15/12/2025

$F = (F_1, F_2, F_3)$ campo vettoriale costante

$$W = F \cdot \underset{\substack{\uparrow \\ \text{spostamento}}}{\Delta s} = F_1 \Delta x_1 + F_2 \Delta x_2 + F_3 \Delta x_3$$

$F = F(x)$, $F \in C^1(A, \mathbb{R}^3)$

$dW = F_1(x)dx_1 + F_2(x)dx_2 + F_3(x)dx_3$ "lavoro infinitesimale"

$$dx_i(\vec{PQ}) = (Q_i - P_i)$$

$Q = (Q_1, Q_2, Q_3)$

$P = (P_1, P_2, P_3)$

$$W = \int_\gamma (F_1(x)dx_1 + F_2(x)dx_2 + F_3(x)dx_3)$$

γ curva regolare

$x(t) = (x_1(t), x_2(t), x_3(t))$

$t \in [a,b]$

$$= \int_a^b [F_1(x(t))x_1'(t) + F_2(x(t))x_2'(t) + F_3(x(t))x_3'(t)]dt$$

DEF. $F_1(x)dx_1 + F_2(x)dx_2 + F_3(x)dx_3 = \omega$ 1-forma differenziale

CAMPO VETTORIALE		1-FORMA DIFFERENZIALE
CONSERVATIVO	$\longleftrightarrow$	ESATTA
IRROTAZIONALE	$\longleftrightarrow$	CHIUSA

DEF. $\omega \in C^1(A)$, $A \subseteq \mathbb{R}^3$ aperto se $\exists\, U \in C^2(A)$ tale che

$\partial_i U(x) = F_i(x) \quad \forall i = 1,2,3$

$\Rightarrow \omega$ si dice esatta

$dU = F_1 dx_1 + F_2 dx_2 + F_3 dx_3 = \omega$

[U si dice primitiva di ω]

$dU = \partial_1 U dx_1 + \partial_2 U dx_2 + \partial_3 U dx_3$

$F(x) = (F_1(x), F_2(x), F_3(x)) \in C^1(A, \mathbb{R}^3)$ e' irrotazionale se
$\mathrm{rot}(F)(x) = 0$

$$\mathrm{rot}(F)(x) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1) = 0$$

$$\Rightarrow \left.\begin{array}{l} \partial_2 F_3(x) = \partial_3 F_2(x) \\ \partial_1 F_3(x) = \partial_3 F_1(x) \\ \partial_1 F_2(x) = \partial_2 F_1(x) \end{array}\right\} \Rightarrow \partial_i F_j(x) = \partial_j F_i(x) \quad i \neq j$$

DEF. $\omega = F_1(x)dx_1 + F_2(x)dx_2 + F_3(x)dx_3 \in C^1(A)$ 1-forma differenziale
ω si dice chiusa se

$$\partial_i F_j(x) = \partial_j F_i(x) \quad \forall i \neq j$$

OSS. se ω e' esatta $\Rightarrow$ ω e' chiusa

ω e' esatta $\Rightarrow$ $\omega \in C^1(A)$ e $F(x) = \nabla U(x)$, $U \in C^2(A)$

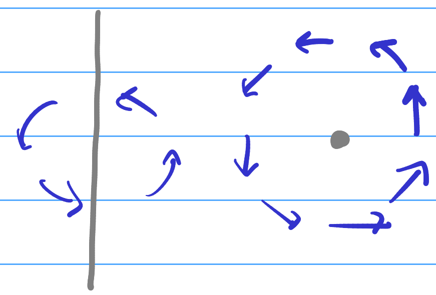
$$\partial_i F_j(x) = \partial_i \partial_j U(x) = \partial_{ij} U(x) = \partial_{ji} U(x) = \partial_j F_i(x)$$

ESEMPIO

$$\omega = -\frac{x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2 + 0\,dx_3$$

$$B(x) = k\left(-\frac{x_2}{x_1^2+x_2^2}, +\frac{x_1}{x_1^2+x_2^2}, 0\right) \quad x \in \mathbb{R}^3 \setminus \{x_1 = x_2 = 0\}$$

- ω e' chiusa



$$\partial_1\left(\frac{x_1}{x_1^2+x_2^2}\right) = \partial_2\left(-\frac{x_2}{x_1^2+x_2^2}\right)$$

$$\frac{x_1^2+x_2^2-x_1(2x_1)}{(x_1^2+x_2^2)^2} = \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}, \quad \frac{-x_1^2-x_2^2+x_2\cdot 2x_2}{(x_1^2+x_2^2)^2} = \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}$$

$\gamma : \{ x(t) = (R\cos(t), R\sin(t), 0) \ , t \in [0, 2\pi] \}$

$$\int_\gamma \omega = \int_0^{2\pi} k \left[\left(- \frac{R\sin(t)}{R^2\cos^2(t) + R^2\sin^2(t)} \right) \cdot (-R\sin(t)) + \left(\frac{R\cos(t)}{R^2} \right) \cdot (R\cos(t)) \right] dt$$

$$= k \int_0^{2\pi} \frac{R^2}{R^2} dt = 2\pi k \neq 0$$

$\Rightarrow$ la 1-forma differenziale associata al campo magnetico B è chiusa ma non è esatta!

ESEMPIO $\omega = - \frac{GMm}{r^3} (x_1 dx_1 + x_2 dx_2 + x_3 dx_3)$

$\gamma \ (x : [a,b] \to \mathbb{R}^3)$ curva regolare a tratti

$$W = \int_\gamma \omega = \int_a^b \left[\sum_{i=1}^3 F_i(x(t)) \cdot x_i'(t) \right] dt$$

$$= - \frac{GMm}{2} \int_a^b \frac{2\, x_1(t)x_1'(t) + x_2(t)x_2'(t) + x_3(t)x_3'(t)}{[x_1^2(t) + x_2^2(t) + x_3^2(t)]^{3/2}} dt$$

$$= + GMm \int_a^b \frac{d}{dt} \left[\frac{1}{[x_1^2(t) + x_2^2(t) + x_3^2(t)]^{1/2}} \right] dt$$

$$= GMm \left[\frac{1}{\|x(b)\|_2} - \frac{1}{\|x(a)\|_2} \right]$$

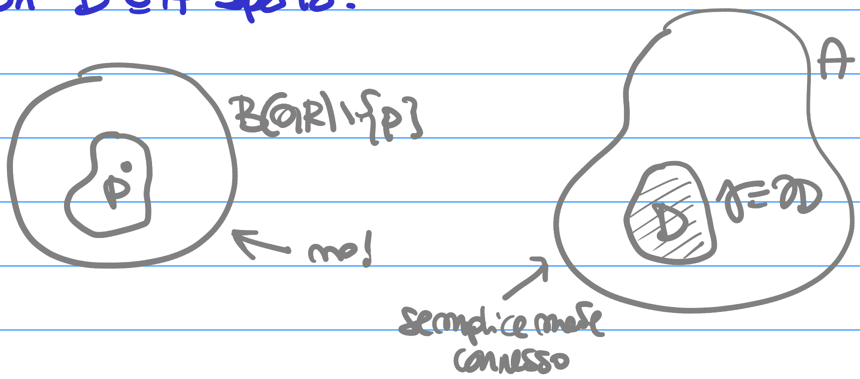
TEOREMA ω 1-forma differenziale di classe $C^1(A)$, $A \subseteq \mathbb{R}^3$ aperto sono equivalenti:

i. ω è esatta,

ii. $\forall \gamma_1, \gamma_2$ regolari a tratti con gli stessi estremi allora $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$,

iii. $\forall \gamma$ curva regolare chiusa vale $\oint_\gamma \omega = 0$.

DEF. $A \subseteq \mathbb{R}^2$ aperto si dice semplicemente connesso se $\forall \gamma$ curva chiusa, semplice e regolare a tratti è $\gamma = \partial D$ con $D \subseteq A$ aperto.



ESEMPI Sono insiemi semplicemente connessi i cerchi, i poligoni e in generale gli insiemi convessi.

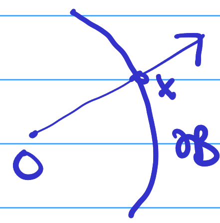
DEF. $A \subseteq \mathbb{R}^3$ aperto, si dice semplicemente connesso se $\forall \gamma$ curva regolare a tratti, semplice e chiusa vale che $\gamma = \partial \Sigma$ con Σ superficie regolare tale che $\Sigma \subseteq A$.

TEOREMA sia $\omega \in C^1(A)$ 1-forma differenziale, $A \subseteq \mathbb{R}^3$ aperto semplicemente connesso allora ω chiusa $\Rightarrow$ ω è esatta

ESERCIZIO 3. *Sia* $F_\alpha(x) = kr^\alpha(x_1, x_2, x_3)$, *con* $r^2 = x_1^2 + x_2^2 + x_3^2 = \|x\|_2^2$, $k \in \mathbb{R}$ *e* $\alpha \in \mathbb{R}$, *un campo centrale, si determini il flusso uscente dalla sfera* $B(O, R)$ *al variare del parametro* α *e si verifichi che* $\mathrm{rot}(F_\alpha) = \mathbf{O}$.

$$F_\alpha(x) = \frac{k\,(x_1, x_2, x_3)}{(x_1^2 + x_2^2 + x_3^2)^{-\alpha/2}} = k\,(x_1^2 + x_2^2 + x_3^2)^{\alpha/2}(x_1, x_2, x_3)$$

$$\Phi_{\partial B(0,R)}(F_\alpha) = \int_{\partial B(0,R)} F_\alpha(x) \cdot N(x)\, d\sigma = \int_{\partial B(0,R)} F_\alpha(x) \cdot \frac{x}{R}\, d\sigma$$



$$= \int_{\partial B(0,R)} kR^\alpha x \cdot \frac{x}{R}\, d\sigma = \int_{\partial B(0,R)} kR^{\alpha+1}\, d\sigma = kR^{\alpha+1} \int_{\partial B(0,R)} d\sigma$$

$$= 4\pi R^{\alpha+3} k$$

$$\mathrm{rot}(F_\alpha) = \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ kr^\alpha x_1 & kr^\alpha x_2 & kr^\alpha x_3 \end{pmatrix}$$

$$= \left(\partial_2(kr^\alpha x_3) - \partial_3(kr^\alpha x_2);\ \partial_3(kr^\alpha x_1) - \partial_1(kr^\alpha x_3);\ \partial_1(kr^\alpha x_2) - \partial_2(kr^\alpha x_1)\right)$$

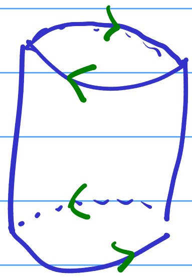
$$= k\left(\alpha r^{\alpha-2}(x_3x_2 - x_2x_3);\ \alpha r^{\alpha-2}(x_1x_3 - x_3x_1);\ \alpha r^{\alpha-2}(x_1x_2 - x_2x_1)\right) = 0$$

$$\underset{i\neq j}{\partial_i}(r^\alpha x_j) = x_j\,\partial_i(r^\alpha) = x_j\,\partial_i\left((x_1^2+x_2^2+x_3^2)^{\alpha/2}\right)$$

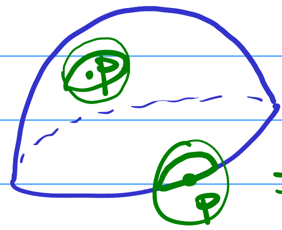
$$= x_j\left(\frac{\alpha}{2}(x_1^2+x_2^2+x_3^2)^{\frac{\alpha}{2}-1}\cdot 2x_i\right)$$

$$= x_j(\alpha r^{\alpha-2}x_i) = \alpha r^{\alpha-2}x_ix_j$$

TEOREMA DEL ROTORE ... $\Rightarrow \displaystyle\int_\Sigma \mathrm{rot}(F)\cdot N\,d\sigma = \int_{\partial\Sigma^+} (F\cdot T)\,ds$



$C =$ cilindro



$S^2 \cap \{x_3 > 0\}$
$= \Sigma$
$\Rightarrow \partial\Sigma = \begin{cases} x_1^2 + x_2^2 = 1 \\ x_3 = 0 \end{cases}$

OSS. $x(u)$ parametrizzazione di un superficie regolare, $u \in K \subseteq \mathbb{R}^2$
$\Rightarrow \partial\Sigma \subseteq x(\partial K)$
$\Sigma = x(K)$

Esercizio 5. *Si calcoli la circuitazione del campo vettoriale* **F** = (a, b, c) *lungo il cammino* $\gamma = \partial\Sigma$, *con* Σ *superficie regolare a tratti.*

$$\oint_{\partial\Sigma=\gamma}(F\cdot T)\,ds \underset{\text{Stokes}}{=} \int_\Sigma \underbrace{\mathrm{rot}(F)}_{=0}\cdot N\,d\sigma = 0$$

ESEMPIO.

$$x(u)=\left(\left[1-\frac{1}{2}u_2\cos(u_1/2)\right]\cos(u_1),\left[1-\frac{1}{2}u_2\cos\left(\frac{u_1}{2}\right)\right]\operatorname{sen}(u_1),-\frac{1}{2}u_2\operatorname{sen}\left(\frac{u_1}{2}\right)\right)$$

$$u=(u_1,u_2)\in[0,2\pi]\times[-1,1]$$

$$N(0,0)=\frac{1}{2}e_3 \qquad N(2\pi,0)=-\frac{1}{2}e_3$$

PROPOSIZIONE Ogni superficie regolare Σ contenuta in $\partial\Omega$ (con $\Omega\subseteq\mathbb{R}^3$ aperto, connesso e limitato) è orientabile.

M25SMRAU codice OPIS

OSSERVAZIONE

$F\in C^1(A,\mathbb{R}^3)$, $A\subseteq\mathbb{R}^3$ aperto

$D\subseteq A$ aperto con frontiera regolare (a tratti)

$$\int_D \operatorname{div}(F)(x)\,dx=\int_{\partial D}[F(x)\cdot N(x)]\,d\sigma$$

$$B_\varepsilon=B(p,\varepsilon)\subseteq D$$

$$\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\operatorname{div}(F)(x)\,dx=\frac{1}{|B_\varepsilon|}\int_{\partial B_\varepsilon}[F(x)\cdot N(x)]\,d\sigma \qquad \forall\varepsilon\leq\varepsilon_0$$

$\operatorname{div}(F)\in C^0$: $\forall\varepsilon>0\ \exists\,\delta=\delta(\varepsilon)$ tale che

$$\operatorname{div}(F)(p)-\delta\leq\operatorname{div}(F)(x)\leq\operatorname{div}(F)(p)+\delta \qquad \forall x\in B_\varepsilon$$

$$\underbrace{\int_{B_\varepsilon}[\operatorname{div}(F)(p)-\delta]\,dx}_{[\operatorname{div}(F)(p)-\delta]\cdot|B_\varepsilon|}\leq\int_{B_\varepsilon}\operatorname{div}(F)(x)\,dx\leq\underbrace{\int_{B_\varepsilon}(\operatorname{div}(F)(p)+\delta)\,dx}_{[\operatorname{div}(F)(p)+\delta]\cdot|B_\varepsilon|}$$

$$\Rightarrow [\operatorname{div}(F)(p)-\delta]\leq\frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\operatorname{div}(F)(x)\,dx\leq[\operatorname{div}(F)(p)+\delta]$$

$$\downarrow \varepsilon\to0^+ \qquad \downarrow \qquad \downarrow \varepsilon\to0^+$$

$$\operatorname{div}(F)(p) \qquad ? \qquad \operatorname{div}(F)(p)$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} \operatorname{div}(F)(x)\,dx = \operatorname{div}(F)(p) \qquad B_\varepsilon = B(p,\varepsilon)$$

$$\Rightarrow \exists \lim_{\varepsilon\to 0^+}\frac{1}{|B_\varepsilon|}\int_{\partial B_\varepsilon} F(x)\cdot N(x)\,d\sigma = \underset{\substack{\uparrow\\ \text{densità di volume}\\ \text{di } \underline{\Phi}(F)}}{\operatorname{div}(F)(p)}$$

OSSERVAZIONE

E campo elettrico, B campo magnetico, ε dielettrico, μ permeabilità, λ conduttività, c velocità della luce

Biot Savart: $$\int_{\gamma=\partial\Sigma^+}(B\cdot T)\,ds = \iint_\Sigma \frac{1}{c}\left[\lambda E+\varepsilon\,\partial_t E\right]\cdot N\,d\sigma$$

Faraday: $$\int_{\gamma=\partial\Sigma^+}(E\cdot T)\,ds = -\frac{1}{c}\iint_\Sigma\left[\mu\,\partial_t B\cdot N\right]d\sigma$$

$$\int_{\gamma=\partial\Sigma^+}(E\cdot T)\,ds = \iint_\Sigma \Big[\underset{\substack{\parallel\\ \nabla\wedge E}}{\operatorname{rot}(E)}\cdot N\Big]d\sigma$$

$$\Rightarrow \iint_\Sigma\left[\nabla\wedge E+\frac{1}{c}\mu\,\partial_t B\right]\cdot N\,d\sigma=0$$

se $(\nabla\wedge E+\frac{1}{c}\mu\,\partial_t B)(p)\neq 0 \Rightarrow \overset{\substack{C^0\\ \cup}}{\Big[}(\nabla\wedge E+\frac{1}{c}\mu\,\partial_t B)(p)\Big]_i = k_0>0\ \ (<0)$ $\quad i=1,2,3$

$$\underset{\substack{\uparrow\\ \text{TEOREMA}\\ \text{PERMANENZA}\\ \text{DEL SEGNO}}}{\Rightarrow} \Big[(\nabla\wedge E+\frac{1}{c}\mu\,\partial_t B)(x)\Big]_i \geq \frac{1}{2}k_0>0 \quad \forall x\in B(p,r_0)$$

scelgo $\Sigma_0 = B(p,r_0)\cap\{x_i=p_i\}$

$$\Rightarrow \iint_{\Sigma_0}\Big[(\nabla\wedge E+\frac{1}{c}\mu\,\partial_t B)(x)\Big]\cdot \underset{\substack{\uparrow\\ e_i}}{N}\,d\sigma \geq k_0\frac{1}{2}\cdot\pi r_0^2>0$$

$$(F) \Rightarrow \quad \mu \partial_t B(x,t) = -c(\nabla \wedge E)(x,t)$$

$$(BS) \Rightarrow \quad \varepsilon \partial_t E(x,t) + \lambda E(x,t) = c(\nabla \wedge B)(x,t)$$

$$\Phi_{\partial D}(E) = \frac{Q}{\varepsilon} = \frac{1}{\varepsilon}\int_D \rho(x,t)\,dx$$

$$\parallel$$

$$\int_{\partial D} (E \cdot N)\,d\sigma = \int_D (\nabla \cdot E)(x,t)\,dx$$

$$\Rightarrow \int_D \left[(\nabla \cdot E)(x,t) - \frac{1}{\varepsilon}\rho(x,t)\right] dx = 0 \qquad \forall D \subseteq A$$

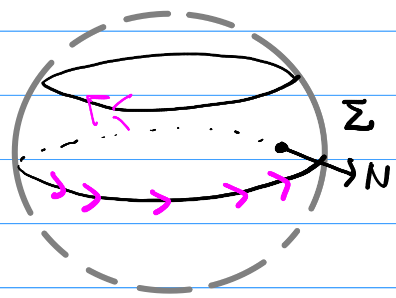
$$\Rightarrow (\nabla \cdot E)(x,t) = \frac{1}{\varepsilon}\rho(x,t)$$

$$(\nabla \cdot B)(x,t) = 0$$

ESERCIZIO.

$F(x) = (x_2x_3, -x_1x_3, x_1x_2)$, $\Sigma = \{x_1^2 + x_2^2 + x_3^2 = 4;\ 0 \le x_3 \le 1\}$

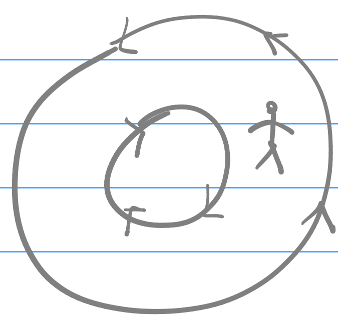
calcolare $\Phi_\Sigma(\mathrm{rot}(F))$



$$\Phi_\Sigma(\mathrm{rot}(F)) = \int_\Sigma \mathrm{rot}(F)(x) \cdot N(x)\,d\sigma$$

$$\parallel$$

$$\oint_{\partial\Sigma^+} (F \cdot T)\,ds$$



$$\begin{cases} x_1 = 2\,\mathrm{sen}(\phi)\cos(\theta) \\ x_2 = 2\,\mathrm{sen}(\phi)\,\mathrm{sen}(\theta) \\ x_3 = 2\cos(\phi) \end{cases} \qquad (\phi,\theta)\in R=\left[\tfrac{\pi}{3},\tfrac{\pi}{2}\right]\times[0,2\pi]$$

$$\partial_1 X(\phi,\theta) = 2\big(\cos(\phi)\cos(\theta), \cos(\phi)\,\mathrm{sen}(\theta), -\mathrm{sen}(\phi)\big)$$

$$\partial_2 X(\phi,\theta) = 2\big(-\mathrm{sen}(\phi)\,\mathrm{sen}(\theta), \mathrm{sen}(\phi)\cos(\theta), 0\big)$$

$$(\partial_1 X \wedge \partial_2 X) = 4\big(\mathrm{sen}^2(\phi)\cos(\theta), \mathrm{sen}^2(\phi)\,\mathrm{sen}(\theta), \mathrm{sen}(\phi)\cos(\phi)\big)$$

$$\Phi_\Sigma(\mathrm{rot}(F)) = \iint_\Sigma 2(x_1,0,-x_3)\cdot N(x)\,d\sigma$$

$$\mathrm{rot}(F)(x) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ x_2x_3 & -x_1x_3 & x_1x_2 \end{vmatrix} = (x_1+x_1, -x_2+x_2, -x_3-x_3)$$

$$= 2(x_1, 0, -x_3)$$

$$= \iint_R 2\big(2\,\mathrm{sen}(\phi)\cos(\theta), 0, -2\cos(\phi)\big)\cdot 4\big(\mathrm{sen}^2(\phi)\cos(\theta), \mathrm{sen}^2(\phi)\,\mathrm{sen}(\theta), \mathrm{sen}(\phi)\cos(\phi)\big)\,d\phi\,d\theta$$

$$= \int_{\pi/3}^{\pi/2}\int_0^{2\pi} 16\left[\mathrm{sen}^3(\phi)\cos^2(\theta) - \mathrm{sen}(\phi)\cos^2(\phi)\right]d\phi\,d\theta$$

$$= 16\int_{\pi/3}^{\pi/2}\left[\pi\,\underbrace{\mathrm{sen}^3(\phi)}_{\mathrm{sen}(\phi)(1-\cos^2(\phi))} - 2\pi\,\mathrm{sen}(\phi)\cos^2(\phi)\right]d\phi$$

$$= 16\pi\int_{\pi/3}^{\pi/2}\left[\mathrm{sen}(\phi) - 3\,\mathrm{sen}(\phi)\cos^2(\phi)\right]d\phi$$

$$= 16\pi\left[-\cos(\phi)+\cos^3(\phi)\right]_{\pi/3}^{\pi/2} = 16\pi\left[\frac{1}{2}-\frac{1}{8}\right] = 6\pi$$

$\gamma_0 \quad x(t) = (2\cos(t), 2\,\mathrm{sen}(t), 0) \qquad t \in [0, 2\pi]$

$\gamma_1 \quad y(t) = (\sqrt{3}\cos(t), -\sqrt{3}\,\mathrm{sen}(t), 1) \qquad t \in [0, 2\pi]$

$$\int_{\gamma_0} (F \cdot T)\, ds = \int_0^{2\pi} (0, 0, 4\,\mathrm{sen}(t)\cos(t)) \cdot (-2\,\mathrm{sen}(t), 2\cos(t), 0)\, dt = 0$$

$$T\,ds = \underbrace{\frac{x'(t)}{\cancel{\|x'(t)\|_2}}}_{T} \cdot \underbrace{\cancel{\|x'(t)\|_2}\, dt}_{ds}$$

$$\int_{\gamma_1} (F \cdot T)\, ds = \int_0^{2\pi} (-\sqrt{3}\,\mathrm{sen}(t), -\sqrt{3}\cos(t), -3\,\mathrm{sen}(t)\cos(t)) \cdot (-\sqrt{3}\,\mathrm{sen}(t), -\sqrt{3}\cos(t), 0)$$

$$= \int_0^{2\pi} [3\,\mathrm{sen}^2(t) + 3\cos^2(t)]\, dt = 3 \cdot 2\pi = 6\pi$$

ESERCIZIO

Dato $F(x) = (x_2, x_1, x_3\sqrt{x_1^2 + x_2^2})$ calcolare $\Phi_{\partial C}(F)$

dove $C = \{x_1^2 - 2x_1 + x_2^2 \leq 0,\ 0 \leq x_3 \leq 1\} \subseteq \mathbb{R}^3$

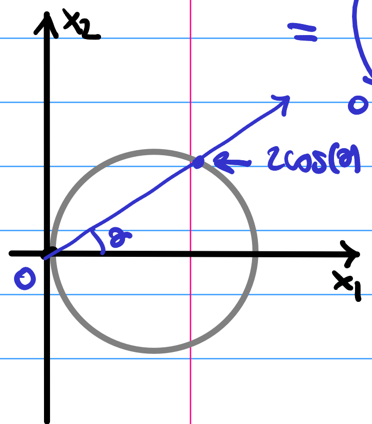
$C = \{(x_1 - 1)^2 + x_2^2 \leq 1\} \times [0, 1]$

$\mathrm{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 = \sqrt{x_1^2 + x_2^2}$

$$\int_C \mathrm{div}(F)\, dx = \int_C [x_1^2 + x_2^2]^{1/2}\, dx \qquad \begin{cases} x_1 = \rho\cos(\vartheta) \\ x_2 = \rho\,\mathrm{sen}(\vartheta) \\ x_3 = t \end{cases} \qquad \det(J) = \rho$$

$$= \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\vartheta)} \rho^2\, d\rho\, d\vartheta\, dt$$

$$\text{"}C\text{"} = \underset{\rho}{[0, 2\cos(\vartheta)]} \times \underset{\vartheta}{\left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]} \times \underset{t}{[0, 1]}$$



$(\rho\cos(\vartheta) - 1)^2 + \rho^2\,\mathrm{sen}^2(\vartheta) \leq 1$

$\rho^2 - 2\rho\cos(\vartheta) \leq 0 \to \rho(\rho - 2\cos(\vartheta)) \leq 0 \to \rho \leq 2\cos(\vartheta)$

$$= \int_0^1 \left[\int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^3(\theta)\, d\theta \right] dt = \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos^3(\theta)\, d\theta$$

$$= \frac{8}{3}\left[\operatorname{sen}(\theta) - \frac{1}{3}\operatorname{sen}^3(\theta) \right]_{-\pi/2}^{\pi/2} = \frac{32}{9}$$

$$\Phi_{\partial C}(F) = \Phi_{S1.}(F) + \Phi_{C_0}(F) + \Phi_{C_1}(F)$$

$$\Phi_{S1.}(F) = \int_0^{2\pi}\int_0^1 \left(\operatorname{sen}(\theta), 1+\cos(\theta), t\left[\operatorname{sen}^2(\theta)+\cos^2(\theta)+2\cos(\theta)+1\right]^{1/2} \right) \cdot$$

$$\cdot \left(\cos(\theta), \operatorname{sen}(\theta), 0 \right) d\theta\, dt$$

$$x(\theta,t) = (1+\cos(\theta), \operatorname{sen}(\theta), t) \qquad \theta\in[0,2\pi],\ t\in[0,1]$$

$$\partial_1 x = (-\operatorname{sen}(\theta), \cos(\theta), 0)$$

$$\partial_2 x = (0, 0, 1)$$

$$\partial_1 x \wedge \partial_2 x = (\cos(\theta), \operatorname{sen}(\theta), 0)$$

$$= \int_0^1\int_0^{2\pi} \left[\operatorname{sen}(\theta)\cos(\theta) + \operatorname{sen}(\theta)(1+\cos(\theta)) \right] d\theta\, dt$$

$$= \int_0^{2\pi} \left[\operatorname{sen}(\theta) + 2\operatorname{sen}(\theta)\cos(\theta) \right] d\theta = \int_0^{2\pi} \left[\operatorname{sen}(\theta) + \operatorname{sen}(2\theta) \right] d\theta = 0$$

$$\Phi_{C_0}(F) = \int_0^{2\pi}\int_0^1 (\rho\operatorname{sen}(\theta), 1+\rho\cos(\theta), 0)\cdot(0,0,\rho)\, d\rho\, d\theta = 0$$

$$C_0 : \ y(\rho,\theta) = (\rho\cos(\theta)+1, \rho\operatorname{sen}(\theta), 0) \qquad (\rho,\theta)\in[0,1]\times[0,2\pi]$$

$$\partial_1 y = (\cos(\theta), \operatorname{sen}(\theta), 0)$$

$$\partial_2 y = (-\rho\operatorname{sen}(\theta), \rho\cos(\theta), 0)$$

$$\partial_1 y \wedge \partial_2 y = (0, 0, \rho)$$

$$\Phi_{C_1}(F) = \int_0^{2\pi}\int_0^1 \left(\rho \operatorname{sen}(\theta), 1+\rho\cos(\theta), \left[\rho^2 \operatorname{sen}^2(\theta) + (1+\rho\cos(\theta))^2\right]^{1/2}\right) \cdot (0,0,\rho)\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \rho\left[\rho^2 + 2\rho\cos(\theta) + 1\right]^{1/2} d\rho\, d\theta$$

Esercizio 10. *Si dimostri che le seguenti affermazioni sono vere o si costruisca un controesempio*
i. $C^0[0,1] \subseteq L^1(0,1)$,
ii. $C^0[0,1] \subseteq L^2(0,1)$,
iii. $C^0[0,1] \subseteq L^\infty(0,1)$,
iv. $C^0(\mathbb{R}) \subseteq L^1(\mathbb{R})$.

i. $f \in C^0[0,1]$ f è continua quindi misurabile, perché se A è aperto $f^{-1}(A)$ è aperto quindi misurabile

osservo che $|f(x)|$ è una funzione continua
$[0,1]$ è compatto $\Rightarrow \exists\, M$ t.c. $|f(x)| \leq M \ \forall x \in [0,1]$

$$\Rightarrow \int_0^1 |f(x)|dx \leq \int_0^1 M\,dx = \int_0^1 M\chi_{[0,1]}(x)dx = M\, m([0,1])$$

$$\Rightarrow f \in L^1(0,1) \Rightarrow C^0[0,1] \subseteq L^1(0,1)$$

ii. se $f \in C^0[0,1] \Rightarrow |f(x)|^2 \leq M^2 \Rightarrow \int_0^1 |f(x)|^2 dx \leq \int_0^1 M^2 dx$

$$\Rightarrow \|f\|_{L^2}^2 \leq M^2$$

iii. se $f \in C^0[0,1] \Rightarrow |f(x)| \leq M \ \forall x \in [0,1]$ (per Weierstrass)
$\Rightarrow f \in L^\infty(0,1)$

iv. $f \in C^0(\mathbb{R}) \Rightarrow f(x) = 1 \Rightarrow 1 \notin L^1(\mathbb{R})$
$f(x) = x \Rightarrow \int_{\mathbb{R}} |x|\, dx = +\infty$

$$\Rightarrow C^0(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$$

Esercizio 8. *Date le seguenti successioni di funzioni*

$$f_k(x) = \cos(kx) \qquad g_k(x) = \frac{1}{k}\sin(kx) \qquad h_k(x) = xe^{-kx} \qquad q_k(x) = \sqrt{k}\,\chi_{(0,1/k)}(x)$$

si determini quali convergono o q.o. in $(0,1)$, *o nello spazio* $L^1(0,1)$ *o in* $L^2(0,1)$, *alla funzione nulla.*

$f_k(x) = \cos(kx) \quad x \in (0,1) \qquad f_k(x) \not\to L \quad \forall x \in (0,1]$

quindi ~~non~~ non ho convergenza q.o.

$g_k(x) = \frac{1}{k}\,\mathrm{sen}(kx) \qquad |g_k(x)| \le \frac{1}{k} \to 0$

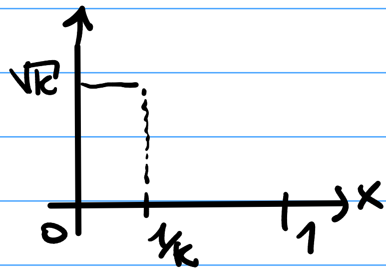
$\Rightarrow g_k(x) \to 0$ q.o. in $(0,1)$

$\|g_k\|_{L^1} = \int_0^1 \frac{1}{k}|\mathrm{sen}(kx)|\,dx \to 0$

$\|g_k\|_{L^1} = \int_0^1 \frac{1}{k}|\mathrm{sen}(kx)|\,dx \le \frac{1}{k}\int_0^1 dx = \frac{1}{k}\, m((0,1)) \to 0$

$\|g_k\|_{L^2}^2 = \int_0^1 \frac{1}{k^2}|\mathrm{sen}(kx)|^2\,dx \le \frac{1}{k^2}\, m((0,1)) \to 0$

$q_k(x) = \sqrt{k}\,\chi_{(0,\frac{1}{k})}(x)$



$\|q_k\|_{L^1} = \int_0^1 |q_k(x)|\,dx = \int_0^{1/k} \sqrt{k}\,dx = \frac{k^{1/2}}{k} = \frac{1}{k^{1/2}} \to 0$

$\|q_k\|_{L^2}^2 = \int_0^1 |q_k(x)|^2\,dx = \int_0^{1/k} k\,dx = \frac{k}{k} = 1 \underset{k\to+\infty}{\not\to} 0$

ESERCIZIO.

$$\begin{cases} u'(x) = u(x) \\ u(0) = 1 \end{cases}$$

Ipotesi $\quad u(x) = \sum_{k=0}^{+\infty} u_k x^k \qquad (u_k \in \mathbb{R},\ \forall k \in \mathbb{N})$

(*)

$$u'(x) = \left(\sum_{k=0}^{+\infty} u_k x^k\right)' \underset{(*)}{=} \sum_{k=0}^{+\infty} (u_k x^k)' = \sum_{k=0}^{+\infty} k u_k x^{k-1}$$

$$\longrightarrow \sum_{k=0}^{+\infty} k u_k x^{k-1} = \sum_{k=0}^{+\infty} u_k x^k$$

$k = j+1$

$$\sum_{j=0}^{+\infty} (j+1) u_{j+1} x^j - \sum_{j=0}^{+\infty} u_j x^j = 0$$

$$\sum_{j=0}^{+\infty} \left[(j+1) u_{j+1} - u_j \right] x^j = 0$$

$$\Rightarrow \boxed{\begin{array}{l} u_j = (j+1) u_{j+1} \quad \forall j \in \mathbb{N} \\ u_0 = 1 \end{array}}$$

$$u_1 = u_0 = 1 \ , \quad u_2 = \frac{u_1}{2} = \frac{1}{2} \ , \quad u_3 = \frac{u_2}{3} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$¿\, u_j = \frac{1}{j!} \, ? \Rightarrow (j+1) u_{j+1} = u_j = \frac{1}{j!} \Rightarrow u_{j+1} = \frac{1}{(1+j)\, j!} = \frac{1}{(j+1)!} \quad (**)$$

$$u(x) = \sum_{j \geq 0} \frac{1}{j!} x^j = e^x$$