$\boxed{08/01/2026}$

$F(t, x(t), \dots, x^{(k)}(t)) = 0$ ← ED in forma implicita

$x : \underset{\mathbb{R}}{\underset{\cap}{(a,b)}} \longrightarrow \mathbb{R}^n$

$m x''(t) = f(t, x(t), x'(t))$

$x''(t) = \frac{1}{m} f(t, x(t), x'(t))$ ← Equazione differenziale in forma normale

$$\begin{cases} x'(t) = y(t) \\ y'(t) = \cancel{x''}(t) = \frac{1}{m} f(t, x(t), y(t)) \end{cases}$$

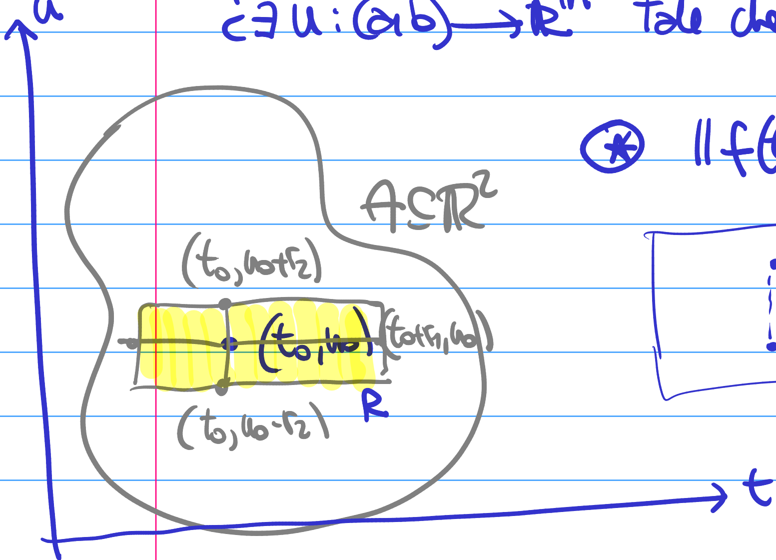
TEOREMA 6.3 (DELLE ITERAZIONI SUCCESSIVE (C.E. PICARD & E.L. LINDELÖF)) *Sia* $A \subseteq \mathbb{R}^2$ *un insieme aperto con* $(t_0, u_0) \in A$ *e* $f \in C(A, \mathbb{R})$. *Siano* $r_1, r_2 > 0$ *due costanti reali tali che il rettangolo* $R = [t_0 - r_1, t_0 + r_1] \times [u_0 - r_2, u_0 + r_2]$ *sia contenuto nell'aperto A e che esista* $L > 0$ *tale che*

$$|f(t,u) - f(t,w)| \leq L|u - w|$$

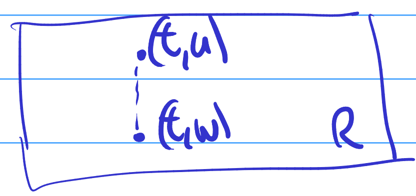
per ogni $t \in [t_0 - r_1, t_0 + r_1]$ *e* $u, w \in [u_0 - r_2, u_0 + r_2]$.
Posto $M = \max_R |f(t,u)|$, *esiste* $\varepsilon > 0$ *tale che il problema di Cauchy (6.1) possiede un'unica soluzione* $u \in C^1[t_0 - \varepsilon, t_0 + \varepsilon]$, *con* $\varepsilon = \min\{r_1, r_2/M\}$.

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$ problema di Cauchy

¿$\exists\, u : (a,b) \longrightarrow \mathbb{R}^n$ Tale che $\big)$?



(*) $\|f(t,u) - f(t,w)\|_2 \leq L \|u - w\|_2 \quad \forall (t,u),(t,w) \in R$



$$\begin{cases} u'(t) = f(t,u(t)) \\ u(t_0) = u_0 \end{cases} \longleftrightarrow \quad u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds$$

$$u(t) = u(t_0) + \int_{t_0}^{t} u'(s)ds = u_0 + \int_{t_0}^{t} f(s,u(s))ds$$

$$\begin{cases} u_0(t) = u_0 \\ u_{k+1}(t) = u_0 + \int_{t_0}^{t} f(s,u_k(s))ds \end{cases} \qquad \circledast \quad \{u_k\}_{k\in\mathbb{N}} \subseteq C^0(I,\mathbb{R}^n)$$

$$u_k \xrightarrow[\text{unif.te}]{} \bar{u} \quad \text{t.c.}$$

$$\searrow k \to +\infty$$

$$\bar{u}(t) = u_0 + \int_{t_0}^{t} f(s,\bar{u}(s))ds$$

$\Rightarrow$ $\bar{u}$ risolve il problema di Cauchy.

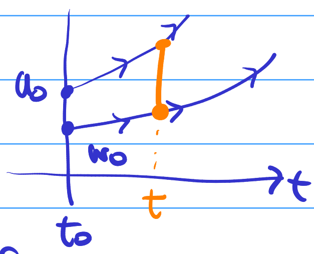
TEOREMA 6.2 (T.H. GRONWALL) *Siano c una costante reale non negativa e u, v : (a, b) → ℝ due funzioni continue e non negative tali che*

$$v(t) \leq c + \left| \int_{t_0}^{t} u(s)v(s)ds \right| \qquad \forall t \in (a,b)$$

Allora

$$v(t) \leq c e^{|U(t,t_0)|} \qquad \text{dove} \qquad U(t,t_0) = \int_{t_0}^{t} u(s)ds$$

$$\begin{cases} u'(t) = f(t,u(t)) \\ u(t_0) = u_0 \end{cases} \qquad \begin{cases} w'(t) = f(t,w(t)) \\ w(t_0) = w_0 \end{cases}$$



$$\downarrow \qquad\qquad \downarrow$$

$$u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds \qquad w(t) = w_0 + \int_{t_0}^{t} f(s,w(s))ds$$

$$|u(t) - w(t)| = \left| u_0 + \int_{t_0}^{t} f(s,u(s))ds - w_0 - \int_{t_0}^{t} f(s,w(s))ds \right|$$

$$\leq |u_0 - w_0| + \left| \int_{t_0}^{t} |f(s,u(s)) - f(s,w(s))| ds \right|$$

Lipschitz $\to \circledast$

$$\leq |u_0 - w_0| + L \left| \int_{t_0}^{t} |u(s) - w(s)| ds \right|$$

Gronwall

$$\Rightarrow \quad |u(t) - w(t)| \leq |u_0 - w_0| e^{L|t-t_0|}$$

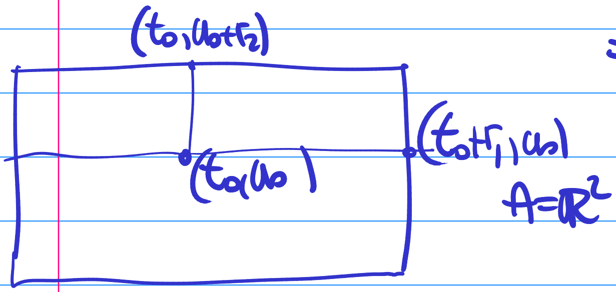
ESEMPI

$$\begin{cases} u'(t) + a(t)u(t) = g(t) \\ u(t_0) = u_0 \end{cases} \qquad a, g \in C^0(\mathbb{R})$$

$$u'(t) = g(t) - a(t)u(t) = f(t, u(t))$$

$$f(t,p) = g(t) - a(t)p \in C^0(\mathbb{R}^2)$$

$$|f(t,p) - f(t,q)| = |g(t) - a(t)p - g(t) + a(t)q|$$



$$= |a(t)| \cdot |p-q| \leq L|p-q|$$

$$L = \max_{t \in [t_0 - r_1, t_0 + r_1]} |a(t)|$$

$$e^{A(t)} \cdot \left[u'(t) + a(t)u(t) = g(t) \right] \qquad A(t) = \int_{t_0}^{t} a(s)\,ds$$

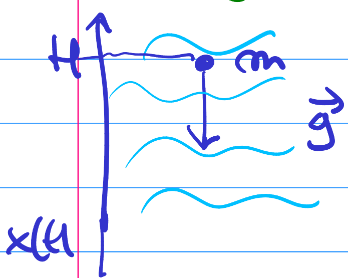
$$e^{A(t)}u'(t) + a(t)e^{A(t)}u(t) = \left(e^{A(t)}u(t)\right)'$$

$$\left[e^{A(t)}u(t)\right]' = g(t)e^{A(t)}$$

$$e^{A(t)}u(t) = c + \int_{t_0}^{t} g(s)e^{A(s)}\,ds$$

$$u(t) = e^{-A(t)}\Big[\underset{\substack{\uparrow \\ u_0}}{c} + \int_{t_0}^{t} g(s)e^{A(s)}\,ds\Big]$$

ESEMPIO 2



$$mx''(t) = -g - ax'(t)$$

$$u(t) = x'(t) \qquad x(t_0) = H$$

$$mu'(t) = -g - \alpha u(t) \qquad x'(t_0) = 0$$

$$t_0 = 0$$

$$u'(t) + \underset{\substack{\uparrow\\ a(t)}}{\frac{\alpha}{m}} u(t) = \underset{\substack{\uparrow\\ g(t)}}{-\frac{1}{m} g}$$

$$A(t) = \int_{t_0}^{t} a(s)\,ds$$

$$\Downarrow$$

$$\int_{t_0}^{t} \frac{\alpha}{m} ds = \frac{\alpha}{m}(t-t_0)$$

$$u(t) = e^{-A(t)}\left[c + \int_{t_0}^{t} g(s) e^{A(s)} ds\right]$$

$$= e^{-\frac{\alpha}{m}(t-t_0)}\left[c + \int_{t_0}^{t} -\frac{1}{m} g\, e^{\frac{\alpha}{m}(s-t_0)} ds\right]$$

$$= c e^{-\frac{\alpha}{m}(t-t_0)} - g/\alpha \left[e^{\frac{\alpha}{m}(s-t_0)}\right]_{t_0}^{t} e^{-\frac{\alpha}{m}(t-t_0)}$$

$$\Rightarrow \quad x'(t) = c e^{-\frac{\alpha}{m}(t-t_0)} - g/\alpha\left[1 - e^{-\frac{\alpha}{m}(t-t_0)}\right] \quad (= u(t))$$

$$x'(t) = -\frac{g}{\alpha} + \underbrace{\left(c + \frac{g}{\alpha}\right)}_{k_0} e^{-\frac{\alpha}{m}(t-t_0)}$$

$$\underset{\substack{\uparrow\\ \text{dati iniziali}}}{\Rightarrow} \quad x'(t) = -\frac{g}{\alpha} + \underset{\substack{\uparrow\\ x'(0) = -\frac{g}{\alpha} + k_0 = 0}}{k_0} e^{-\alpha t} = \frac{g}{\alpha}\left[e^{-\alpha t} - 1\right]$$

ESEMPIO.

$$\begin{cases} u'(t) = u^2(t) \\ u(0) = 1 \end{cases} \qquad f(t,p) = \underset{\substack{\uparrow\\ a(t)}}{1} \cdot \underset{\substack{\uparrow\\ b(p)}}{p^2}$$

$$\int_{t_0}^{t}\left[\frac{u'(t)}{u^2(t)} = 1\right] dt$$

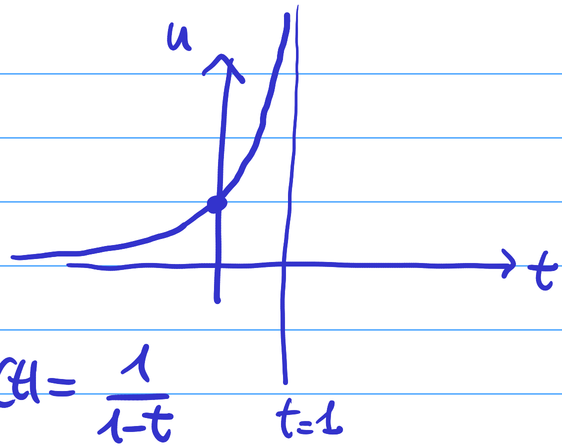
$$\int_{t_0}^{t} 1\, dt = (t - t_0)$$

$$\int_{t_0}^{t} \frac{u'(t)}{u^2(t)} dt \underset{\substack{\uparrow\\ u(t) = w\\ u'(t)dt = dw}}{=} \int_{u(t_0)}^{u(t)} \frac{1}{w^2}\, dw = \left[-w^{-1}\right]_{u(t_0)}^{u(t)} = -\frac{1}{u(t)} + \frac{1}{u(t_0)}$$

$$\Rightarrow -\frac{1}{u(t)} + \frac{1}{\underset{\overset{\shortparallel}{1}}{u(t_0)}} = t - \underset{\overset{\shortparallel}{0}}{t_0}$$

$$\Rightarrow -\frac{1}{u(t)} + 1 = t$$

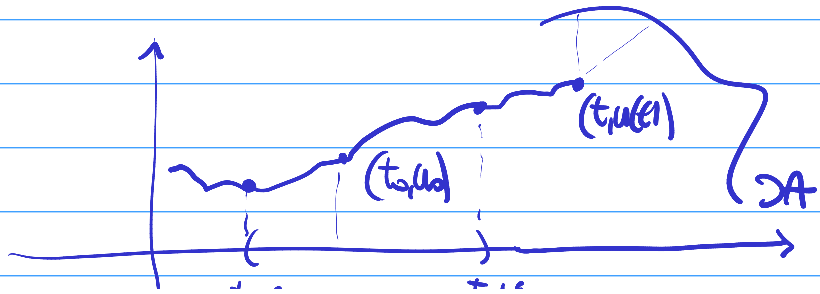
$$\frac{1}{u(t)} = 1 - t \Rightarrow u(t) = \frac{1}{1-t}$$



Teorema 14. *Supponiamo che* $f \in C^0 \cap \mathrm{Lip}_{loc,2}(A, \mathbb{R})$ *e sia* $u : (T_*, T^*) \longrightarrow \mathbb{R}$ *una soluzione massimale di* (7.1). *Se non è vero che* $T^* = +\infty$ *allora*

$$\lim_{t \longrightarrow T^{*-}} \left[|u(t)| + \frac{1}{d((t, u(t)), \partial A)} \right] = +\infty \tag{7.2}$$

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$



TEOREMA 6.5 *Consideriamo il problema di Cauchy (6.1) e sia la funzione f definita in* $A = (a, b) \times \mathbb{R} \subseteq \mathbb{R}^2$*, supponiamo inoltre che per ogni compatto* $K \subseteq (a, b)$ *esistano due costanti* $c_i = c_i(K)$*, con* $i = 1, 2$*, tali che*

$$|f(t, u)| \leq c_1 + c_2|u| \qquad \textit{per ogni } t \in K \textit{ e per ogni } u \in \mathbb{R}$$

Allora la soluzione è prolungabile ad una soluzione definita in tutto (a, b) *(si noti che non è richiesto che l'intervallo* (a, b) *sia limitato!).*

TEOREMA 6.6 *Sia u una soluzione massimale di (6.1) definita su* (a, b)*. Per ogni compatto* $K \subseteq A \subseteq \mathbb{R}^2$ *esiste* $\delta = \delta(K) > 0$ *tale che per ogni* $t \notin (a + \delta, b - \delta)$ *il punto* $(t, u(t))$ *non appartiene a K.*

TEOREMA 6.7 *Sia u una soluzione del problema di Cauchy (6.1) e sia la funzione* $f \in C^1(A)$ *con* $A = (a, b) \times \mathbb{R} \subseteq \mathbb{R}^2$*, supponiamo che esista* $c > 0$ *tale che*

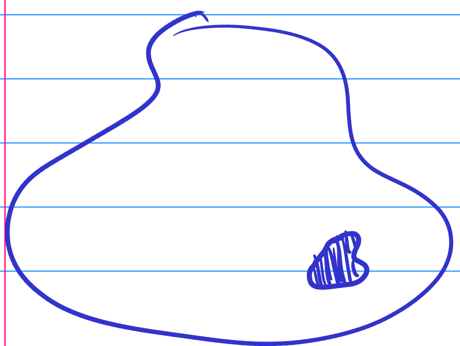
$$|u(t)| \leq c \qquad \textit{per ogni } t$$

allora la soluzione è prolungabile ad una soluzione definita in tutto (a, b).

ESERCIZIO

$$\begin{cases} u'(t) = u(t)(1 - u(t)) \\ u(0) = \lambda \in \mathbb{R} \end{cases} \qquad f(t, p) = p(1-p) \in C^\infty(\mathbb{R}^2) = p - p^2$$

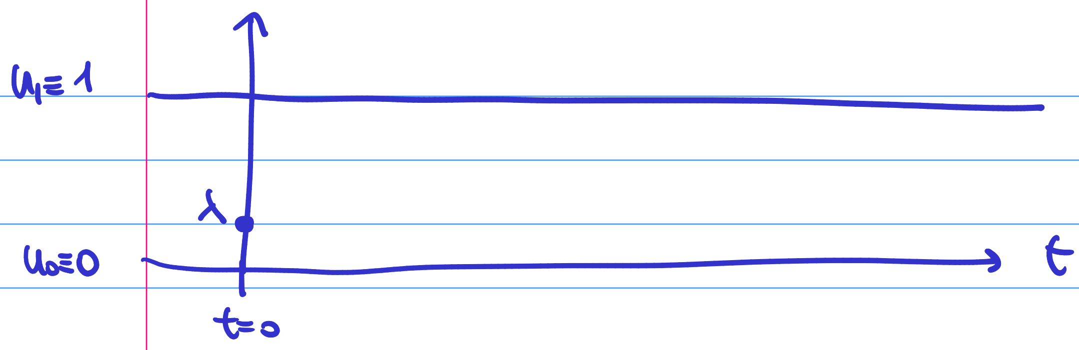
equazione logistica



$u(t) = \frac{z(t)}{N}$ % zombies

$1 - u(t)$ % "sani"

$1 > \lambda > 0$



$$u'(t)=u(t)(1-u(t))$$

$$\int_0^t \left(\frac{u'(t)}{u(t)(1-u(t))}=1\right)dt$$

$$\int_0^t \frac{u'(t)}{u(t)(1-u(t))}\,dt=\int_{u(0)}^{u(t)}\frac{1}{w(1-w)}\,dw=\int_\lambda^{u(t)}\left[\frac{1}{w}+\frac{1}{1-w}\right]dw$$

$w=u(t)$

$dw=u'(t)dt$

$$=\Big[\ln|w|-\ln|1-w|\Big]_\lambda^{u(t)}=\ln\left(\frac{w}{1-w}\right)\Big]_\lambda^{u(t)}$$

$$\Rightarrow \quad \ln\left(\frac{u(t)}{1-u(t)}\right)-\ln\left(\frac{\lambda}{1-\lambda}\right)=t$$

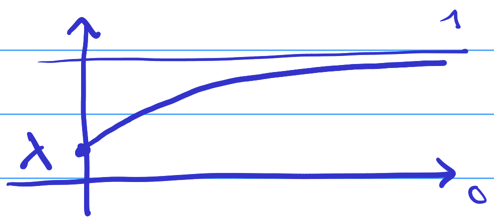
$$\ln\left(\frac{u(t)}{1-u(t)}\cdot\frac{1-\lambda}{\lambda}\right)=t$$

$$\frac{u(t)}{1-u(t)}\cdot\frac{1-\lambda}{\lambda}=e^t$$

$$u(t)=\left[\frac{\lambda}{1-\lambda}\right]e^t\,(1-u(t))$$

$$u(t)\left[1+\frac{\lambda}{1-\lambda}e^t\right]=\frac{\lambda}{1-\lambda}e^t$$

$$u(t)=\frac{\lambda e^t}{(1-\lambda)+\lambda e^t}$$



LUNEDI' 12/01 8-11 aula 4

MARTEDI' 13/01 10-12 aula 3