quantitative definition. Since the notion of "probability" is fundamentally intertwined with the definition of risk, the next section addresses the precise meaning adopted in this paper for the term "probability." In particular, at this point, we carefully draw a distinction between "probability" and "frequency." Then, using this distinction, we return to the idea of risk, and give a "second-level" definition (of risk which generalizes the first-level definition) and is large enough and flexible enough to include at least all the aspects and subtleties of risk that have been encountered in the authors' experience.

2. QUALITATIVE ASPECTS OF THE NOTION OF RISK

The subject of risk has become very popular in the last few years and is much talked about at all levels of industry and government. Correspondingly, the literature on the subject has grown very large [see for example refs. (1–3)]. In this literature the word "risk" is used in many different senses. Many different kinds of risk are discussed: business risk, social risk, economic risk, safety risk, investment risk, military risk, political risk, etc. Now one of the requirements for an intelligible subject is a uniform and consistent usage of words. So we should like to begin sorting things out by drawing some distinctions in

[1]Kaplan & Associates, Inc., 17840 Skypark Blvd. Irvine, CA 92714.
[2]Pickard, Lowe and Garrick, Inc.

0272-4332/81/0300-0011$03.00/1

meaning between various of these words as we shall use them. We begin with "risk" and "uncertainty."

2.1. The Distinction Between Risk and Uncertainty

Suppose a rich relative had just died and named you as sole heir. The auditors are totaling up his assets. Until that is done you are not sure how much you will get after estate taxes. It may be $1 million or $2 million. You would then certainly say you were in a state of uncertainty, but you would hardly say that you were facing risk. The notion of risk, therefore, involves both uncertainty and some kind of loss or damage that might be received. Symbolically, we could write this as:

$$\text{risk} = \text{uncertainty} + \text{damage}.$$

This equation expresses our first distinction. As a second, it is of great value to differentiate between the notions of "risk" and "hazard." This is the subject of the next section.

2.2. The Distinction Between Risk and Hazard

It is very useful, especially in understanding the public controversies surrounding energy production and transport facilities, to draw a distinction between the ideas of risk and hazard.

In the dictionary[4] we find hazard defined as "a source of danger." Risk is the "possibility of loss or injury" and the "degree of probability of such loss." Hazard, therefore, simply exists as a source. Risk includes the likelihood of conversion of that source into actual delivery of loss, injury, or some form of damage. This is the sense in which we use the words. As an example, the ocean can be said to be a hazard. If we attempt to cross it in a rowboat we undergo great risk. If we use the *Queen Elizabeth*, the risk is small. The *Queen Elizabeth* thus is a device that we use to safeguard us against the hazard, resulting in small risk. As in Sec. 2.1., we express this idea symbolically in the form of an equation:

$$\text{risk} = \frac{\text{hazard}}{\text{safeguards}}.$$

This equation also brings out the thought that we may make risk as small as we like by increasing the safeguards but may never, as a matter of principle, bring it to zero. Risk is never zero, but it can be small.

Included under the heading "safeguards" is the idea of simple awareness. That is, awareness of risk reduces risk. Thus, if we know there is a hole in the road around the corner, it poses less risk to us than if we zip around not knowing about it.

2.3. Relativity of Risk

Connected to this thought is the idea that risk is relative to the observer. We had a case in Los Angeles recently that illustrates this idea. Some people put a rattlesnake in a man's mailbox. Now if you had asked that man: "Is it a risk to put your hand in your mailbox?" He would have said, "Of course not." We however, knowing about the snake, would say it is very risky indeed.

Thus risk is relative to the observer. It is a subjective thing—it depends upon who is looking. Some writers refer to this fact by using the phrase "perceived risk." The problem with the phrase is that it suggests the existence of some other kind of risk—other than perceived. It suggests the existence of an "absolute risk." However, under attempts to pin it down, the notion of absolute risk always ends up being somebody else's perceived risk. This brings us in touch with some fairly deep philosophical matters, which incidentally are reminiscent of those raised in Einstein's theory of the relativity of space and time.

This subject will become clear after we have given precise, quantitative definitions of "risk" and "probability." We begin this process in the next section by giving the definition of risk. We postpone the definitions of probability until Sec. 4. This order of presentation departs a little from the logical order because the definition of risk uses the term probability. This works out all right, however, since the reader already has a good intuitive grasp of the meaning of probability. The earlier discussions of risk will then serve to motivate the detailed attention given to the subtleties of the definition of probability.

So, qualitatively, risk depends on what you do and what you know and what you do not know. Let us proceed now to put the idea on a quantitative basis.

3. QUANTITATIVE DEFINITION OF RISK (FIRST LEVEL)

3.1. "Set of Triplets Idea"

In analyzing risk we are attempting to envision how the future will turn out if we undertake a certain course of action (or inaction). Fundamentally, there-

Table I. Scenario List

Scenario	Likelihood	Consequence
S_1	p_1	x_1
S_2	p_2	x_2
$\vdots$	$\vdots$	$\vdots$
S_N	p_N	x_N

fore, a risk analysis consists of an answer to the following three questions:

(i) What can happen? (i.e., What can go wrong?)
(ii) How likely is it that that will happen?
(iii) If it does happen, what are the consequences?

To answer these questions we would make a list of outcomes or "scenarios" as suggested in Table I. The ith line in Table I can be thought of as a triplet:

$$\langle s_i, p_i, x_i \rangle$$

where s_i is a scenario identification or description;
p_i is the probability of that scenario; and
x_i is the consequence or evaluation measure of that scenario, i.e., the measure of damage.

If this table contains all the scenarios we can think of, we can then say that it (the table) is the answer to the question and therefore is the risk. More formally, using braces, { }, to denote "set of" we can say that the risk, R, "is" the set of triplets:

$$R = \{\langle s_i, p_i, x_i \rangle\}, \qquad i = 1, 2, \ldots, N.$$

This definition of risk as a set of triplets is our first-level definition. We shall refine and enlarge it later.[3] For now let us show how to give a pictorial representation of risk.

3.2. Risk Curves

Imagine now, in Table I, that the scenarios have been arranged in order of increasing severity of damage. That is to say, the damages x_i obey the ordering relationship:

$$x_i \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_N.$$

By adding a fourth column in which we write the cumulative probability, adding from the bottom, we have Table II.

[3]Having defined risk as a set of triplets, we may now, in line with section 2.2, define hazard as a set of doublets thus: $H = \{\langle s_i, x_i \rangle\}$.

Table II. Scenario List with Cumulative Probability

Scenario	Likelihood	Consequences	Cumulative probability
s_1	p_1	x_1	$P_1 = P_2 + p_1$
s_2	p_2	x_2	$P_2 = P_3 + p_2$
$\vdots$	$\vdots$	$\vdots$	
s_i	p_i	x_i	$P_i = P_{i+1} + p_i$
$\vdots$	$\vdots$	$\vdots$	
s_{N-1}	p_{N-1}	x_{N-1}	$P_{N-1} = P_N + p_{N-1}$
s_N	p_N	x_N	$P_N = p_N$

If we now plot the points $\langle x_i, P_i \rangle$ we obtain the staircase function shown as a dashed line, in Fig. 1.

Let us next note that what we called "scenarios" in Table I are really categories of scenarios. Thus for example, the scenario "pipe break" actually includes a whole category of different kinds and sizes of breaks that might be envisioned, each resulting in a slightly different damage, x.[4] Thus we can argue ourselves into the view that the staircase function should be regarded as a discrete approximation to a continuous reality. Thus if we draw in a smoothed curve, $R(x)$, through the staircase, we can regard that curve as representing the actual risk. Hence we call it the "risk curve."

Probably the most well-known examples of such curves were published in the *Reactor Safety Study, Wash 1400* [ref. (5)]. Figure 2 is an example taken from that study. Note in this example that the curves are plotted on log-log scale which results in the characteristic concave downward shape. In this case the asymptotes, as shown in Fig. 3, have the interpretation of "maximum possible damage" and "probability of any damage at all."

3.3. Comments on the Definition

One often hears it said that "risk is probability *times* consequence." We find this definition misleading and prefer instead, in keeping with the set of triplets idea, to say that "risk is probability *and* consequence." In the case of a single scenario the probability times consequence viewpoint would equate a low-probability high-damage scenario with a high-probability low-damage scenario—clearly not the same thing at all.

In the case of multiple scenarios the probability times consequence view would correspond to saying

[4]The categories of scenarios, incidentally, should of course be chosen so that they are mutually exclusive and the same event does not show up in more than one category.

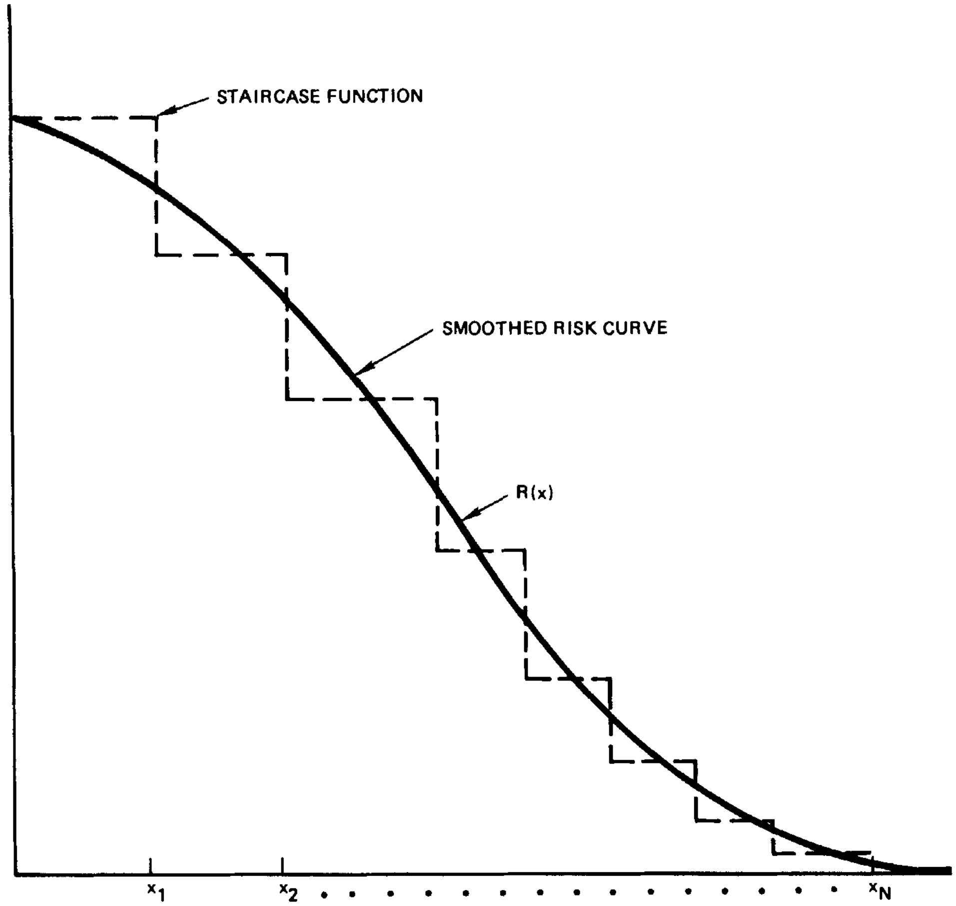

Fig. 1. Risk curve.

that the risk is the expected value of damage, i.e., the mean of the risk curve. We say it is not the mean of the curve, but the curve itself which is the risk. A single number is not a big enough concept to communicate the idea of risk. It takes a whole curve.

Now the truth is that a curve is not a big enough concept either. It takes a whole family of curves to fully communicate the idea of risk. This is the basis for the level 2 definition to which we shall come shortly. First we pick up some further points in connection with level 1.

3.4. Multidimensional Damage

In many applications, it is appropriate to identify different types of damage, e.g., loss of life and loss of property. In these cases, the damage, x, can be regarded as a multidimensional or vector quantity rather than a single scalar. The risk curve now becomes a risk surface over the multidimensional space as suggested in Fig. 4.

In this case the ordinate, $R(x, y)$, over the point x, y is the probability that damage type 1 will be greater than x and damage type 2 will be greater than y.

An example of a risk surface, presented in tabular form, is shown in Table III taken from a hearing on railroad transport of spent nuclear fuel,[6] and modeled after a similar table given in ref. (7). This table at any box lists the probability that N people or more will receive a dose of D mr or more as a result of a shipment of spent fuel.

3.5. Completion of the Scenario List

One of the criticisms that has been made of the *Reactor Safety Study* may be paraphrased essentially as follows:

> A risk analysis is essentially a listing of scenarios. In reality, the list is infinite. Your analysis, and any analysis, is perforce finite, hence incomplete. Therefore no matter how thoroughly and carefully you have done your work, I am

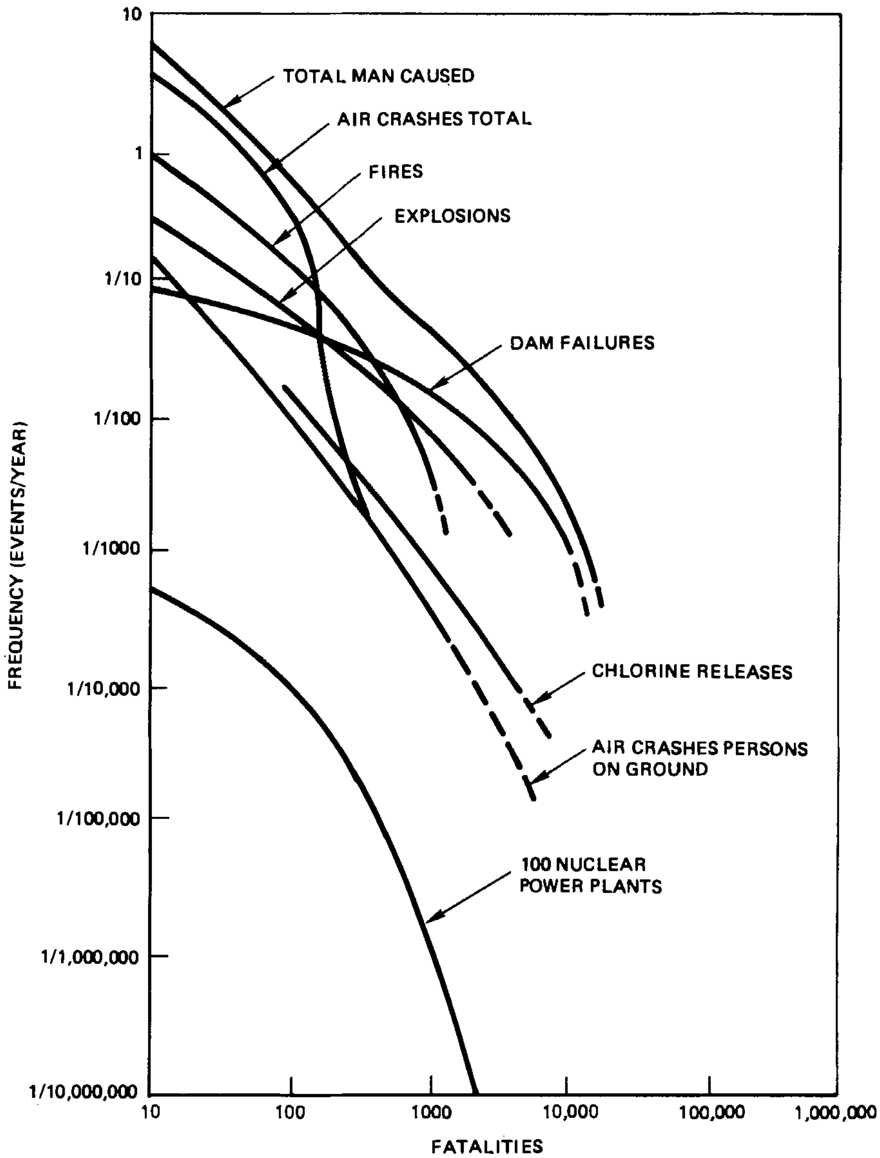

Fig. 2. Frequency of fatalities due to man-caused events.

not going to trust your results. I'm not worried about the scenarios you have identified, but about those you haven't thought of. Thus I am never going to be satisfied.

The critic here has a valid point about risk analysis. The implied conclusion, that we should not build nuclear reactors, is not valid. For whatever course of action, or nonaction, is proposed in place of building reactors must also be subject to a risk analysis. That risk analysis will also have the same inherent limitation as the *Reactor Safety Study*. That limitation in itself, therefore, cannot be used to argue for one branch of the decision tree over another since it applies to all branches.

Nevertheless, the critic has made a good point about the risk analysis formalism. Let us see therefore what can be done to improve the formalism to address this point.

One tactic that comes to mind, in light of the fact that the s_i are categories of scenarios, is to include another category, s_{N+1}, to the list. We may call this category the "other" category. By definition, it contains all scenarios not otherwise included in the list. Correspondingly we would now say that a risk analysis is a set of triplets:

$$R = \{\langle s_i, p_i, x_i \rangle\}, x_i = 1, 2, \ldots, N+1$$

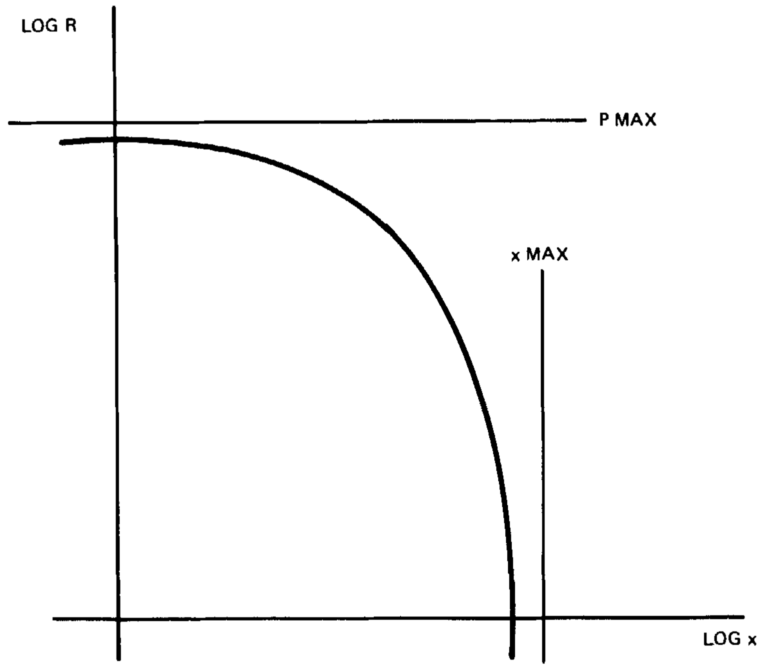

Fig. 3. Risk curve on a log-log scale.

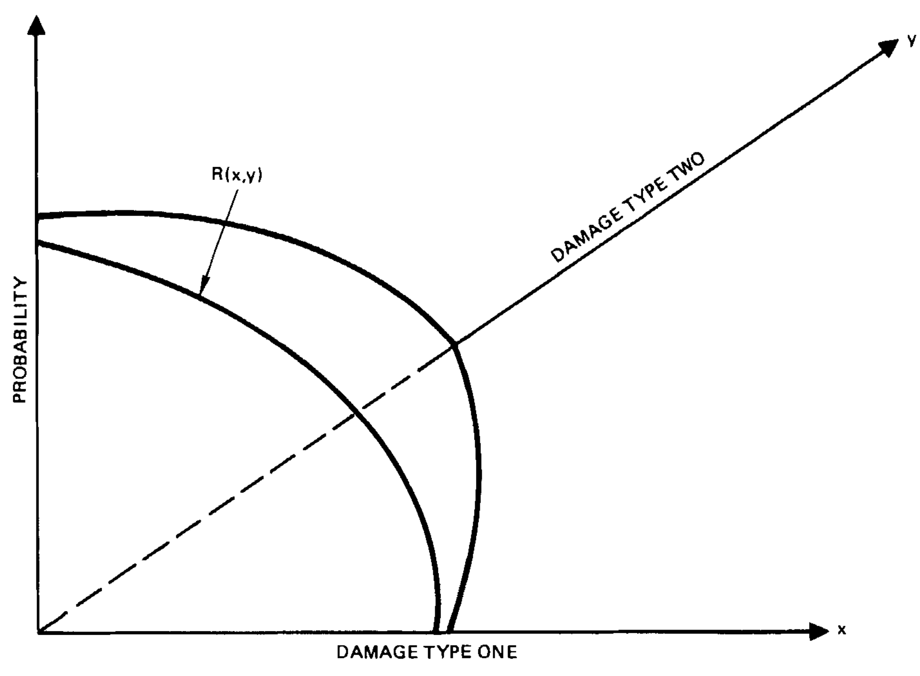

Fig. 4.

Table III. Probability of Human Exposure to Radiation[a, b]

Number of people, N/D	Dose, mr 1	10	10^2	10^3	10^4	10^5	10^6
1	1.17×10^{-5}	1.17×10^{-5}	1.17×10^{-5}	1.16×10^{-5}	9.00×10^{-6}	4.24×10^{-6}	1.69×10^{-6}
10	1.17×10^{-5}	1.17×10^{-5}	1.15×10^{-5}	8.90×10^{-6}	4.54×10^{-6}	1.79×10^{-6}	4.15×10^{-7}
10^2	1.17×10^{-5}	1.17×10^{-5}	8.65×10^{-6}	5.05×10^{-6}	1.03×10^{-6}	5.45×10^{-7}	7.74×10^{-8}
10^3	1.17×10^{-5}	1.14×10^{-5}	6.05×10^{-6}	2.63×10^{-6}	6.95×10^{-7}	1.50×10^{-7}	4.27×10^{-9}
10^4	1.03×10^{-5}	7.40×10^{-6}	3.60×10^{-6}	1.24×10^{-6}	2.96×10^{-7}	1.64×10^{-8}	2.26×10^{-11}
10^5	8.45×10^{-6}	5.95×10^{-6}	2.31×10^{-6}	6.70×10^{-7}	1.14×10^{-7}	2.85×10^{-10}	

[a] In a 1,000-mile-long train shipment of spent nuclear fuel, probability that N or more persons will receive D or more dose to the whole body from gross fission products which are released in an accident during this shipment and which deposit (i.e., Fallout) on the ground.
[b] $P(\geqslant N, \geqslant D)$.

which includes all the scenarios we have thought of, and also an allowance for those we have not thought of.

Thus extended, the set of scenarios may be said to be logically complete.

It seems at first glance that what we have done here is simply a logical trick which does not address the fundamental objection. It is a little bit more than a trick, however. For one thing, it takes the argument out of the verbal realm and into the quantitative realm. Instead of the emotional question, "What about the things that you have not thought of?" "What probability should we assign to the residual category s_{N+1}?"

Once the question has been phrased in this way, we can proceed like rational people, in the same way we do to assign any probability. We ask what evidence do we have on this point? What knowledge, what relevant experience? In particular, we note that one piece of evidence is always present—namely that scenarios of the type s_{N+1} have not occurred yet, otherwise we would have included them elsewhere on the list.

How much is this piece of evidence worth? This is a question that can be answered rationally within the framework of the theory of probability using Bayes' theorem. We shall return to this point in Sec. 6. It is timely now to explain the sense in which we are using the word probability.

4. PROBABILITY

People have been arguing about the meaning of probability for at least 200 years, since the time of Laplace and Bayes. The major polarization of the argument is between the "objectivist" or "frequentist" school who view probability as something external, the result of repetitive experiments, and the "subjectivists" who view probability as an expression of an internal state—a state of knowledge or state of confidence.

In this paper we adopt the point of view that both schools are right; they are just talking about two different ideas. Unfortunately, they both use the same word—which seems to be the source of most of the confusion. We shall, therefore, assign each idea the dignity of its own name.

4.1. The Definition of Probability and Distinction Between Probability and Frequency

What the objectivists are talking about we shall call "frequency." What the subjectivists are talking about we shall call "probability." Thus, "probability" as we shall use it is a numerical measure of a state of knowledge, a degree of belief, a state of confidence. "Frequency" on the other hand refers to the outcome of an experiment of some kind involving repeated trials. Thus frequency is a "hard" measurable number. This is so even if the experiment is only a thought experiment or an experiment to be done in the future. At least in concept then, a frequency is a well-defined, objective, measurable number.

Probability, on the other hand, at first glance is a notion of a different kind. Defined, essentially, as a number used to communicate a state of mind, it thus seems "soft" and changeable, subjective—not measureable, at least not in the usual way.

The cornerstone of our approach is the idea that given two meaningful statements (or propositions or events), it makes sense to say that one is more (less, equally) likely than the other. That is, we accept as an axiom the comparability of uncertainty. Since two uncertain statements can be compared, the next logical step is to devise a scale to calibrate uncertainty.

This can be done in several ways. The most direct, however, is to use frequency in the following way.[5] Suppose we have a lottery basket containing coupons numbered from 1 to 1000. Suppose the basket is to be thoroughly mixed, and that you are about to draw a coupon blindfolded. We ask: Will you draw a coupon numbered 632 or less? With respect to this question you experience a certain state of confidence. Similarly, I experience a state of confidence with respect to this same question. Let us agree to call this state of confidence, "probability 0.632." Now we both know exactly what we mean by $p = 0.632$. So if you now say that the probability of your horse winning tomorrow is 0.632, I know exactly what your experiential state of confidence is. We have communicated!

In the same way, we may define or "calibrate" the entire probability scale using frequency as a standard of reference. Note that the process is entirely parallel to the way by which we define "red," "chair," "seventeen," and all words or other symbols.

This method of definition shows the intimate connection between probability and frequency. This connection needs to be recognized always and at the same time not allowed to obscure the fundamental difference. Frequency is used to calibrate the probability scale in a "bureau of standards" sense. Once the calibration is established, we then use probability to discuss our state of confidence in areas where we are dealing with one time events and have no frequency information at all.

In this way we liberate ourselves from the restrictions of the relative frequency school of thought (e.g., that only mass repetitive phenomena can be analyzed probabilistically) and create for ourselves a systematic, disciplined theory and language for dealing with rare events, for quantifying risks, and making decisions in the face of the uncertainties attendant to these events.

This then is the definition adopted in this paper. For additional insight we quote the following paragraph, from unpublished notes by E. T. Jaynes:

> Probability theory is an extension of logic, which describes the inductive reasoning of an idealized being who represents degrees of plausibility by real numbers. The numerical value of any probability (A/B) will in general depend not only on A and B, but also on the entire background of other propositions that this being is taking into account. A probability assignment is 'subjective' in the sense that it describes a state of knowledge rather than any property of the 'real' world; but it is completely 'objective' in the sense that it is independent of the personality of the user; two beings faced with the same total background of knowledge must assign the same probabilities.

and, as further elaboration cite the following paragraph by A. DeMorgan[6]:

> We have lower grades of knowledge, which we usually call degrees of belief, but they are really degrees of knowledge.... It may seem a strange thing to treat knowledge as a magnitude, in the same manner as length, or weight, or surface. This is what all writers do who treat of probability, and what all their readers have done, long before they ever saw a book on the subject.... By degree of probability we really mean, or ought to mean, degree of belief.... Probability then, refers to and implies belief, more or less, and belief is but another name for imperfect knowledge, or it may be, expresses the mind in a state of imperfect knowledge.

4.2. Distinction between Probability and Statistics

Corresponding to the above definitions of frequency and probability, as numbers, we may say that statistics, as a subject, is the study of frequency type information. That is, it is the science of handling data. On the other hand probability, as a subject, we might say is the science of handling the lack of data.

Thus, one often hears people say that we cannot use probability because we have insufficient data. In light of our current definitions, we see that this is a misunderstanding. When one has insufficient data, there is nothing else one can do *but* use probability.

4.3. "Probability of Frequency" Framework

Now there are two ways we could talk about the flipping of coins, corresponding to two different questions. We could first ask: What is the probability of a head on the next toss? Alternately we could say: I am going to toss the coin 10,000 times. What is the frequency, i.e., the percentage of heads going to be?

In the first method we answer simply with a number, our state of confidence on the prospect of a head on the next toss, as reflected for example in the odds we would take in a bet.

In the second method we are asked to predict the outcome, ϕ, of an experiment to be done in the

[5] Modeled after ref. (10), p. 161.

[6] Further discussion of the foundations of the subjectivistic theory can be found in refs. (8–11).

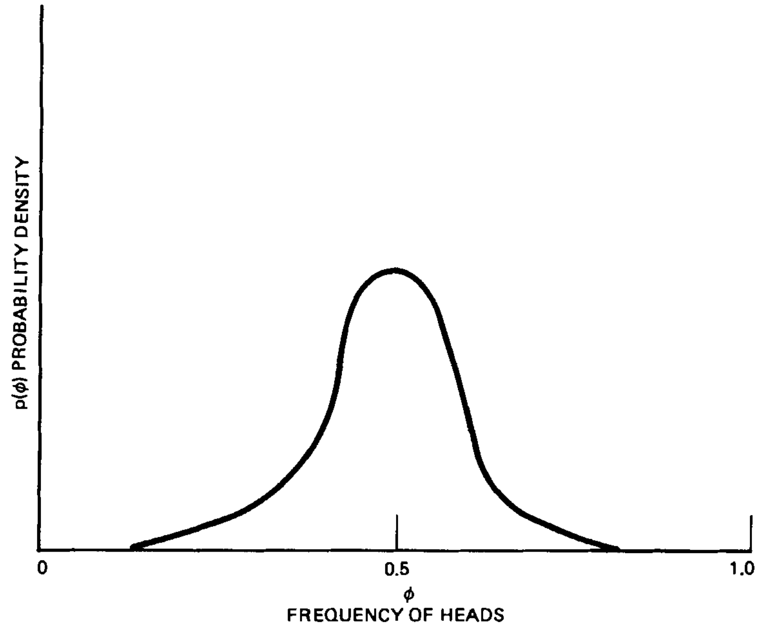

Fig. 5. Probability curve.

future. Since we do not know this outcome we express our prediction in the form of a probability curve against it (Fig. 5). Thus in the second method we are led to the notion of a probability curve against frequency as a way of, or a framework for, expressing our state of knowledge.

This notion of probability of frequency will be of use to us in the next section in expanding the definition of risk. Before proceeding to this, we note, coming back to the coins, that the answer to the first question can be derived from the answer to the second. Thus, having given the probability of frequency curve $p(\phi)$ we would then, consistent with that, express our probability of heads on the next try as:

$$p(\text{heads}) = \int_0^{10} \phi p(\phi) d\phi$$

Thus the second method includes or encompasses the first. The reverse cannot be said, and thus the second method intuitively comes across as a fuller, more complete discussion of the situation.

5. LEVEL 2 DEFINITION OF RISK

When one presents a risk curve as the result of an analysis, one of the things that invariably happens is that someone asks: "How confident are you in the curve?"[7] In view of our usage of the term probability, the risk curve already expresses our state of confidence. It appears thus as if the question is asking: "How confident are you in your state of confidence?" In this form the question seems undefined and unanswerable. However, there is a valid thought behind it. What we need to do, therefore, is to expand our framework somehow, in such a way that within the enlarged framework the question can be given a precise meaning and then be answered.

5.1. Risk Curves in Frequency Format

For this purpose we make use of the probability of frequency idea in the following way. We imagine a thought experiment in which we undertake the proposed course of action, or inaction, many, many times. At the end of this experiment we will be able to look back at the records and ask: "How frequently did scenario s_i occur?" This frequency will then be an experimentally measured number. Let us denote it by ϕ_i. Its units are occurrences per trial.

At the end of the experiment, therefore, we will have the set of numbers, ϕ_i, and the set of triplets:

$$\{\langle s_i, \phi_i, x_i \rangle\} \quad i = 1, \ldots, N+1.$$

[7]For example, the major criticism by the Lewis Committee[12] of the *Reactor Safety Study*[5] had to do with the uncertainty of the risk curves.

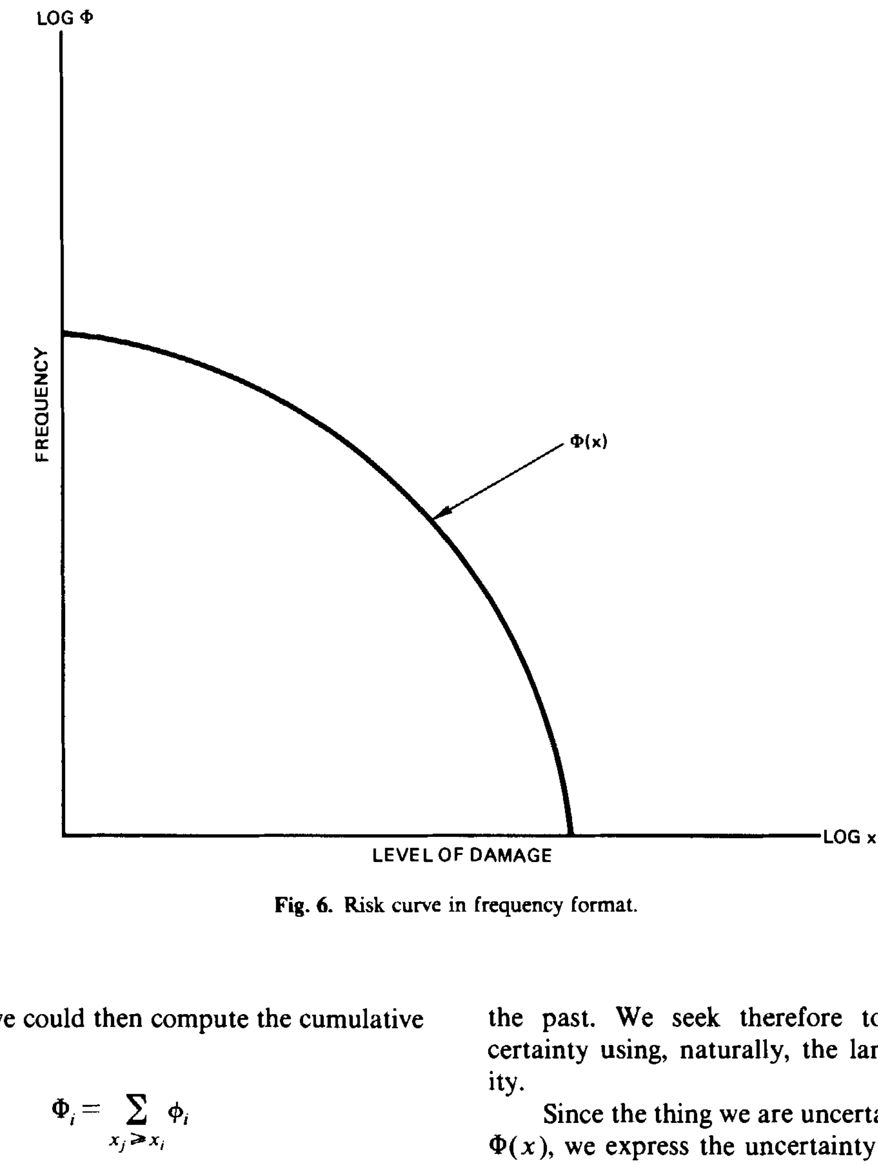

Fig. 6. Risk curve in frequency format.

As in Sec. 3.2. we could then compute the cumulative frequency:

$$\Phi_i = \sum_{x_j \geq x_i} \phi_i$$

(where the sum is over all scenarios having damage equal to or greater than x_i). Also as in Sec. 3.2, we could now plot Φ vs. x, obtaining Fig. 6 which we refer to as a risk curve in frequency format. This whole curve may be regarded as the outcome of our thought experiment.

5.2. Inclusion of Uncertainty

Now since we have not yet actually done the thought experiment of the previous section, we have uncertainty about what its outcome would be. The degree of uncertainty depends upon our total state of knowledge as of right now; upon all the evidence, data, and experience with similar courses of action in the past. We seek therefore to express this uncertainty using, naturally, the language of probability.

Since the thing we are uncertain about is a curve, $\Phi(x)$, we express the uncertainty by embedding this curve in a space of curves and erecting a probability distribution over this space.

Pictorially, this is represented by a diagram of the form of Fig. 7. This figure is what we call a risk curve in probability of frequency format. It consists of a family of curves, $\Phi_P(x)$, with the parameter being the cumulative probability. To use this diagram we would, for example, enter with a specific x value and choose say the curve $P=0.90$. The ordinate of this curve $\Phi_{0.90}(x)$ is then the 90th percentile frequency of x. That is to say, we are 90% confident that the frequency with which damage level x or greater occurs, is not larger than $\Phi_{0.90}(x)$.

Figure 7 is the pictorial form of our level 2 definition of risk. It is of interest to express this definition also in terms of the set of triplets idea.

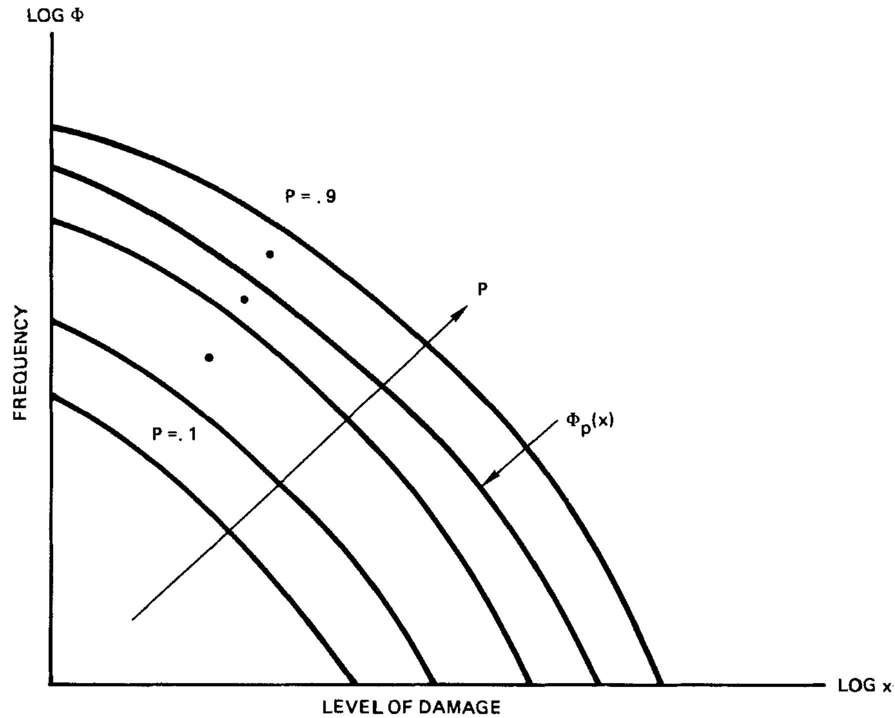

Fig. 7. Risk curve in probability of frequency format.

5.3. Set of Triplets Including Uncertainty

In listing our set of triplets for a proposed course of action, suppose that we now acknowledge that, to tell the truth about it, we do not know the frequency with which scenario category s_i occurs. We would then express our state of knowledge about this frequency with a probability curve $p_i(\phi_i)$ is the probability density function for the frequency ϕ_i, of the ith scenario. Thus we now have a set of triplets in the form:

$$R = \{\langle s_i, p_i(\phi_i), x_i \rangle\} \tag{1}$$

which set of triplets, we could say, is the risk including uncertainty in frequency.

From set (1) we can construct the risk family Fig. 7 by cumulating frequencies from the bottom in a manner entirely parallel to that used in Sec. 3.2.

Similarly, if there is uncertainty in the damage also, we would have the set of triplets:

$$R = \{\langle s_i, p_i(\phi_i), \zeta_i(x_i) \rangle\}, \tag{2}$$

or more generally,

$$R = \{\langle s_i, p_i(\phi_i, x_i) \rangle\} \tag{3}$$

using a joint distribution on ϕ_i, x_i. In this case also we can construct the risk family of curves, Fig. 7. The method for doing this is for our present purposes a mechanical detail and is outlined in the Appendix.

5.4. Comments on the Level 2 Definition

Figure 7, or equivalently equations (1), (2), or (3), constitutes our expanded level 2 definition of risk. We observe that it includes the level 1 definition in the sense that the expected frequency, $\overline{\Phi}(x)$, at any x is the probability $P(x)$ at that X. Thus we have lost nothing in going from level 1 to level 2 and have gained the ability to explicitly include uncertainty in the calculation of risk. This is particularly important in risk analyses, such as in ref. (5), where scenarios are identified using fault trees and event trees, and where the fundamental input data on failure rates of components is uncertain.

The explicit inclusion of uncertainty also allows us to avoid the awkward notion of "relative risk"[12, 13] which was introduced to compare the risk of different designs or systems when there was little confidence in the calculations of the individual risks themselves. In the level 2 point of view, the uncertainty is an intrinsic part of the risk, as it should be, and the comparison of two systems is readily done by viewing Fig. 7 for the two systems side by side.

The use of the term "relative" in the preceding paragraph refers to comparing two things being looked at; i.e., alternate plant designs or courses of action, etc. In Sec. 2.3, relativity of risk was used in another sense, in comparing who it is that is doing the looking. In the latter sense, risk is always relative to the observer. It is subjective just as is probability itself. It depends on what the observer knows.

On the other hand, however, as Jaynes points out in the last sentence of his definition (Sec. 4.1) any two rational observers given the same totality of information must calculate the same Fig. 7, and thus agree on the quantification of risk.

In this sense, we may say that the level 2 definition of risk is "absolute" and "objective." It depends upon the evidence at hand, but other than that is independent of the personality of the user. Two rational beings given the identical evidence must assess the risk identically.

6. ASSESSMENT OF THE FREQUENCIES OF SCENARIOS, INCLUDING THE "OTHER" SCENARIO—THE USE OF BAYES' THEOREM

We have now said, essentially, that risk is a listing of scenarios, and that any two rational observers, given the same total background of information and evidence, must assign the same frequency to those scenarios. More precisely, they will assign the same probability of frequency curves, $p_i(\phi_i)$, to those scenarios. In the present section we shall say a few words on the use of Bayes' theorem in this connection.

6.1. Example

Suppose for example that the scenario under consideration is the occurrence of a certain event, a turbine trip, at a specific power plant, plant m. We wish to know the frequency, ϕ_m, of this event.[8]

The information we have relating to this point may be regarded as falling into three categories:

(1) Our general background knowledge of the design and manufacture of the turbine, operating procedures of the plant, and so on.

(2) The experience we have had with our specific plant so far.

(3) Our experience with similar turbines in similar plants.

For example, at our specific plant, we may have had k_m occurrences in T_m operating years. Similarly, the type[(3)] data would consist of a set of doublets:

$$\begin{gathered} \langle k_1, T_1 \rangle \\ \langle k_2, T_2 \rangle \\ \vdots \\ \langle k_j, T_j \rangle \end{gathered}$$

giving the experience of all plants which are deemed to be "similar" to plant m.

The question now is how to combine these three types of information into a probability curve, $p(\phi_m/E)$ expressing our state of knowledge about ϕ_m. The fundamental conceptual tool admirably suited to this purpose is Bayes' theorem, which we write as follows:

$$p(\phi_m/E) = p(\phi_m)\left[\frac{p(E/\phi_m)}{p(E)}\right]$$

where $p(\phi_m/E)$, the "posterior," is the probability we assign to ϕ_m after having evidence E; $p(\phi_m)$, the "prior," is the probability we would assign to ϕ_m before learning the evidence E; $p(E/\phi_m)$, the "likelihood," is the conditional probability that evidence E would be observed if the true frequency were actually ϕ_m; and $p(E)$, is the prior probability of the evidence E.

Now to use Bayes' theorem, we would express, or encode, the information of types (1) and (3) in the prior, $p(\phi_m)$. This could now be called the "generic" prior. The plant specific experience, (2), would constitute the evidence and enters the calculation through the likelihood function:

$$p(E/\phi_m) = \left(\frac{(\phi_m T_m)^{k_m}}{k_m!}\right) e^{-\phi_m T_m}.$$

The denominator $p(E)$ is then the sum, or integral, of the numerator

$$p(E) = \int_0^\infty p(\phi_m) p(E/\phi_m) d\phi_m$$

[8]Harking back to Sec. 5.1, we mean by ϕ_m here, the average occurrence rate, occurrences per year, in a thought experiment in which we operate plant m millions, and millions of years.

and ensures that the normalization of $p(\phi_m/E)$ is correct.

Further details of this application of Bayes' theorem may be found in numerous published examples[14–17].

6.2. "Two-Stage" Use of Bayes' Theorem

A more recent development consists of a "two-stage" use of Bayes' theorem in which information type (1) alone is used as prior and type (3) as evidence to generate a population variability curve. This curve expresses the fact that different plants in the population have different frequencies of occurrence of this event. The population variability curve is then used as the prior in a second application of Bayes' theorem with information (2) as evidence. Details of this two stage process will be presented in a subsequent paper.

6.3. Application to the "Other" Scenarios

The same reasoning as above may be applied to s_{N+1}, the category of scenarios we have not yet thought of (see Sec. 3.5). In this case $k_m = 0$. In fact $k_j = 0$ for all plants, j, since if such a scenario had occurred, we would have added it to the list explicitily. Thus the evidence, E, for scenario s_{N+1} is that it has not yet occurred in all the years of experience so far. But this is a perfectly good piece of evidence like any other and Bayes' theorem applies exactly as before.

Thus, the category of scenarios not yet thought of, may be handled by the same process as any other scenario category: The relevant evidence is assembled and quantitatively assessed using Bayes' theorem. Thus, this aspect of the risk controversy may be brought closer to a rational, more unemotional treatment.

7. ON "ACCEPTABLE" RISK

We now explore whether our definitions offer any insights into the perennial question of "What is the Acceptable risk?"[1,3,18,19] At the outset we remark that when a question has been debated as long and as unsuccessfully as this one, perhaps that is a clue that we are asking the wrong question, or asking the question in the wrong way.

There are two difficulties with the notion of acceptable risk: one major and one minor. The minor difficulty is that it implies that risk is linearly comparable. It implies that one can say that the risk of course of action A or design A is greater or less than that of design B. The difficulty with this is evident at the level 1 definition. For example, in Fig. 8, the risks are clearly different, yet we cannot readily say that one is bigger than the other. They are not linearly comparable. The situation is even more difficult in the corresponding level 2 definition, Fig. 9, for example.

Of course it is possible to reduce these risk curves, or families of curves, to single numbers, for example in level 1 by introducing a utility function, $U(x)$, against x and performing an expected value operation:

$$\bar{U} = -\int_0^\infty U(x)\frac{dP}{dx}(x)\,dx.$$

In level 2 we could proceed similarly by regarding the set of curves in Fig. 7 as a discrete probability distribution[20] over the function space of curves $\Phi(x)$

$$\{\langle p_i, \Phi_i(x)\rangle\}.$$

For each such discrete curve, Φ_i, then we calculate an expected utility as:

$$\bar{U}_i = \int_0^\infty U(x)\frac{d\Phi_i(x)}{dx}\,dx$$

and then set

$$\bar{U} = \sum_i p_i \bar{U}_i.$$

These figures of demerit are scalars and thus linearly comparable. The risks could then be said to have been made linearly comparable, but only at the cost of a great loss of information in the expectation operation. One wonders whether it would not be better, rather than defining an explicit utility function, to simply look at Fig. 9 and say: "Design B will cost, say, Δ dollars more than A. Is it worth it to you?"

Assuming we prefer risk B to risk A, to decide whether it is worth Δ dollars more we would have to ask ourselves what else we could do with these Δ dollars. Perhaps we could obtain a much larger risk

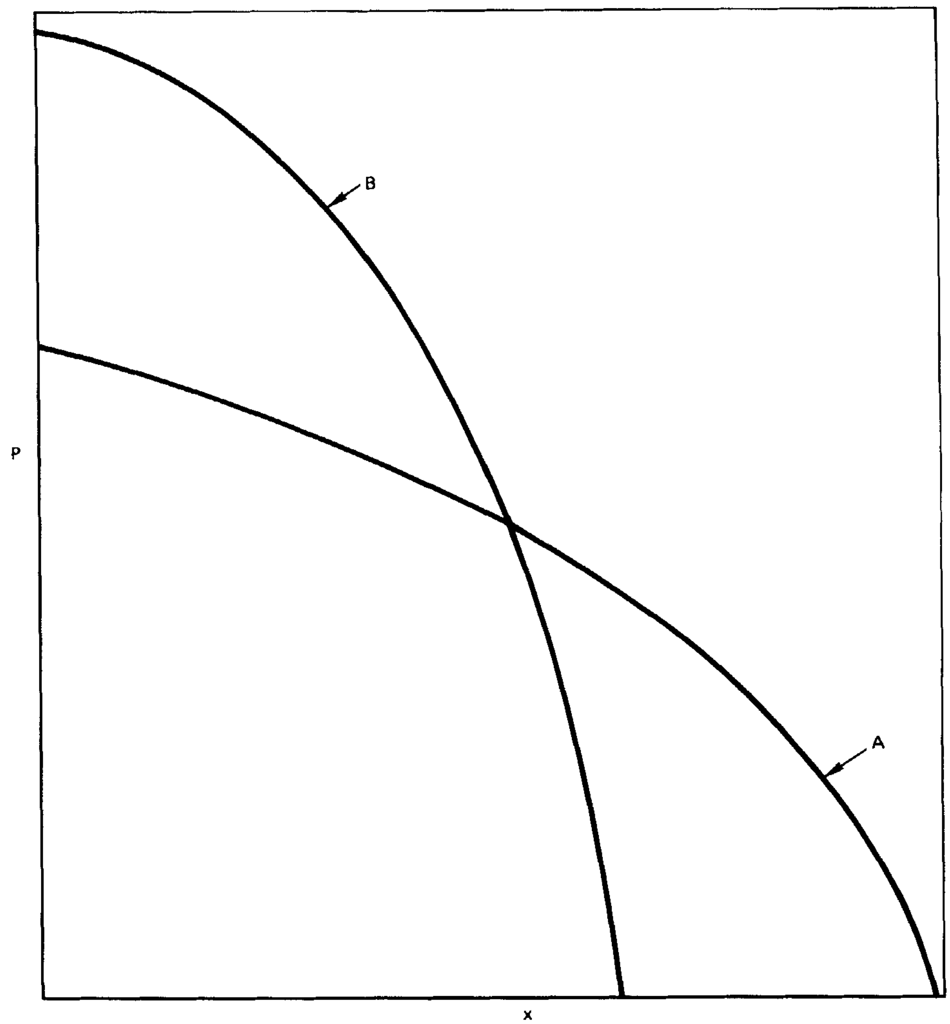

Fig. 8. Risks for two designs.

reduction by spending these dollars on automobile safety, or drinking water, or SO_2 emissions, etc.[9]

This brings us to the major difficulty with the notion of acceptable risk. That is, that risk cannot be spoken of as acceptable or not in isolation, but only in combination with the costs and benefits that are attendant to that risk. Considered in isolation, no risk is acceptable! A rational person would not accept any risk at all except possibly in return for the benefits that come along with it.

Even then, if a risk is acceptable on that basis, it is still not acceptable if it is possible to obtain the same benefit in another way with less risk. Or, if it is possible to reduce the risk at small cost, then the risk is unacceptable. Conversely, a much larger risk may be perfectly acceptable if it brings with it a substantially reduced cost or increased benefit.

Thus one cannot talk about risk in isolation. One needs to adopt a decision theory point of view and ask: "What are my options, what are the costs, benefits, and risks of each?" That option with the optimum mix of cost, benefit, and risk is selected. The risk associated with that option is acceptable. All others are unacceptable.

8. CONCLUSION

In this paper we have first attempted to pull apart some qualitative considerations related to the idea of risk and then presented two ways of quantita-

[9]Perhaps even, as Cohen & Lee suggest, by organizing computer dating services [see ref. (2), p. 146].

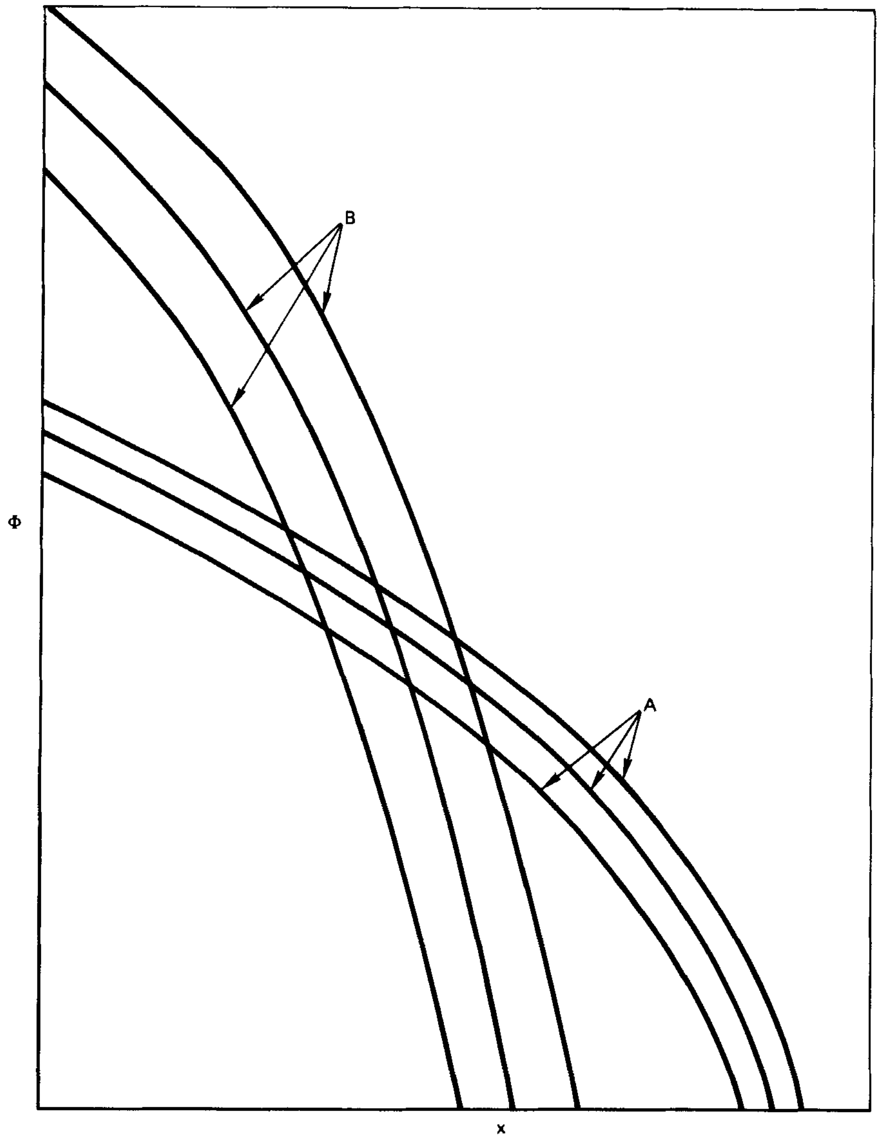

Fig. 9. Risk curves for two designs in probability of frequency format.

tively defining risk. The second of these, including uncertainty, appears to be comprehensive enough to deal with whatever questions and subtleties arise in the course of risk assessment.

In light of these definitions we have discussed the question of completeness of a risk analysis, and the notions of relative risk, relativity of risk, and acceptability of risk. We have argued that the question of completeness can be handled in a rational way by introducing a category of "other" scenarios, and assessing the frequency of occurrence of this category using the existing evidence and Bayes' theorem, just as for any "ordinary" category of scenarios.

We have argued that a single number is not a big enough concept to communicate risk. It takes a whole curve, or actually a family of curves, to communicate the idea of risk. Notwithstanding this, we have indicated how the family of curves can be reduced to a single number, but have urged caution in the use of this reduction since it inevitably involves a great loss of information.

Finally let us emphasize that the purpose of risk analysis and risk quantification is always to provide input to an underlying decision problem which involves not just risks but also other forms of costs and benefits. Risk must thus be considered always within a decision theory context. Within this context, that risk is acceptable, which comes along with the optimum decision option, all other risks are unacceptable, even if smaller.

APPENDIX. CONSTRUCTING THE RISK FAMILY FROM THE SET OF TRIPLETS IN THE CASE OF UNCERTAINTY

Let us begin with the set (1)

$$R=\{\langle s_i, p_i(\phi_i), x_i\rangle\}.$$

Let

$\Phi_i =$ frequency of damage x_i or greater,

and

$\Pi_i(\Phi_i)=$ probability density function for Φ_i.

Then since

$$\Phi_i=\Phi_{i+1}+\phi_i$$

the distributions for Φ_i may be obtained by working up from the bottom as in Table II, doing a probabilistic addition at each step.

The probabilistic addition is expressed by the convolution operation:

$$\Pi_i(\Phi_i)=\int_0^{\Phi_i}\Pi_{i+1}(\Phi_{i+1})p_i(\Phi_i-\Phi_{i+1})\,d\Phi_{i+1}$$

or, if there is dependency, by

$$\Pi_i(\Phi_i)=\int_0^{\Phi_i}\Pi_{i+1}(\Phi_{i+1})p_i(\Phi_i-\Phi_{i+1}|\Phi_{i+1})\,d\Phi_{i+1}.$$

In the case of the set (3)

$$R=\{\langle s_i p_i(\phi_i, x_i)\rangle\}$$

of which (2) is a special case, we define

$$P_i(\phi, x)=\int_0^{\infty}p(\phi,\xi)\,d\xi, \phi>0.$$

This is the probability, per unit ϕ, that the frequency of scenario s_i is ϕ and that the damage is greater than x. We next augment this definition with

$$P_i(0,x)=1-\int_{0+\varepsilon}^{\infty}P_i(\phi,x)\,d\phi$$

so that now the integral of $P_i(\phi, x)$ over ϕ, including $\phi=0$ in a Lebesgue sense, is unity.

If we now define:

$$\Pi_{N-1}(\Phi,x)=\int_0^{\infty}P_{N-1}(\phi,x)P_N(\Phi-\phi,x)\,d\phi$$

$$\Pi_{N-2}(\Phi,x)=\int_0^{\Phi}P_{N-2}(\phi,x)\Pi_{N-1}(\Phi-\phi,x)\,d\phi$$

$$\vdots$$

$$\Pi_1(\Phi,x)=\int_0^{\Phi}P_1(\phi,x)\Pi_2(\Phi-\phi,x)\,d\phi,$$

the function $\Pi_1(\Phi, x)$ can now be plotted as the family of risk curves, Fig. 7. We have thus shown how to compute Fig. 7 from the sets of triplets (1), (2), or (3).

It may appear that this is a complicated computation. In fact it can be done quite simply using the concepts of discrete probability distribution (DPD) arithmetic[20]. This subject will be included in a future paper devoted to DPD methodology.

ACKNOWLEDGMENT

The ideas in this paper are the authors' current synthesis of numerous thoughts and viewpoints which they have picked up over many years of working on a large variety of risk problems with many different colleagues all of whom contributed. It is difficult in this evolutionary process to pinpoint who contributed which part of which idea. Nevertheless, certain people stand out for special acknowledgement: E. T. Jaynes and M. Tribus for their unpublished notes for a UCLA short course; N. Rasmussen and the WASH 1400 group for their pioneering study; P. Bieniarz for contributing the germ of Sec. 6.3; M. Griesmeyer for a conversation that led to Appendix I; and G. E. Apostolakis for encouragement, inspiration, and many technical discussions and viewpoints which have found their way into this paper in places too numerous to mention.

REFERENCES

1. W. D. Rowe, *An Anatomy of Risk* (John Wiley & Sons, New York, 1977).
2. B. L. Cohen and I-S. Lee, *A Catalogue of Risks Conference Record*, IEEE Standards Workshop on Human Factors and Nuclear Safety (1979).
3. D. Okrent, Comment on societal risk, *Science* **208**, (April 25, 1980).

4. *Webster's New Collegiate Dictionary* (G. & C. Merriam Co., Springfield, 1977).
5. U.S. Nuclear Regulatory Commission, *Reactor Safety Study: An Assessment of Accident Risks in U.S. Commercial Nuclear Power Plants*, WASH-1400 (NUREG-75/014) (October 1975).
6. B. J. Garrick and S. Kaplan, "Cost-Benefit Estimate of Transporting Spent Nuclear Fuel by Special Trains," ANS Probabilistic Analysis of Nuclear Reactor Safety, Topical Meeting (May 1978).
7. U.S. Environmental Protection Agency, *Transportation Accident Risks in the Nuclear Power Industry, 1975–2020*, NSS 8191.1.
8. B. de Finetti, *Theory of Probability*, Vols. 1 and 2 (John Wiley & Sons, New York, 1974).
9. A. DeMorgan, *Formal Logic*, (Taylor and Walton, London, 1847).
10. D. V. Lindley, *Introduction to Probability and Statistics from a Bayesian Viewpoint* (Cambridge University Press, 1970).
11. G. Apostolakis, Probability and risk assessment: The subjectivistic viewpoint and some suggestions, *Nuclear Safety* **19** (3), 305–315 (May–June 1978).
12. H. W. Lewis *et al.*, "Risk Assessment Review Group Report to the U.S. Nuclear Regulatory Commission," NUREG/CR-0400 (September 1978).
13. R. J. Breen, "Defense in Depth: Approach to Safety in Light of TMI-2," *Proceedings of American Nuclear Society*, 26th Annual Meeting, Las Vegas, Nevada (June 10, 1980).
14. S. Kaplan and B. J. Garrick, On the use of a Bayesian reasoning in safety and reliability decisions— Three examples," *Nuclear Technology* **44** (July 1979).
15. G. Apostolakis, S. Kaplan, B. Garrick, and R. Duphily, Data specialization for plant specific risk studies," *Nuclear Engineering and Design* **56** (1980).
16. G. Apostolakis, S. Kaplan, B. Garrick, and W. Dickter, Assessment of the frequency of failure to scram in light-water reactors," *Nuclear Safety* (1979).
17. S. Kaplan, B. J. Garrick, and P. P. Bieniarz, On the use of Bayes' theorem in assessing the frequency of anticipated transients," *Nuclear Engineering and Design*, to be published.
18. C. Starr, Social benefit vs. technological risk, *Science* **165**, 1232–1238 (1969).
19. W. W. Lawrence, *Of Acceptable Risk* (William Kaufman Inc., Los Altos, 1976).
20. S. Kaplan, *Notes on the LLL/NRC Waste Disposal Study II, Probabilistic Arithmetic*, Lawrence Livermore Lab, UCRL 15005 (Also KAI-2).