Calcolare $\displaystyle\int x^2\sqrt{16+x^6}\,dx$

$$\frac{1}{3}\int 3x^2\sqrt{16+x^6}\,dx = \frac{1}{3}\int\sqrt{16+t^2}\,dt$$

$$x^3=t$$
$$3x^2\,dx=dt$$

Da qui varie possibilità:

1° per parti)

$$\int\sqrt{16+t^2}\,dt = t\sqrt{16+t^2} - \int\frac{(t^2+16)-16}{\sqrt{16+t^2}}\,dt = (*)$$

$$f'(t)=1 \Rightarrow f(t)=t$$
$$g(t)=\sqrt{16+t^2} \Rightarrow g'(t)=\frac{\cancel{2}t}{\cancel{2}\sqrt{16+t^2}}$$

$$= t\sqrt{16+t^2} - \int\sqrt{t^2+16}\,dt + \int\frac{16\,dt}{\sqrt{16+t^2}}$$

$$\Rightarrow \int\sqrt{16+t^2}\,dt = \frac{1}{2}t\sqrt{16+t^2} + 8\int\frac{dt}{\sqrt{16+t^2}} =$$

$$= \quad '' \quad + 2\int\frac{dt}{\sqrt{1+\left(\frac{t}{4}\right)^2}}$$

$$= \quad '' \quad + 2\cdot 4\,\text{settsh}\left(\frac{t}{4}\right)$$

$$= \frac{1}{2}t\sqrt{16+t^2} + 8\,\text{settsh}\left(\frac{t}{4}\right) + c$$

$$(*) = \frac{1}{6}x^3\sqrt{16+x^6} + \frac{8}{3}\log\left(\frac{x^3}{4}+\sqrt{1+\frac{x^6}{16}}\right) + c$$

$$= \frac{1}{6}x^3\sqrt{16+x^6} + \frac{8}{3}\log\left(x^3+\sqrt{16+x^6}\right)\boxed{-\frac{8}{3}\log 4} + c_1$$

2° modo

$$\int \sqrt{16+t^2}\, dt =$$

$\sqrt{16+t^2} = t+s \qquad s = \sqrt{16+t^2} - t > 0$

$16 + \cancel{t^2} = \cancel{t^2} + 2st + s^2$

$t = \frac{16-s^2}{2s}, \quad dt = \frac{(-2s^2)-(16-s^2)}{2s^2} ds$

$= - \frac{s^2+16}{2s^2} ds$

$\sqrt{16+t^2} = s + \frac{16-s^2}{2s} = \frac{s^2+16}{2s}$

$$= \int \frac{s^2+16}{2s} \left(- \frac{(s^2+16)}{2s^2} \right) ds$$

$$= -\frac{1}{4} \int \frac{(s^2+16)^2}{s^3} ds = -\frac{1}{4} \int \frac{s^4 + 32s^2 + 16^2}{s^3} ds =$$

$$= -\frac{1}{4} \int \left(s + \frac{32}{s} + \frac{16^2}{s^3} \right) ds$$

$$= -\frac{1}{4} \left(\frac{s^2}{2} + 32 \log s - \frac{256}{2s^2} \right) + c$$

$$= -\frac{s^2}{8} - 8 \log s + \frac{32}{s^2} + c$$

$s = \sqrt{16+t^2} - t$

$= \sqrt{16+x^6} - x^3$

3° modo

$$\int \sqrt{16+t^2}\, dt =$$

$t = 4 \sinh s$

$\sqrt{16+t^2} = \sqrt{16 + 16 \sinh^2 s} =$

$= 4 \sqrt{1+\sinh^2 s} = 4 \cosh s$

$dt = 4 \cosh s \, ds$

$s = \text{settsh } \frac{t}{4}$

$$= \int 4 \cosh s \cdot 4 \cosh s \, ds =$$

$$= 16 \int \cosh^2 s \, ds$$

$$= 16\int\left(\frac{e^s+e^{-s}}{2}\right)^2 ds = \frac{16}{4}\int \left(e^{2s}+2+e^{-2s}\right)ds =$$

$$= 4\left(\frac12 e^{2s}+2s-\frac12 e^{-2s}\right)$$

$$s = \text{settsh}\,\frac{t}{4} = \log\left(\frac{t}{4}+\sqrt{\frac{t^2}{16}+1}\right)$$

$$e^s = \left(\frac{t}{4} + \frac{\sqrt{t^2+16}}{4}\right)$$

4° modo Stessa sost. $t = 4\sinh s$, ma poi per parti.

$$\int \cosh^2 s\, ds = \int \cosh s \cosh s\, ds =$$

$$f'(s) = \cosh s \Rightarrow f(s) = \sinh s$$

$$g(s) = \cosh s \Rightarrow g'(s) = \sinh s$$

$$= \sinh s \cosh s - \int \sinh^2 s\, ds =$$

$$\cosh^2 s - \sinh^2 s = 1$$

$$\sinh^2 s = 1 + \cosh^2 s$$

$$= \quad '' \quad - \int 1\, ds - \boxed{\int \cosh^2 s\, ds}$$

$$\Rightarrow \int \cosh^2 s\, ds = \frac12 \sinh s \cosh s - \frac12 s$$

$$s = \text{settsh}\,\frac{t}{4} = \log\left(\frac{t}{4}+\sqrt{1+\frac{t^2}{16}}\right)$$

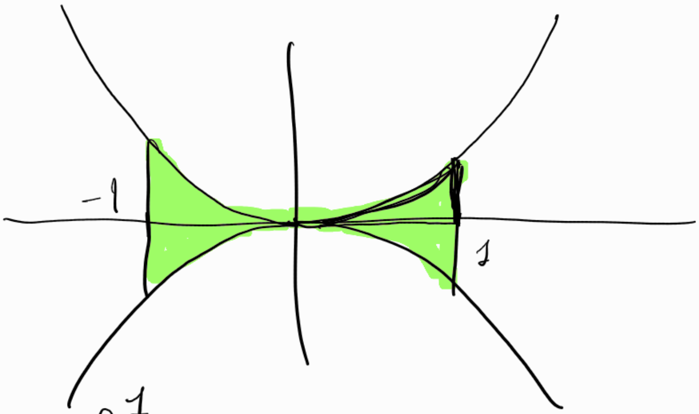
$$= \frac12 \underbrace{\sinh s}_{t/4}\ \underbrace{\cosh s}_{\sqrt{1+\sinh^2 s}\ =\ \sqrt{1+\frac{t^2}{16}}} - \frac12 \log\left(\frac{t}{4}+\sqrt{1+\frac{t^2}{16}}\right) + c$$

$$= \frac{1}{32}\, t\sqrt{16+t^2} - \frac12 \log\left(t+\sqrt{16+t^2}\right) + c_1$$

Calcolare l'area dell'insieme

$$D = \{(x,y): |x| \le 1,\ |y| \le x^2\sqrt{16+x^6}\}$$

$$= \{(x,y): -1 \le x \le 1,\ -x^2\sqrt{16+x^6} \le y \le \underbrace{x^2\sqrt{16+x^6}}_{g(x)}\}$$



$$\text{Area } D = 4\int_0^1 x^2\sqrt{16+x^6}\,dx =$$

$$= 4\left(\frac{1}{6}x^3\sqrt{16+x^6} + \frac{8}{3}\log\left(x^3+\sqrt{16+x^6}\right)\right)\Bigg|_0^1 =$$

$$= 4\left(\frac{1}{6}\sqrt{17} + \frac{8}{3}\left(\log\left(1+\sqrt{17}\right) - \log 4\right)\right)$$

Trovare una formula iterativa per

$$I_n = \int_0^1 (x^4-2)^n \, dx$$

$$I_n = \int_0^1 1\cdot(x^4-2)^n \, dx =$$

$$\begin{cases} f'(x) = 1 \Rightarrow f(x) = x \\ g(x) = (x^4-2)^n \Rightarrow g'(x) = n\,(x^4-2)^{n-1}\,4x^3 = 4n\,x^3(x^4-2)^{n-1} \end{cases}$$

$$= \underbrace{x(x^4-2)^n\Big|_0^1}_{(-1)^n} - 4n\int_0^1 (x^4-2+2)(x^4-2)^{n-1}\,\cancel{dx} =$$

a 1° membro

$$= (-1)^n - 4n\int_0^1 (x^4-2)^n dx - 8n\, I_{n-1}$$

$$\Rightarrow (1+4n)\, I_n = (-1)^n - 8n\, I_{n-1}$$

$$I_n = \frac{1}{1+4n}\left[(-1)^n - 8n\, I_{n-1}\right]$$

Calcolare l'area della regione piana

$$E = \{(x,y) : x \geq 0 \;,\; \log(3+x^2) \leq y \leq \log(3+2x)\}$$

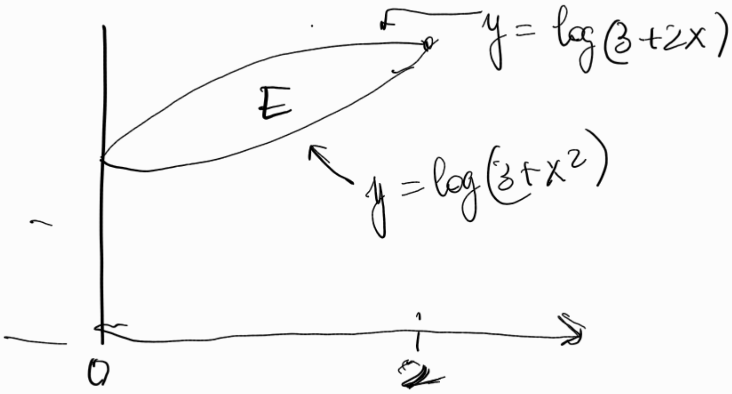
Cerco per quali x $\quad \log(3+x^2) \leq \log(3+2x)$

$$\Updownarrow$$

$$\cancel{3} + x^2 \leq \cancel{3} + 2x$$

$$\Updownarrow$$

$$x^2 - 2x \leq 0$$



$\Updownarrow$

$0 \leq x \leq 2$.

$$\text{Area } E = \int_0^2 \left[\log(3+2x) - \log(3+x^2)\right] dx =$$

$$\int_0^2 1\cdot\log(3+2x)\,dx = x\log(3+2x)\Big|_0^2 - \int_0^2 \frac{2x+3-3}{3+2x}\,dx$$

$$f'(x)=1 \qquad f(x)=x$$

$$g(x)=\log(3+2x) \Rightarrow g'(x) = \frac{2}{3+2x}$$

$$= 2\log 7 - \int_0^2 \left(1 - \frac{3}{3+2x}\right) dx$$

$$= 2\log 7 - 2 + \frac{3}{2}\log(3+2x)\Big|_0^2$$

$$= 2\log 7 - 2 + \frac{3}{2}\left(\log 7 - \log 3\right) =$$

$$\int_0^2 \log(3+x^2)\,dx = x\log(3+x^2)\Big|_0^2 - 2\int \frac{x^2+3-3}{3+x^2}\,dx =$$

$$f'(x)=1 \Rightarrow f(x)=x$$

$$g(x)=\log(3+x^2) \Rightarrow g'(x) = \frac{2x}{3+x^2}$$

$$= 2\log 7 - 2\int_0^2 \left(1 - \frac{3}{3+x^2}\right) dx$$

$$= 2\log 7 - 4 + 6\int_0^2 \frac{dx}{3+x^2} =$$

$$= 2\log 7 - \cancel{4} + \frac{6}{3}\int_0^2 \frac{dx}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx =$$

$$= 2\log 7 - \cancel{4} + 2\sqrt{3}\ \operatorname{arctg}\left(\frac{x}{\sqrt{3}}\right)\Big|_0^2$$

$$= 2\log 7 - \cancel{4} + 2\sqrt{3}\ \operatorname{arctg}\left(\frac{2}{\sqrt{3}}\right)$$

Alla fine viene

$$\text{Area } E = 2 + \frac{3}{2}\log\left(\frac{7}{3}\right) - 2\sqrt{3}\ \operatorname{arctg}\left(\frac{2}{\sqrt{3}}\right)$$

$$\frac{1}{4}\int 4x^3 \log\left(x^8+5x^4+6\right)dx \qquad x^4 = t \qquad 4x^3dx = dt$$

$$= \frac{1}{4}\int \log\left(t^2+5t+6\right)dt$$

$$= \frac{1}{4}\int \log\left((t+2)(t+3)\right)dt$$

$$= \frac{1}{4}\int\left(\log|t+2| + \log|t+3|\right)dt$$

in realtà i moduli non servono perchè $t = x^4 > 0$

$$\int \log|t+2|\,dt \qquad t+2 = s \qquad dt = ds$$

$$= \int \log|s|\,ds = s\log|s| - \int \frac{s}{s}\,ds = s\log|s| - s + c$$

$$= s\left(\log|s| - 1\right) + c$$

$$= (t+2)\left(\log|t+2| - 1\right) + c$$

$$\int \log(t+3)\,dt = (t+3)\left(\log(t+3) - 1\right)$$

L'integrale cercato vale

$$\frac{1}{4}\left[(t+2)\left(\log(t+2)-1\right)+(t+3)\left(\log(t+3)-1\right)\right]\Bigg|_{t=x^4}+c$$

Formula iterativa per

$$I_n=\int_0^1 e^{-2x}\sin^n(\pi x)\,dx =$$

$$f'(x)=e^{-2x} \Rightarrow f(x)=-\frac{1}{2}e^{-2x}$$

$$g(x)=\sin^n(\pi x) \Rightarrow g'(x)=n\pi\sin^{n-1}(\pi x)\cos(\pi x)$$

$$=\underbrace{-\frac{1}{2}e^{-2x}\sin^n(\pi x)\Big|_0^1}_{=0}+\frac{n\pi}{2}\int_0^1 e^{-2x}\sin^{n-1}(\pi x)\cos(\pi x)\,dx$$

$$f'(x)=e^{-2x} \Rightarrow f(x)=-\frac{1}{2}e^{-2x}$$

$$g(x)=\sin^{n-1}(\pi x)\cos(\pi x)$$

$$\Rightarrow g'(x)=\pi(n-1)\sin^{n-2}(\pi x)\cos^2(\pi x)-\underline{\pi\sin^n(\pi x)}$$

$$=\frac{n\pi}{2}\Bigg[\cancel{-\frac{1}{2}e^{-2x}\sin^{n-1}(\pi x)\cos(\pi x)\Big|_0^1}$$

$$+\frac{\pi}{2}(n-1)\int_0^1 e^{-2x}\sin^{n-2}(\pi x)\overbrace{\cos^2(\pi x)}^{1-\sin^2(\pi x)}\,dx$$

$$-\frac{\pi}{2}\underbrace{\int_0^1 e^{-2x}\sin^n(\pi x)\,dx}_{I_n}\Bigg]=$$

$$=-n(n-1)\frac{\pi^2}{4}\left(I_{n-2}-I_n\right)-\frac{\pi^2}{4}n\,I_n$$

$$\int \underbrace{e^{2x}}_{e^x e^x} \log(e^{2x}-2e^x+2)\,dx \qquad e^x = t$$

$$e^x\,dx = dt$$

$$= \int t \log(t^2-2t+2)\,dt =$$

Per parti:

$$f'(t) = t \Rightarrow f(t) = \frac{t^2}{2}$$

$$g(t) = \log(t^2-2t+2) \Rightarrow g'(t) = \frac{2(t-1)}{t^2-2t+2}$$

$$= \frac{t^2}{2}\log(t^2-2t+2) - \int \frac{t^2(t-1)}{t^2-2t+2}\,dt$$

Faccio la divisione

$$= \frac{t^2}{2}\log(\quad) - \int\left(t+1 - \frac{2}{\underbrace{t^2-2t+2}_{(t-1)^2+1}}\right)dt =$$

$$\left[\begin{array}{l|l} t^3 - t^2 & t^2-2t+2 \\ \hline -t^3+2t^2-2t & t+1 \\ \hline \quad t^2-2t & \\ \quad -t^2+2t-2 & \\ \hline \qquad\quad -2 & \end{array}\right]$$

$$= \frac{t^2}{2}\log(\quad) - \frac{t^2}{2} - t + 2\int \frac{dt}{(t-1)^2+1}$$

$$= \quad '' \quad '' \quad '' \quad + 2\,\mathrm{arctg}(t-1) + c$$

$$t = e^x$$

$$\int \frac{dx}{(1+x)\sqrt{|x^2-1|}}$$

Facciamolo separatamente per $x \in (-1,1)$

e $x \in (-\infty,-1) \cup (1,+\infty)$.

Nel 1° caso $x \in (-1,1)$

$$\int \frac{dx}{(1+x)\sqrt{1-x^2}} = *$$

$x = \cos t$ $\quad t \in (0,\pi)$ $\quad t = \arccos x$

$dx = -\sin t\, dt$

$\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = |\sin t| = \sin t$

$$* = \int \frac{-\cancel{\sin t}\, dt}{(1+\cos t)\cancel{\sin t}} = -\int \frac{dt}{1+\cos t} =$$

due possibilità $\tau = \operatorname{tg}\frac{t}{2}$

$$-\int \frac{dt}{1+\cos t} \cdot \frac{1-\cos t}{1-\cos t} = \int \frac{-1+\cos t}{\sin^2 t}\, dt =$$

$$= \left(-\int \frac{1}{\sin^2 t}\, dt\right) + \boxed{\int \frac{\cos t}{\sin^2 t}\, dt}$$

$\left(-\int \frac{1}{\sin^2 t}\, dt\right) = \operatorname{cotg} t$

$\int \frac{\cos t}{\sin^2 t}\, dt = \int \frac{ds}{s^2} = -\frac{1}{s} = -\frac{1}{\sin t}$

$\sin t = s$

$\cos t\, dt = ds$

$$\operatorname{cotg}' t = \left(\frac{\cos t}{\sin t}\right)' = \frac{-\sin^2 t - \cos^2 t}{\sin^2 t} = -\frac{1}{\sin^2 t}$$

Se invece $|x|>1$.

$$\int \frac{dx}{(1+x)\sqrt{x^2-1}}$$

$\sqrt{x^2-1} = x+t$

$x = \begin{cases} \cosh t & \text{se } x>1 \\ -\cosh t & \text{se } x<-1 \end{cases}$

$\sqrt{x^2-1} = \sqrt{\cosh^2 t - 1} = |\sinh t|$