$$f(x) = \operatorname{arctg}\left(\frac{\cos x}{1-\sin x}\right) - \frac{x}{2}$$

Dominio: $x \neq \frac{\pi}{2} + 2k\pi$.

Non ci sono simmetrie apparenti.

Non è periodica, tuttavia $f(x+2\pi) = f(x) - \pi$

Conviene studiarla in un intervallo di ampiezza 2π, per es. $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$

Limiti $\quad f$ continua nel suo dominio.

$$\lim_{x\to\left(-\frac{3\pi}{2}\right)^+} f(x) = \quad \dots \text{ dopo.}$$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \quad \dots \text{ dopo.}$$

$$\lim_{x\to+\infty} f(x) = -\infty \qquad (\text{limitato} + (-\infty))$$

$$\lim_{x\to-\infty} f(x) = +\infty$$

La funzione non ammette asintoti obliqui perché è pari a $\frac{x}{2}$ + funzione periodica non costante.

Derivata prima $\quad f$ è derivabile nel suo dominio

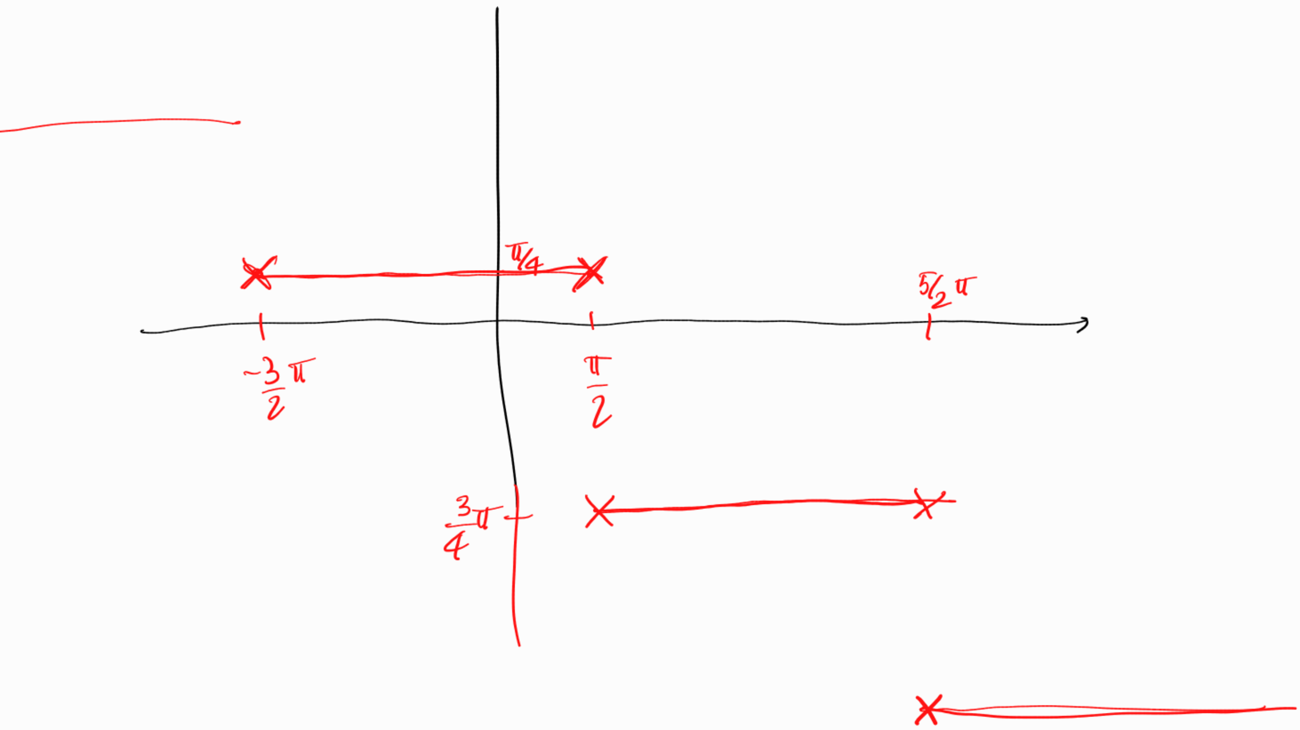
$$f(x) = \operatorname{arctg}\left(\frac{\cos x}{1-\sin x}\right) - \frac{x}{2}$$

$$f'(x) = \frac{1}{1+\frac{\cos^2 x}{(1-\sin x)^2}} \cdot \frac{-\sin x(1-\sin x) + \cos^2 x}{(1-\sin x)^2} - \frac{1}{2}$$

$$= \frac{-\sin x + 1}{(1-\sin x)^2 + \cos^2 x} - \frac{1}{2} = \frac{-\sin x + 1}{1+1-2\sin x} - \frac{1}{2} =$$

$$= \frac{1-\sin x}{2-2\sin x} - \frac{1}{2} = 0 \Rightarrow$$ f è costante in ogni intervallo in cui è definita.

$\forall x \in \left(-\frac{3}{2}\pi, \frac{\pi}{2}\right) \qquad f(x) = f(0) = \operatorname{arctg} 1 = \frac{\pi}{4}$



$$\int \operatorname{arctg}\left(4\sqrt{x}-1\right) dx =$$

sost. $4\sqrt{x}-1 = t \qquad \sqrt{x} = \frac{t+1}{4} \qquad x = \frac{(t+1)^2}{16}$

$$dx = \frac{2(t+1)}{16}\, dt = \frac{t+1}{8}\, dt$$

$$= \frac{1}{8}\int (t+1)\operatorname{arctg} t \, dt =$$ per parti

$t+1 = f'(t) \Rightarrow f(t) = \frac{(t+1)^2}{2}$

$g(t) = \operatorname{arctg} t \Rightarrow g'(t) = \frac{1}{1+t^2}$

$$= \frac{1}{8}\left[\frac{(t+1)^2}{2}\operatorname{arctg} t - \frac{1}{2}\int \frac{(t+1)^2}{1+t^2}\, dt\right] =$$

$$= \frac{1}{16}\left[(t+1)^2 \operatorname{arctg} t - \int \frac{t^2+1+2t}{1+t^2}\,dt\right] =$$

$$= \frac{1}{16}\left[(t+1)^2 \operatorname{arctg} t - t - \int \frac{2t}{1+t^2}\,dt\right]$$

$$= \frac{1}{16}\left[(t+1)^2 \operatorname{arctg} t - t - \log(1+t^2)\right]\Bigg|_{t=4\sqrt{x}-1} + c$$

$$= \frac{1}{16}\left[16\,x \operatorname{arctg}(4\sqrt{x}-1) - 4\sqrt{x}+1 - \log\left(1+(4\sqrt{x}-1)^2\right)\right] + c$$

$$= x \operatorname{arctg}(4\sqrt{x}-1) - \frac{1}{4}\sqrt{x} - \frac{1}{16}\log\left(16x - 8\sqrt{x}+2\right) + c_1$$

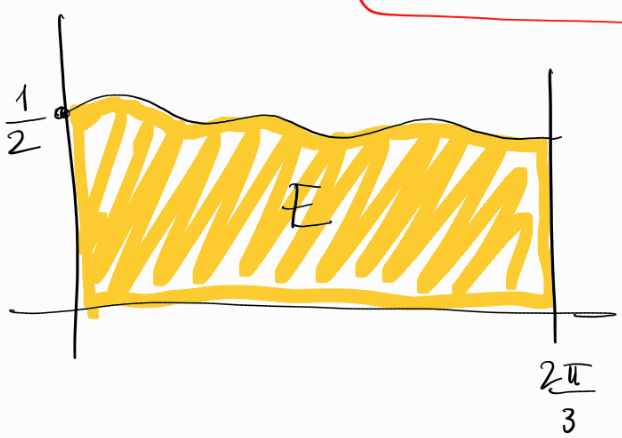
Oppure subito per parti

$$\int \operatorname{arctg}(4\sqrt{x}-1)\,dx = x \operatorname{arctg}(4\sqrt{x}-1) - \int \frac{x}{1+(4\sqrt{x}-1)^2}\,\frac{4}{2\sqrt{x}}$$

$$= x \operatorname{arctg}(4\sqrt{x}-1) - 2\underbrace{\int \frac{\sqrt{x}}{1+(4\sqrt{x}-1)^2}\,dx}_{\text{si pone } \sqrt{x}=t}$$

Calcolare l'area della regione limitata del primo quadrante delimitata dagli assi coordinati, dalla retta $x = \frac{2\pi}{3}$ e del grafico di

$$f(x) = \frac{1}{3-\cos x+2\sin x} \quad > 0$$



$$\text{Area } E = \int_0^{\frac{2\pi}{3}} \frac{dx}{3-\cos x+2\sin x} =$$

$$\left[\begin{array}{ll} \text{So st } t = \operatorname{tg}\frac{x}{2} & x \neq \pi + 2k\pi \\ \cos x = \frac{1-t^2}{1+t^2} & \sin x = \frac{2t}{1+t^2} \\ \frac{x}{2} = \operatorname{arctg} t & dx = \frac{2\,dt}{1+t^2} \end{array}\right.$$

$x=0 \Rightarrow t=0$; $x=\frac{2\pi}{3} \Rightarrow t=\sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{1}{3-\frac{1-t^2}{1+t^2}+\frac{4t}{1+t^2}} \frac{2\,dt}{1+t^2} =$$

$$= 2\int_0^{\sqrt{3}} \frac{dt}{3+3t^2-1+t^2+4t} = 2\int_0^{\sqrt{3}} \frac{dt}{\underbrace{4t^2+4t+2}_{4t^2+4t+1+1 \,=\, (2t+1)^2+1}} =$$

$\Delta < 0$

$$= \int_0^{\sqrt{3}} \frac{2\,dt}{(2t+1)^2+1} =$$

$$= \cancel{2}\, \frac{\operatorname{arctg}(2t+1)}{\cancel{2}} \Big|_0^{\sqrt{3}} = \operatorname{arctg}(2\sqrt{3}+1) - \frac{\pi}{4}$$

$$\int \frac{dt}{2t^2+2t+1} = \int \frac{dt}{\left(2t^2+2t+\frac{1}{2}\right)+\frac{1}{2}} = \int \frac{dt}{\left(\frac{2t+1}{\sqrt{2}}\right)^2+\frac{1}{2}} =$$

$$\left(\sqrt{2}t+\frac{1}{\sqrt{2}}\right)^2$$

$$= 2\int \frac{dt}{(2t+1)^2+1}$$

$$|\bar{z}-2| - |z|^2 - 2\,\mathrm{Im}(z) = 3i$$

$z = x+iy \qquad x,y\in\mathbb{R}$

$|\bar{z}-2| = |x-iy-2| = \sqrt{(x-2)^2+y^2}$

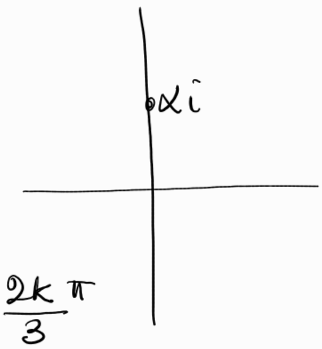
$|z|^2 = x^2+y^2$

$\sqrt{(x-2)^2+y^2} - x^2 - y^2 - 2y = 3i \qquad$ impossibile !

$\dfrac{w^3}{i}\in\mathbb{R} \iff w^3 = i\,\alpha \qquad \alpha\in\mathbb{R}$

Distinguiamo $\alpha=0$, $\alpha>0$, $\alpha<0$.

$\boxed{\alpha=0} \qquad w^3=0 \iff w=0$



$\alpha>0 \qquad \alpha i = \alpha e^{i\frac{\pi}{2}}$

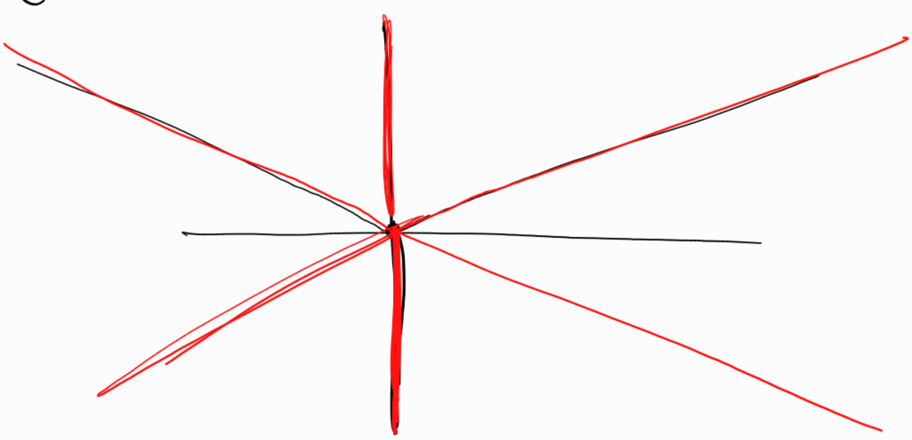
$W_k = \sqrt[3]{\alpha}\, e^{i\varphi_k} \qquad \varphi_k = \dfrac{\frac{\pi}{2}+2k\pi}{3} = \dfrac{\pi}{6}+\dfrac{2k\pi}{3}$

$k=0,1,2.$

$W_0 = \sqrt[3]{\alpha}\, e^{i\frac{\pi}{6}}$

$W_1 = \sqrt[3]{\alpha}\, e^{i\frac{5}{6}\pi} \qquad \alpha$ varia in $(0,+\infty)$

$W_2 = \sqrt[3]{\alpha}\, e^{i\frac{3}{2}\pi} \qquad \Rightarrow$ sono 3 semirette.



Se invece $\alpha < 0$, allora
$$\alpha i = (-\alpha)\, e^{-i\frac{\pi}{2}}$$

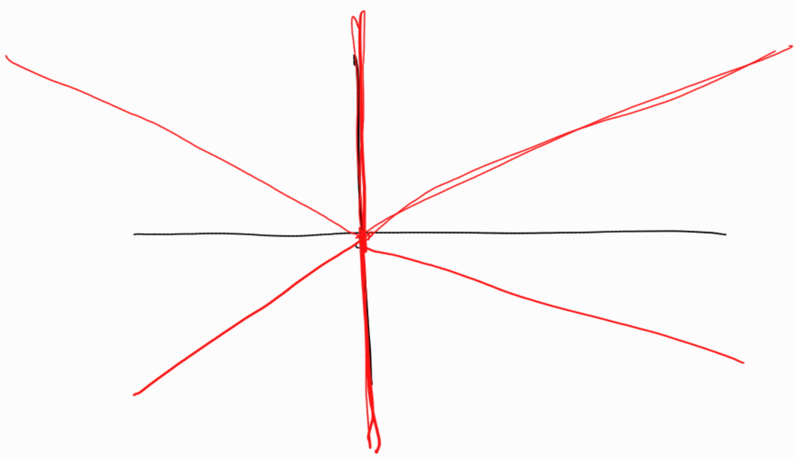
$$W_k = \sqrt[3]{-\alpha}\, e^{i\varphi_k}$$
$$\varphi_k = -\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{7\pi}{6}$$

$$\varphi_k = -\frac{\pi}{6} + \frac{2k\pi}{3}$$
$$\varphi_0 = -\frac{\pi}{6}$$
$$\varphi_1 = \frac{\pi}{2}$$
$$\varphi_2 = \frac{7\pi}{6}$$

Alla fine vengono fuori 6 semirette $= 3$ rette nel piano complesso.

Sinteticamente:
$$w = |w|\, e^{i\varphi}$$
$$w^3 = |w|^3 e^{i3\varphi}$$
$$\frac{w^3}{i} = |w|^3 e^{i\left(3\varphi - \frac{\pi}{2}\right)} \in \mathbb{R}$$
$$\iff 3\varphi - \frac{\pi}{2} = k\pi$$
$$3\varphi = \frac{\pi}{2} + k\pi$$
$$\varphi = \frac{\pi}{6} + \frac{k\pi}{3} \qquad k = 0,1,2,3,4,5$$

Inoltre va aggiunta la soluzione $w=0$



Trovare le radici ottave di 16 e disegnarle nel piano complesso

$$16 = 16 e^{i0}$$

$$z_k = \underset{\substack{\shortparallel \\ \sqrt{2}}}{\sqrt[8]{16}}\, e^{i\varphi_k} \qquad \varphi_k = \frac{0+2k\pi}{8} = \frac{k\pi}{4}$$

$k = 0, 1, \ldots 7$

$$z_0 = \sqrt{2}\, e^{i0} = \sqrt{2}$$

$$z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$$

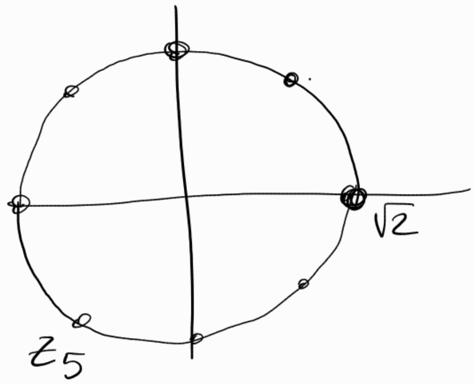
$$z_2 = \sqrt{2}\, e^{i\frac{\pi}{2}} = \sqrt{2}\, i$$

$$z_3 = \sqrt{2}\, e^{i\frac{3\pi}{4}} = -1+i$$

$$z_4 = -\sqrt{2}$$

$$z_5 = -z_1 = \overline{z_3}$$

$\vdots$



Sapendo che $z = 1-i$ è radice del polinomio

$$P(z) = z^4 - 2z^3 + 6z^2 - 8z + 8,$$

trovare le altre radici.

$P(z)$ è a coeff^{ti} reali $\Rightarrow$ se $z = 1-i$ è uno zero, anche $\bar{z} = 1+i$ è uno zero.

$$P(z) = \underbrace{(z-(1-i))(z-(1+i))}\, Q_2(z) = \cdot$$

$$((z-1)+i)((z-1)-i)$$

$$(z-1)^2 + 1 = z^2 - 2z + 2$$

$$
\begin{array}{rrrrr|l}
z^4 & -2z^3 & +6z^2 & -8z & +8 & z^2-2z+2 \\ \hline
-z^4 & +2z^3 & -2z^2 & & & z^2+4 \\ \hline
/ & / & 4z^2 & -8z & +8 & \\
 & & -4z^2 & +8z & -8 & \\ \hline
 & & & / & &
\end{array}
$$

$Q_2(z) = z^2+4.$

Le radici sono: $1-i$, $1+i$, $2i$, $-2i$

$$z^2 - |z|^2 + \frac{4z}{i} = 0 \qquad z = x+iy, \quad x,y \in \mathbb{R}$$

$$\cancel{x^2} - y^2 + 2ixy - \cancel{x^2} - y^2 + 4y - 4ix = 0$$

$$\frac{z}{i} = \frac{z\cdot(-i)}{1} = -iz = -i(x+iy) = y - ix$$

$$\begin{cases} -2y^2 + 4y = 0 \\ 2xy - 4x = 0 \end{cases} \qquad \begin{cases} 2y - y^2 = 0 \\ x(y-2) = 0 \end{cases}$$

$y = 0 \Rightarrow x = 0$

$y = 2$ $\quad x$ qualsiasi.

Le soluzioni sono: $z = 0$, e $z = x + 2i$ con x qualsiasi.

$$\left|\frac{w-2}{w+i}\right| = 2 \iff \left|\frac{w-2}{w+i}\right|^2 = 4 \iff \boxed{w \neq -i}$$

$w = x + iy$

$$|w-2|^2 = 4\,\underbrace{|w+i|^2}_{x^2+(y+1)^2}$$

$$\overset{\shortparallel}{(x-2)^2+y^2}$$

$$(x-2)^2 + y^2 = 4\left(x^2 + (y+1)^2\right)$$

$$x^2 - 4x + \cancel{4} + y^2 = 4x^2 + 4y^2 + 8y + \cancel{4}$$

$3x^2 + 3y^2 + 8y + 4x = 0$ è una circonf.

$$x^2 + y^2 + \frac{8}{3}y + \frac{4}{3}x = 0$$

$$\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + \left(y^2 + \frac{8}{3}y + \frac{16}{9}\right) = \frac{4}{9} + \frac{16}{9}$$

$$\left(x + \frac{2}{3}\right)^2 + \left(y + \frac{4}{3}\right)^2 = \frac{20}{9}$$

Circonf. di centro $\left(-\frac{2}{3}, -\frac{4}{3}\right)$ e raggio $\frac{2\sqrt{5}}{3}$

$f(x) = x^\alpha - \beta x$

Trovare $\alpha, \beta \in \mathbb{R}$ t.c.

1) $x = 1$ sia max. locale.

$f'(x) = \alpha x^{\alpha - 1} - \beta$

$f'(1) = 0 \iff \alpha - \beta = 0$ $\boxed{\alpha = \beta}$

Vediamo $f''(1) = \alpha(\alpha - 1) x^{\alpha - 2}\Big|_{x=1} = \alpha(\alpha - 1)$

se $f''(1) > 0 \iff (\alpha < 0) \vee (\alpha > 1)$
allora $x = 1$ è min. locale stretto

$f''(1) < 0 \iff 0 < \alpha < 1$
allora $x = 1$ è pto di max locale stretto.

se $f''(1) = 0$, cioè $\alpha = 0$ o $\alpha = 1$.

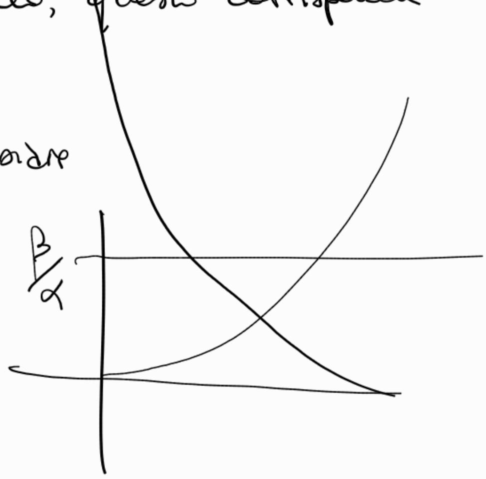
se $\alpha = 0 = \beta$ $f(x) \equiv 1$ Ok!

se $\alpha = 1 = \beta$ $f(x) = x - x = 0$ Ok!

Soluzione $\boxed{0 \leq \alpha \leq 1, \quad \beta = \alpha}$

2) f invertibile in $(0, +\infty)$, cioè iniettiva.
Poiché f è continua in un intervallo, questo corrisponde a f strett. monotòna.

$f'(x) = \alpha x^{\alpha - 1} - \beta$. non deve cambiare segno.



In particolare è vero se $f'(x)$ non si annulla mai.

$f'(x) = 0 \iff x^{\alpha-1} = \frac{\beta}{\alpha}$

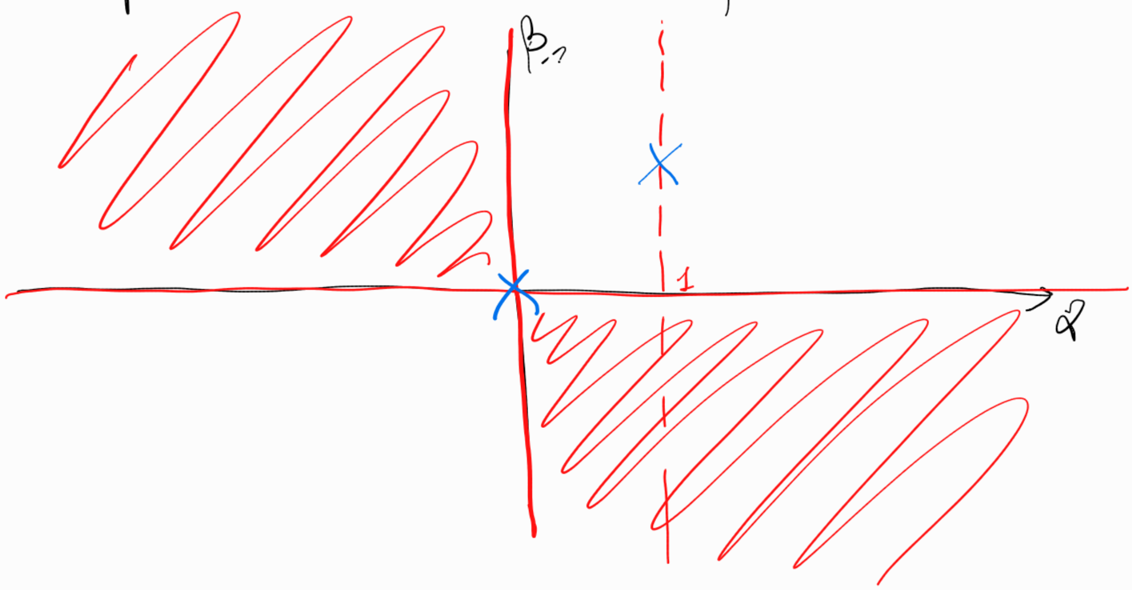
Se β e α sono discordi, f è iniettiva.

Se β e α sono concordi, f' cambia segno $\Rightarrow$ non è iniettiva
e $\alpha \neq 1$

se $\beta = 0$ $\quad f(x) = x^{\alpha}$ sempre iniettiva in $(0, +\infty)$
tranne se $\alpha = 0$.

Se $\alpha = 0$, $f(x) = -\beta x$ sempre ~~iniettiva~~ tranne quando $\beta = 0$

se $\alpha = 1$, $f(x) = x - \beta x = (1-\beta)x$
sempre iniettiva tranne se $\beta = 1$.



f convessa in $(0,+\infty)$

$$f'(x) = \alpha x^{\alpha-1} - \beta$$

$$f''(x) = \alpha(\alpha-1)\underbrace{x^{\alpha-2}}_{\vee\, 0}$$

f converse $\Leftrightarrow$ $\alpha(\alpha-1) \geq 0 \Leftrightarrow (\alpha \leq 0) \vee (\alpha \geq 1)$

$$\sum_{k=0}^{\infty} \boxed{\sqrt[3]{\frac{k!}{(k+5)!}}} =$$

$a_k > 0$

$$\frac{a_{k+1}}{a_k} = \sqrt[3]{\frac{(k+1)!}{(k+6)!} \cdot \frac{(k+5)!}{k!}} = \sqrt[3]{\frac{k+1}{k+6}} \to 1$$

$$\sqrt[3]{\frac{k!}{(k+5)!}} = \sqrt[3]{\frac{\cancel{k!}}{\cancel{k!}(k+1)(k+2)(k+3)(k+4)(k+5)}} =$$

$$= \frac{1}{\sqrt[3]{(k+1)(k+2)\cdots(k+5)}} \sim \frac{1}{k^{5/3}}$$