$$f(x) = \begin{cases} \dfrac{\text{sen}(2x)}{x} & \text{se} \quad x<0 \\ a(x^5+1)+bx^2 & \text{se} \quad x \geq 0. \end{cases}$$

Trovare tutti i valori $a, b \in \mathbb{R}$ t.c.:

a) f è continua in $\mathbb{R}$

b) f def^{te} crescente per $x \to +\infty$

c) f ha un flesso in $x_0 = 1$

d) f ammette massimo locale in $x_0 = 1$

e) f è derivabile in $x_0 = 0$.

a) f continua in $\mathbb{R} \setminus \{0\}$.

Continuità in $x_0 = 0$.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[a(x^5+1)+bx^2\right] = a = f(0)$$

continua da destra!

$$\lim_{x \to 0^-} f(x) = \boxed{\lim_{x \to 0^-} \frac{\sin(2x)}{x}} \overset{?}{=} f(0) = a$$

$$\boxed{\lim_{x \to 0^-} \frac{\sin(2x)}{x}} = 2$$

f continua $\Leftrightarrow$ $a = 2$, b qualsiasi.

b) f def^{te} crescente per $x \to +\infty$

$$f'(x) \underset{x>0}{=} \left(a(x^5+1)+bx^2\right)' = \boxed{5ax^4 + 2bx}$$

La domanda equivale a chiedere quando

$$f'(x) \geq 0 \quad \text{def}^{te} \quad \text{per } x \to +\infty$$

Vero se $a > 0$, $\forall b \in \mathbb{R} \Rightarrow \lim_{x \to +\infty} f'(x) = +\infty$

$$\Rightarrow f'(x) > 0 \quad \text{def}^{te} \text{ per } x \to +\infty.$$

Falso se $a<0$ $\forall b\in\mathbb{R} \Rightarrow \lim_{x\to+\infty} f'(x) = -\infty$

Se $a=0 \Rightarrow f(x) = bx^2 \Rightarrow f'(x) = 2bx$

è debolmente crescente in questo caso $\Leftrightarrow b \geq 0$.

Risp: $\{a>0, b\in\mathbb{R}\} \cup \{a=0, b\geq 0\}$

c) f ha un flesso in $x_0 = 1$

$f(x) = a(x^5+1) + bx^2$ vicino a $x_0 = 1$

$f'(x) = 5ax^4 + 2bx$

$f''(x) = 20ax^3 + 2b$

Deve essere $f''(1) = 0 \Leftrightarrow 20a + 2b = 0$

$\Leftrightarrow b = -10a$

$\Rightarrow f(x) = a(x^5+1-10x^2)$

$f''(x) = 20ax^3 - 20a = 20a(x^3-1)$

$f''(1) = 0$

$f'''(x) = 20a\cdot 3x^2 = 60ax^2$

$f'''(1) = 60a$

se $a>0 \Rightarrow f'''(1)>0 \Rightarrow f'''(x)>0$ in un intorno di $x_0=1$.

$\Rightarrow f''(x)$ strett. crescente in un intorno di $x_0=1$

$\Rightarrow f''(x)<0$ in un intorno sinistro di $x_0=1$

>0 " destro " "

$\Rightarrow x_0 = 1$ è un flesso.

se $a<0 \Rightarrow f'''(1)<0 \Rightarrow f'''(x)<0$ in un intorno di $x_0=1$.

$\Rightarrow$ come prima $\Rightarrow$ f ha un flesso in $x_0 = 1$

se $a = 0 \Rightarrow f(x) \equiv 0$ ha flesso in ogni $x_0 > 0$

Risposta $\boxed{b = -10\,a, \quad a \text{ qualsiasi.}}$

Altrimenti: studio del segno di

$$f''(x) = 20\,a(x^3-1) = \underline{20\,a}\,(x-1)\underbrace{(x^2+x+1)}_{>0 \ \forall x}$$

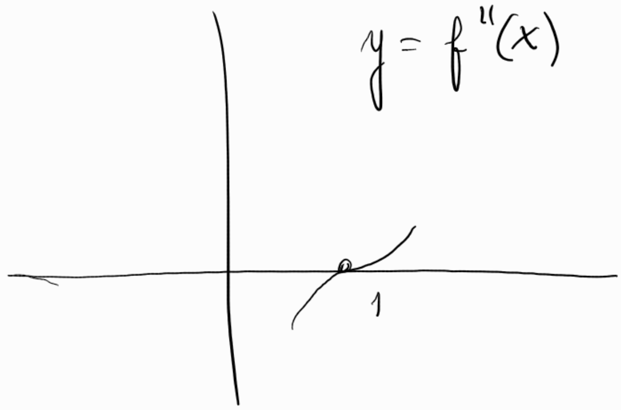
cambia segno attraversando $x_0 = 1$.

se $a > 0 \Rightarrow f''(x) < 0$ a sinistra di 1, > 0 destra di 1 $\Rightarrow$ flesso $x_0 = 1$

$a < 0 \Rightarrow$ al contrario $\Rightarrow$ flesso

$a = 0 \Rightarrow f$ costante $\Rightarrow$ flesso.

$\Rightarrow$ stesse conclusioni.



d) f ha max. locale in $x_0 = 1$.

$f(x) = a(x^5+1) + bx^2$

$f'(x) = 5ax^4 + 2bx$

Deve essere $f'(1) = 0 \Leftrightarrow \quad 5a + 2b = 0 \Leftrightarrow b = -\frac{5}{2}a$

$f'(x) = a(5x^4 - 5x) = 5a(x^4 - x) =$

$= 5a\,\underbrace{x}_{\geq 0}\,(x-1)\,\underbrace{(x^2+x+1)}_{> 0}$

$$f''(x) = 5a\,(4x^3 - 1)$$

$$f''(1) = 15a$$

se $a > 0 \Rightarrow f'(1)=0,\ f''(1)>0 \Rightarrow$ $x_0 = 1$ min. locale stretto.

non è max locale

se $a < 0 \Rightarrow f'(1)=0,\ f''(1)<0 \Rightarrow$ $x_0 = 1$ max. locale stretto.

se $a=0 \Rightarrow f \equiv 0$ è max. locale

Risp. $\quad b = -\frac{5}{2}a, \quad a \leq 0$

e) f derivabile in $x_0 = 0$

Poiché f derivabile $\Rightarrow$ f continua

deve essere $a = 2$

$$f(x) = \begin{cases} \dfrac{\sin(2x)}{x} & \text{se } x<0 \\ 2(x^5+1) + bx^2 & x \geq 0. \end{cases}$$

Poiché f è continua in 0,

$$f'_+(0) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (10x^4 + 2bx) = 0$$

$$f'_-(0) \overset{?}{=} \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \left(\frac{\sin(2x)}{x}\right)' =$$

$$\lim_{x\to 0^-} \frac{2\cos(2x)x - \sin(2x)}{x^2} \underset{H}{\overset{0}{=}} \lim_{x\to 0^-} \frac{-4\sin(2x)\cancel{x} + \cancel{2\cos 2x - 2\cos(2x)}}{2\cancel{x}}$$

$$= 0$$

Quindi la risposta è la stessa che per il punto a)

$$\boxed{a = 2,\ b \text{ qualsiasi}}$$

$$\lim_{x\to 0^-} \frac{2\cos(2x)x - \sin(2x)}{x^2} = \lim_{x\to 0^-} \frac{2x\left(\cancel{1}+o(x)\right) - \cancel{2x} + o(x^2)}{x^2}$$

$$= \lim_{x\to 0^-} \frac{o(x^2)}{x^2} = 0$$

$$\lim_{x\to 0^+} \frac{1-\cosh x \cos x}{x^\alpha} = \lim_{x\to 0^+} \frac{x^4}{6x^\alpha} = \begin{cases} 0 & \text{se } \alpha < 4 \\ \frac{1}{6} & \text{se } \alpha = 4 \\ +\infty & \text{se } \alpha > 4 \end{cases}$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$$

$$1 - \cosh x \cos x = 1 - \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right)$$

$$= \cancel{1} - \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} + o(x^5) - \cancel{\frac{x^2}{2}} + \frac{x^4}{4} - \frac{x^4}{24} =$$

$$= x^4\left(\frac{1}{4} - \frac{1}{12}\right) + o(x^5) = \frac{x^4}{6} + o(x^5) \sim \frac{x^4}{6}$$

$$\left(\frac{1-\sqrt{1-x}}{-1+\sqrt{1+x}}\right)^\alpha - 1 \qquad x \to 0^+ \qquad \alpha > 0$$

$$\frac{1-\sqrt{1-x}}{-1+\sqrt{1+x}} \cdot \frac{-1-\sqrt{1+x}}{-1-\sqrt{1+x}} = \frac{-1-\sqrt{1+x}+\sqrt{1-x}+\sqrt{1-x^2}}{1-(1+x)} =$$

$$= \frac{1+\sqrt{1+x}-\sqrt{1-x}-\sqrt{1-x^2}}{x} =$$

$$= \frac{\cancel{1} + \cancel{1} + \frac{x}{2} - \cancel{\frac{x^2}{8}} + o(x^2) - \left(\cancel{1} - \frac{x}{2} - \cancel{\frac{x^2}{8}}\right) - \left(\cancel{1} - \frac{x^2}{2}\right)}{x}$$

$$= \frac{x + \frac{x^2}{2} + o(x^2)}{x} = 1 + \frac{x}{2} + o(x)$$

$$\left(\frac{1-\sqrt{1-x}}{-1+\sqrt{1+x}}\right)^{\alpha} - 1 = \Big(1 + \underbrace{\frac{x}{2} + o(x)}_{\downarrow\, 0}\Big)^{\alpha} - 1 \sim$$

$$\left[(1+t)^{\alpha} - 1 \sim \alpha t \quad \text{per } t \to 0\right]$$

$$\sim \alpha\left(\frac{x}{2} + o(x)\right) \sim \frac{\alpha x}{2}$$ infinitesimo di ordine 1.

$$\int \sqrt{1+x^2}\, dx = \int 1 \cdot \sqrt{1+x^2} =$$

Per parti:

$$f'(x) = 1 \Rightarrow f(x) = x$$

$$g(x) = \sqrt{1+x^2} \Rightarrow g'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}$$

$$= x\sqrt{1+x^2} - \int \frac{x^2+1-1}{\sqrt{1+x^2}}\, dx$$

$$= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\, dx + \int \frac{dx}{\sqrt{1+x^2}}$$ = settsinh x = $= \log\left(x + \sqrt{1+x^2}\right)$

$$\Rightarrow \quad \cancel{2}\int \sqrt{1+x^2}\, dx = \frac{1}{2}\left(x\sqrt{1+x^2} + \log\left(x + \sqrt{1+x^2}\right)\right) + c$$

$$\int$$

$$\int_{-\pi/2}^{\pi/2} \cos^4 x \, dx \underset{\uparrow}{=} 2 \int_0^{\pi/2} \cos^4 x \, dx =$$

f è pari

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$= 2 \int_0^{\pi/2} \left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(1 + \cancel{2\cos(2x)} + \cos^2(2x)\right) dx =$$

non dà contributo

$$\int_0^{\pi/2} 2\cos(2x) dx = \sin(2x)\Big|_0^{\pi/2} = 0$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(1 + \frac{1+\cos(4x)}{2}\right) dx =$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{3}{2} + \frac{1}{2}\cos(4x)\right) dx =$$

$$= \frac{1}{2}\left(\frac{3}{2} \cdot \frac{\pi}{2} + \cancel{\frac{1}{2} \frac{\sin(4x)}{4}\Big|_0^{\pi/2}}\right) = \frac{3\pi}{8}$$

Oppure per parti:

$$\int_0^{\pi/2} \cos^4 x \, dx = \int_0^{\pi/2} \cos x \cos^3 x \, dx =$$

$$f'(x) = \cos x \Rightarrow f(x) = \sin x$$

$$g(x) = \cos^3 x \Rightarrow g'(x) = -3\cos^2 x \sin x$$

$$= \cancel{\sin x \cos^3 x \Big|_0^{\pi/2}} + 3 \int_0^{\pi/2} \cos^2 x \underbrace{\sin^2 x}_{= 1-\cos^2 x} dx$$

$$= \quad 3\int_0^{\pi/2} \cos^2 x\,dx - 3\int_0^{\pi/2}\cos^4 x\,dx \quad \text{a } 1^{\circ} \text{ membro}$$

$$\Rightarrow \quad 4\int_0^{\pi/2}\cos^4 x\,dx = 3\int_0^{\pi/2}\cos^2 x\,dx =$$

$$= 3\int_0^{\pi/2}\frac{1+\cancel{\cos(2x)}}{2}\,dx = \frac{3}{2}\frac{\pi}{2} = \frac{3\pi}{4}$$

$$\Rightarrow \int_0^{\pi/2}\cos^4 x\,dx = \frac{3\pi}{16}$$

$$\lim_{x\to 0}\frac{e^x - e^{\sin x}}{\operatorname{tg} x - x} = \lim_{x\to 0}\frac{x^3/6}{x^3/3} = \frac{3}{6} = \frac{1}{2}$$

$$D = \operatorname{tg} x - x = \cancel{x} + \frac{x^3}{3} + o(x^3) - \cancel{x} \sim \frac{x^3}{3} \quad \text{per } x\to 0$$

$$N = e^x - e^{\sin x} = \qquad e^t = 1 + t + \frac{t^2}{2} + o(t^2) \quad t\to 0$$

$$= \cancel{1} + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) +$$

$$-\left(\cancel{1} + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + o(x^3)\right) =$$

$$= \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) - \left(\cancel{x} - \frac{x^3}{6} + o(x^3)\right) +$$

$$-\frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 - \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3 =$$

$$= \cancel{\frac{x^2}{2}} + \frac{x^3}{6} + o(x^3) + \cancel{\frac{x^3}{6}} - \cancel{\frac{1}{2}(x^2)} - \cancel{\frac{1}{6}x^3} =$$

$$= \frac{x^3}{6} + o(x^3) \sim \frac{x^3}{6}$$