$$\sum_{k=1}^{\infty} \underbrace{k^2\left(e^{1/k^\alpha}-1\right)}_{a_k}$$ Serie a termini positivi

Se $\alpha \leq 0$, $a_k \not\to 0 \Rightarrow$ la Σ non converge, diverge a $+\infty$

Se $\alpha > 0$ $\frac{1}{k^\alpha} \to 0 \Rightarrow e^{\frac{1}{k^\alpha}} - 1 \sim \frac{1}{k^\alpha}$

$$\Rightarrow a_k = k^2\left(e^{\frac{1}{k^\alpha}}-1\right) \sim \frac{1}{k^{\alpha-2}}$$

Converge $\Leftrightarrow$ $\alpha-2>1$ $\Leftrightarrow$ $\alpha>3$

La serie converge $\Leftrightarrow$ $\alpha > 3$

diverge a $+\infty$ $\Leftrightarrow$ $\alpha \leq 3$

$$\sum_{k=1}^{\infty} k^2\left(e^{\frac{1}{k^3}}-1\right)(\log x)^k$$ $\boxed{x>0}$

Poniamo $\log x = y \Rightarrow$ diventa una Σ di potenze:

$$\sum_{k=1}^{\infty} \underbrace{k^2\left(e^{\frac{1}{k^3}}-1\right)}_{a_k} y^k$$

Trovo il raggio di convergenza.

$$r = \lim_{k\to+\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to+\infty} \frac{a_k}{a_{k+1}} =$$ OSS $a_k \sim \frac{1}{k}$

$$= \lim_{k\to+\infty} \frac{1/k}{1/k+1} = 1.$$

La Σ converge assolutamente per $|y|<1$

Non converge per $|y|>1$.

$y=1 \Rightarrow \sum_k k^2\left(e^{\frac{1}{k^3}}-1\right)$ diverge.

$y=-1 \Rightarrow \sum_k \underbrace{k^2\left(e^{\frac{1}{k^3}}-1\right)}_{a_k}(-1)^k$

Non converge assolutamente. Proviamo con Leibniz.

1) $a_k \to 0$ ok

2) a_k decrescente

$$f(x)=x^2\left(e^{\frac{1}{x^3}}-1\right)$$

$$f'(x)=2x\left(e^{\frac{1}{x^3}}-1\right)+x^2\cdot e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)=$$

$$=2x\left(e^{\frac{1}{x^3}}-1\right)-\frac{3}{x^2}e^{\frac{1}{x^3}}=$$

$$=\frac{1}{x^2}\underbrace{\left(\boxed{\frac{2\cdot\left(e^{1/x^3}-1\right)}{1/x^3}}\boxed{-3e^{\frac{1}{x^3}}}\right)}_{\to -1}<0 \text{ def}^{te} \text{ per } x\to+\infty$$

$$\downarrow 2 \qquad \downarrow -3$$

def$^{te}<0$

$$a_{k+1}=f(k+1)\underset{\text{def}^{te} \text{ per } k\to+\infty}{<}f(k)=a_k$$

Per Leibniz la Σ converge.

In definitiva la Σ converge $\Leftrightarrow -1\le y<1$

cioè $-1\le \log x<1$, cioè $\frac{1}{e}\le x<e$

Al variare di $\alpha \in \mathbb{R}$, studiare la continuità e la derivabilità in $x=0$ della funzione

$$f(x) = \begin{cases} (\log(1+3|x|))^{\alpha} \log(x^4) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Continuità

$$\lim_{x\to 0} (\log(1+3|x|))^{\alpha} \log x^4 \overset{?}{=} 0 = f(0)$$

$$\parallel$$

$$\lim_{x\to 0^+} \underbrace{(\log(1+3x))^{\alpha}}_{3^{\alpha} x^{\alpha}} 4 \log x =$$

$$= 4 \cdot 3^{\alpha} \lim_{x\to 0^+} x^{\alpha} \log x = \begin{cases} 0 & \alpha > 0 \\ -\infty & \alpha \leq 0. \end{cases}$$

Continuità $\Leftrightarrow$ $\alpha > 0$

Derivabilità Per essere derivabile deve essere continua

$\Rightarrow$ $\alpha > 0$

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x} =$$

$$= \lim_{x\to 0} \frac{1}{x} (\log(1+3|x|))^{\alpha} \log(x^4)$$

$$\lim_{x\to 0^+} \frac{1}{x} (\log(1+3x))^{\alpha} 4 \log x =$$

$$= 4 \cdot 3^{\alpha} \lim_{x\to 0^+} x^{\alpha-1} \log x = \begin{cases} 0 & \text{se } \alpha > 1 \\ -\infty & \text{se } \alpha \leq 1 \end{cases}$$

e quindi, per simmetria

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \begin{cases} 0 & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1. \end{cases}$$

f derivabile $\iff$ $\alpha > 1$.

$$\lim_{x\to +\infty} |\cos x|^{2x}$$

Dim. che il limite non esiste con il teorema ponte

Trovo due succ^{ni} a_k e b_k t.c.

$a_k \to +\infty$, $b_k \to +\infty$

$$\lim_{k\to+\infty} f(a_k) \neq \lim_{k\to+\infty} f(b_k)$$

$a_k = \pi k \to +\infty$

$$f(a_k) = |\cos(\pi k)|^{2\pi k} = 1^{2\pi k} = 1 \to 1.$$

$$b_k = \frac{\pi}{2} + \pi k \to +\infty$$

$$f(b_k) = 0^{2\left(\frac{\pi}{2}+\pi k\right)} = 0 \to 0$$

$$\lim_{x\to 0^+} \frac{\log(x+\cos x)}{x^\alpha} = \lim_{x\to 0^+} \frac{x}{x^\alpha} = \begin{cases} 0 & \text{se } \alpha < 1 \\ 1 & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha > 1 \end{cases}$$

$$\log(\underbrace{x+\cos x}_{\downarrow\, 1}) = \log(1 + (\underbrace{\cos x - 1 + x}_{\downarrow\, 0})) \sim$$

$$\sim \quad \cos x - 1 + x = \cancel{1} - \frac{x^2}{2} + o(x^2) - \cancel{1} + x =$$

$$= x + o(x) \sim x$$

$$\lim_{x \to 0^+} \frac{\sin(x^3) - (\sin x)^3}{\left(\cos(x^3) - (\cos x)^3\right) x^\beta}$$

$$\sin(x^3) - (\sin x)^3 = x^3 - \frac{x^9}{6} + o(x^9) +$$

$$- \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6) \right)^3 =$$

$$= \cancel{x^3} - \frac{x^9}{6} + o(x^9) - \left(\cancel{x^3} - \frac{x^5}{2} + o(x^5) \right)$$

$$= \frac{x^5}{2} + o(x^5) \sim \frac{x^5}{2}$$

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^5)$$

per $t \to 0$

$$\cos(x^3) - (\cos x)^3 =$$

$$= 1 - \frac{x^6}{2} + \frac{x^{12}}{24} + o(x^{12}) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right)^3$$

$$= \cancel{1} - \frac{x^6}{2} + \frac{x^{12}}{24} + o(x^{12}) - \left(\cancel{1} - \frac{3x^2}{2} + o(x^2)\right)$$

$$= \frac{3}{2}x^2 + o(x^2) \sim \frac{3}{2}x^2$$

$$\frac{\sin(x^3) - (\sin x)^3}{\left(\cos(x^3) - (\cos x)^3\right) x^\beta} \sim \frac{\frac{x^5}{\cancel{2}}}{\frac{3x^2}{\cancel{2}} \cdot x^\beta} = \frac{x^{3-\beta}}{3} \xrightarrow[x \to 0^+]{}$$

$$\to \begin{cases} 0 & \text{se } \beta < 3 \\ 1/3 & \text{se } \beta = 3 \\ +\infty & \text{se } \beta > 3 \end{cases}$$

$$\lim_{x \to +\infty} \left(\left(\frac{\alpha x + 2}{x} \right)^{x^2} + \cancel{x^4} \right) e^{-2x} = \qquad \alpha > 0$$

$$x^4 e^{-2x} \to 0$$

$$= \lim_{x \to +\infty} e^{x^2 \log\left(\alpha + \frac{2}{x}\right) - 2x}$$

$\underline{\alpha \neq 1}$

$$\underbrace{x^2 \log\left(\alpha + \frac{2}{x}\right)} - 2x = x^2 \left(\boxed{\log\left(\alpha + \frac{2}{x}\right)} - \frac{2}{x} \right) \to$$

$$\log\left(\alpha + \frac{2}{x}\right) \to \log \alpha \qquad \frac{2}{x} \to 0$$

$$\to \log \alpha$$

$$\to \begin{cases} +\infty & \text{se } \alpha > 1 \\ -\infty & \text{se } 0 < \alpha < 1 \end{cases}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{\overline{\qquad}} \to \begin{cases} +\infty & \text{se } \alpha > 1 \\ 0 & \text{se } 0 < \alpha < 1 \end{cases}$$

$\boxed{\alpha = 1}$

$$x^2 \log\left(1 + \frac{2}{x}\right) - 2x$$

$$\frac{2}{x} \to 0$$

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2) \qquad t \to 0$$

$$x^2 \log\left(1 + \frac{2}{x}\right) - 2x = x^2 \left(\cancel{\frac{2}{x}} - \frac{1}{2} \frac{4}{x^2} + o\left(\frac{1}{x^2}\right) \right) - \cancel{2x}$$

$$\to -2$$

$f(x) \to e^{-2}$

$$\left(1+\frac{2}{x}\right)^{x^2} = \left[\left(1+\frac{2}{x}\right)^{x}\right]^{x} \sim e^{2x}$$

NO

Dire per quali $\alpha > 0$ converge l'integrale

$$\int_0^{+\infty} \frac{\operatorname{arctg}(x-2)}{(x^\alpha+\alpha)^2}\,dx$$

e calcolarlo per $\alpha = 1$.

Problemi solo per $x \to +\infty$.

$$\frac{\operatorname{arctg}(x-2)}{(x^\alpha+\alpha)^2} \sim \frac{\pi}{2\,x^{2\alpha}} \qquad x \to +\infty$$

def^{te} a segno positivo.

L'integrale converge $\Leftrightarrow 2\alpha > 1 \Leftrightarrow \boxed{\alpha > 1/2}$.

$\boxed{\alpha = 1}$
$$\int_0^{+\infty} \frac{\operatorname{arctg}(x-2)}{(x+1)^2}\,dx$$

$$\int \frac{\operatorname{arctg}(x-2)}{(x+1)^2}\,dx =$$

$f'(x) = \frac{1}{(x+1)^2}$ $\quad f(x) = -\frac{1}{x+1}$

$g(x) = \operatorname{arctg}(x-2)$, $g'(x) = \frac{1}{1+(x-2)^2}$

$$= -\frac{1}{x+1}\operatorname{arctg}(x-2) + \int \frac{dx}{(x+1)(x^2-4x+5)} = (*)$$

$$\frac{1}{(x+1)(x^2-4x+5)} = \frac{A}{x+1} + \frac{B(2x-4)+C}{x^2-4x+5}$$

$$1 = A(x^2-4x+5) + B(2x-4)(x+1) + C(x+1)$$

$$\begin{cases} 0 = A + 2B & (2) \\ 0 = -4A - 2B + C & (1) \\ 1 = 5A - 4B + C & (0) \end{cases} \qquad \boxed{A = -2B}$$

$$\begin{cases} 8B - 2B + C = 0 \\ -10B - 4B + C = 1 \end{cases} \qquad \begin{cases} 6B + C = 0 \\ -14B + C = 1 \end{cases}$$

$$20B = -1 \qquad \boxed{B = -\frac{1}{20}} \qquad \boxed{C = -6B = +\frac{3}{10}}$$

$$\boxed{A = +\frac{1}{10}}$$

$$(*) = -\frac{1}{x+1}\operatorname{arctg}(x-2) + \frac{1}{10}\int \frac{dx}{x+1} - \frac{1}{20}\int \frac{2x-4}{x^2-4x+5}dx$$

$$+ \frac{3}{10}\int \boxed{\frac{1}{x^2-4x+5}}\, dx = \qquad = \frac{1}{(x-2)^2+1}$$

$$= -\frac{1}{x+1}\operatorname{arctg}(x-2) + \frac{1}{10}\log(x+1) - \frac{1}{20}\log(x^2-4x+5) +$$

$$+ \frac{3}{10}\operatorname{arctg}(x-2) + c_1$$

$$\int_0^{+\infty} f(x)\,dx = \lim_{\omega\to+\infty} \int_0^{\omega} f(x)\,dx =$$

$$= \lim_{\omega\to+\infty} \left[\boxed{-\frac{1}{\omega+1} \operatorname{arctg}(\omega-2)} \xrightarrow{} 0 + \frac{1}{10}\log(\omega+1) - \frac{1}{20}\log(\omega^2-4\omega+5) + \boxed{\frac{3}{10}\operatorname{arctg}(\omega-2)} \right] - \left(-\operatorname{arctg}(-2) - \frac{1}{20}\log 5 + \frac{3}{10}\operatorname{arctg}(-2) \right)$$

$$\downarrow$$

$$\frac{3\pi}{20}$$

$$= \frac{3\pi}{20} + \frac{1}{20}\log\frac{(\omega+1)^2}{\omega^2-4\omega+5} \to 0 - \frac{7}{10}\operatorname{arctg} 2 + \frac{1}{20}\log 5$$

$$I_n = \int_0^1 e^{-x}\sin^n(\pi x)\,dx$$

1) Trovare una formula iterativa che dà I_n in funzione di I_{n-2} $\quad n \geq 2$

2) Usarla per calcolare I_4

$$I_n = \int_0^1 e^{-x}\sin^n(\pi x)\,dx =$$

$$f'(x) = e^{-x}, \quad f(x) = -e^{-x}$$

$$g(x) = \sin^n(\pi x), \quad g'(x) = \pi n \sin^{n-1}(\pi x)\cos(\pi x)$$

$$= \cancel{-e^{-x}\sin^n(\pi x)\Big|_0^1} + \pi n \int_0^1 e^{-x}\sin^{n-1}(\pi x)\cos(\pi x)\,dx$$

$$= (*)$$

$$f'(x) = e^{-x} \qquad f(x) = -e^{-x}$$

$$g(x) = \sin^{n-1}(\pi x)\cos(\pi x)$$

$$g'(x) = \pi(n-1)\sin^{n-2}(\pi x)\underbrace{\cos^2(\pi x)}_{=1-\sin^2(\pi x)} +$$

$$-\pi \sin^n(\pi x)$$

$$= \pi(n-1)\sin^{n-2}(\pi x) +$$

$$-\pi(n-\cancel{1})\sin^n(\pi x) - \cancel{\pi \sin^n(\pi x)}$$

$$= \pi(n-1)\sin^{n-2}(\pi x) - \pi n \sin^n(\pi x)$$

$$(*) = \pi n \left[\cancel{-e^{-x}\sin^{n-1}(\pi x)\cos(\pi x)}\right]_0^1 +$$

$$+\pi(n-1)\underbrace{\int_0^1 e^{-x}\sin^{n-2}(\pi x)\,dx}_{I_{n-2}} +$$

$$-\pi n \underbrace{\int_0^1 e^{-x}\sin^n(\pi x)\,dx}_{I_n}$$

Riscrivo

$$I_n = \pi^2 n(n-1) I_{n-2} - \pi^2 n^2 I_n$$

$$I_n(1+\pi^2 n^2) = \pi^2 n(n-1) I_{n-2}$$

$$\boxed{I_n = \frac{\pi^2 n(n-1)}{1+\pi^2 n^2} I_{n-2}}$$

$$I_4 = \frac{\pi^2 \, 4 \cdot 3}{16\pi^2+1} I_2 = \frac{12\pi^2}{16\pi^2+1} I_2$$

$$I_2 = \frac{2\pi^2}{4\pi^2+1} I_0$$

$$I_4 = \frac{24\pi^4}{(16\pi^2+1)(4\pi^2+1)} I_0 = \frac{24\pi^4}{(16\pi^2+1)(4\pi^2+1)} \left(1-\frac{1}{e}\right)$$

$$I_0 = \int_0^1 e^{-x}\,dx = e^{-x}\Big|_1^0 = 1 - e^{-1}$$

$f(x) = \sin(\arccos x) =$ Dominio : $[-1,1]$.

$\arccos x \in [0,\pi]$

se $\alpha \in [0,\pi]$ $\quad \sin\alpha = +\sqrt{1-\cos^2\alpha}$

$$= \sqrt{1-(\cos(\arccos x))^2} = \sqrt{1-x^2}$$

$z^4 = \alpha(|z|^4+2)$

$\alpha = 2$ impossibile.

$\alpha = 2/3$ 4 soluzioni

$\alpha = -1/3$.

$\boxed{\alpha=2}$ $\quad z^4 = 2(|z|^4+2)$

$z = \rho e^{i\varphi}$

$$\rho^4 e^{i4\varphi} = 2(\rho^4+2)e^{i0}$$

$$\begin{cases}\rho^4 = 2(\rho^4+2) \\ \varphi = \dots\end{cases} \Leftrightarrow \rho^4 = -4 \qquad \text{impossibile}$$

$$\alpha = \frac{2}{3} \qquad z^4 = \frac{2}{3}\left(|z|^4+2\right)$$

$$z = \rho e^{i\varphi} \qquad \rho^4 e^{i4\varphi} = \frac{2}{3}\left(\rho^4+2\right)e^{i0}$$

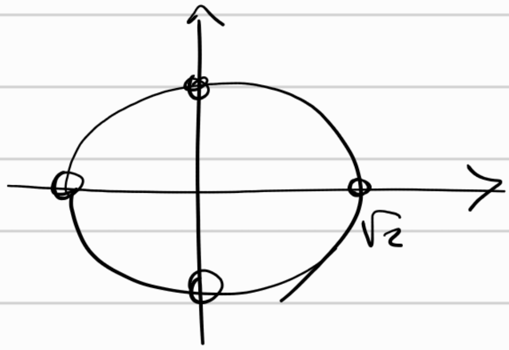
$$\rho^4 = \frac{2}{3}\rho^4 + \frac{4}{3} \Leftrightarrow \frac{\rho^4}{3} = \frac{4}{3} \Leftrightarrow \rho^4 = 4$$

$$\Leftrightarrow \boxed{\rho = \sqrt{2}}$$

$$4\varphi = 0 + 2k\pi$$

$$\varphi_k = \frac{k\pi}{2} \qquad k = 0, 1, 2, 3$$

$$z_k = \sqrt{2}\, e^{i\varphi_k}$$



$$z_0 = \sqrt{2}$$

$$z_1 = \sqrt{2}e^{i\frac{\pi}{2}} = \sqrt{2}\, i$$

$$z_2 = \sqrt{2}e^{i\pi} = -\sqrt{2}$$

$$z_3 = \sqrt{2}e^{i\frac{3\pi}{2}} = -\sqrt{2}\, i$$

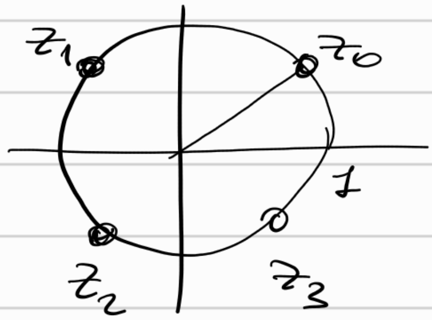
$$\boxed{\alpha = -\frac{1}{3}} \qquad z^4 = -\frac{1}{3}\left(|z|^4+2\right)$$

$$\boxed{z = \rho e^{i\varphi}} \qquad \rho^4 e^{i4\varphi} = \frac{1}{3}\left(\rho^4+2\right)\boxed{e^{i\pi}}$$

$$\rho^4 = \frac{\rho^4}{3} + \frac{2}{3} \Leftrightarrow \frac{2}{3}\rho^4 = \frac{2}{3} \Leftrightarrow \rho = 1$$

$$4\varphi = \pi + 2k\pi$$

$$\varphi_k = \frac{\pi}{4} + \frac{k\pi}{2} \qquad k = 0, 1, 2, 3.$$



$$z_0 = 1 \cdot e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$$

$$z_1 = e^{i\frac{3}{4}\pi} = \frac{-1+i}{\sqrt{2}}$$

$$z_2 = e^{i\frac{5}{4}\pi} = \frac{-1-i}{\sqrt{2}} = -z_0 = \overline{z_1}$$

$$z_3 = e^{i\frac{7}{4}\pi} = \frac{1-i}{\sqrt{2}} = -z_1 = \overline{z_0}$$

Calcolare l'area di

$$E = \left\{ (x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad 0 \leq y \leq -\sqrt{3}\,\mathrm{tg}x - \mathrm{tg}^2 x \right\}$$

$$-\sqrt{3}\,\mathrm{tg}x - \mathrm{tg}^2 x \geq 0$$

$$\mathrm{tg}^2 x + \sqrt{3}\,\mathrm{tg}x \leq 0$$

$$\mathrm{tg}x\,(\mathrm{tg}x + \sqrt{3}) \leq 0 \iff -\sqrt{3} \leq \mathrm{tg}x \leq 0$$

$$\Updownarrow \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$-\frac{\pi}{3} \leq x \leq 0$$

$$\text{Area}(t) = \int_{-\pi/3}^{0} \left(-\sqrt{3}\,\text{tg}\,x - \text{tg}^2 x\right) dx =$$

$$= -\int_{-\pi/3}^{0} \left(\sqrt{3}\,\text{tg}\,x + \text{tg}^2 x\right) dx$$

$$\int \text{tg}\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x\,dx}{\cos x}$$

$\cos x = t$
$-\sin x\,dx = dt$

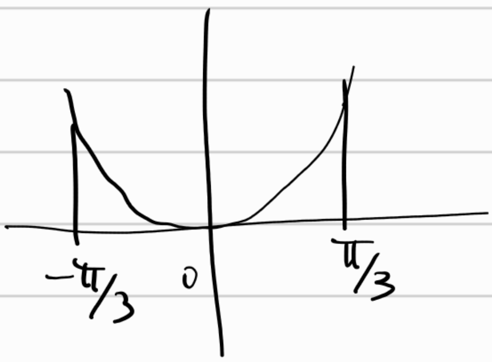
$$= -\int \frac{dt}{t} = -\log|t| = -\log|\cos x|$$

$$\int_{-\pi/3}^{0} \text{tg}\,x = \log|\cos x|\Big|_{0}^{-\pi/3} = \log\frac{1}{2} - \cancel{\log 1} = -\log 2$$

$$\int \text{tg}^2 x\,dx = \int\left(\left(\text{tg}^2 x + 1\right) - 1\right)dx = \text{tg}\,x - x$$

$$\int_{-\pi/3}^{0} \text{tg}^2 x\,dx = \int_{0}^{\pi/3} \text{tg}^2 x\,dx = \left(\text{tg}\,x - x\right)\Big|_{0}^{\pi/3} =$$

$$= \sqrt{3} - \frac{\pi}{3}$$



Ordinare i seguenti infinitesimi, per $x \to 0^+$

$$f(x) = tg^2 x - x^2 + x^5 = \cancel{x^2} + \frac{2}{3}x^4 + o(x^4) - \cancel{x^2} + x^5$$

$$tg\, x = x + \frac{x^3}{3} + o(x^4)$$

$$= \frac{2}{3}x^4 + o(x^4) \sim \frac{2}{3}x^4$$

$$tg^2 x = \left(x + \frac{x^3}{3} + o(x^4)\right)^2 = x^2 + \frac{2}{3}x^4 + o(x^4)$$

$$g(x) = x^5 + 2x^6 + x^5 \log x =$$

$$= x^5 \log x \left(\frac{1}{\log x} + \frac{2x}{\log x} + 1 \right) \sim x^5 \log x$$

$$\frac{1}{\log x} \to 0 \qquad \frac{2x}{\log x} \to 0$$

Inf^{mo} di ordine inferiore rispetto a x^5
ma di ordine superiore risp. a x^α $\forall \alpha < 5$

$$h(x) = \sqrt[3]{1+x^5} - \sqrt{1+x^5}$$

$$\sqrt[3]{1+t} = 1 + \frac{t}{3} + o(t) \qquad t \to 0$$

$$\sqrt[3]{1+x^5} = 1 + \frac{x^5}{3} + o(x^5)$$

$$\sqrt{1+x^5} = 1 + \frac{x^5}{2} + o(x^5)$$

$$\sqrt[3]{1+x^5} - \sqrt{1+x^5} = \cancel{1} + \frac{x^5}{3} + o(x^5) - \cancel{1} - \frac{x^5}{2} =$$

$$= -\frac{x^5}{6} + o(x^5)$$ inf^{mo} di ordine 5

In ordine crescente di infinitesimo :

$$f, g, h$$

$$\sqrt[3]{1+x^5} - \sqrt{1+x^5} = \underbrace{\sqrt[3]{1+x^5}}_{\sim 1} \underbrace{\left(1 - \sqrt[6]{1+x^5}\right)}_{\sim -\frac{x^5}{6}}$$