Oggi lezione Analisi I $9 \to 12$
Esercitazione facoltativa 14–16

Domani $8 \to 9$ "Assaggi di Magistrale" (Aula 7)

verificare
$9 \to 13$ Geometria (Bandinelli)
$13 \to 15$ Analisi I (Bandinelli – Aula 14)

$\int_a^b f(x)\,dx$ è l'integrale di Riemann (integrale definito)

$\int f(x)\,dx$ è l'insieme delle primitive di f (integrale indefinito)

Il primo teorema fondamentale del calcolo integrale dice
Se f è continua in I intervallo, allora

$$\int f(x)\,dx = \int_c^x f(t)\,dt + c_1$$

$(c \in I)$

Il secondo teorema fondamentale dice:

Se f è continua in $[a,b]$,

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]\Bigg|_a^b$$

Esempio:

$$\int_1^2 \frac{1}{x}\,dx = \log x\Big|_1^2 = \log 2 - \cancel{\log 1} = \log 2.$$

$$\int_2^3 (x^3-4x^2+5x-7)\,dx = \left(\frac{x^4}{4}-\frac{4}{3}x^3+\frac{5}{2}x^2-7x\right)\Bigg|_2^3 =$$

$$= \frac{1}{4}(81-16) - \frac{4}{3}(27-8) + \frac{5}{2}(9-4) - 7(3-2)$$

Questo riconduce il calcolo degli integrali di Riemann al calcolo delle primitive (cioè gli integrali indefiniti). Costruiamo una tabella di integrali indefiniti a partire dalle tabelle delle derivate.

$f(x)$	$\int f(x)\,dx$
c costante	$cx + c_1 \quad (c_1 \in \mathbb{R})$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$ Esempio $\int x^{3/2}\,dx = \frac{2}{5}x^{5/2} + c$ (in $[0,+\infty)$)
$\frac{1}{x}$	$\left.\begin{array}{ll}\log x + c & \text{se } x>0 \\ \log(-x) + c & \text{se } x<0\end{array}\right\} \log\lvert x\rvert + c$

$$\int_{-3}^{-2} \frac{dx}{x} = \log|x|\,\Big|_{-3}^{-2} = \log 2 - \log 3 = \log\frac{2}{3}$$

$\sqrt{x}$	$\frac{2}{3}x^{3/2} + c$
$\frac{1}{x^2}$	$-\frac{1}{x} + c$

$\frac{1}{x^3}$	$-\frac{1}{2x^2} + c$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$
e^x	$e^x + c$
a^x	$\frac{a^x}{\log a} + c \qquad \forall a > 0,\ a \neq 1$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\frac{1}{\cos^2 x} = 1 + tg^2 x$	$tg\, x + c \qquad x \neq \frac{\pi}{2} + k\pi$
$\frac{1}{\sin^2 x}$	$-cotg\, x + c = -\frac{\cos x}{\sin x} + c \qquad x \neq k\pi$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c = -\arccos x + c_1$
$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x + c$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sinh x + c$
$\frac{1}{\sqrt{1+x^2}}$	$settsinh\, x + c = \log\left(x + \sqrt{1+x^2}\right) + c$
$\frac{1}{\sqrt{x^2-1}}$	$settcosh\, x + c = \log\left(x + \sqrt{x^2-1}\right) + c$ per $x > 1$
	$settcosh(-x) + c = \log\left(-x + \sqrt{x^2-1}\right) + c$ per $x < -1$

$\lvert x\rvert$	$\frac{x\lvert x\rvert}{2}+c$

$$\int_1^2 \left[(x-1)^3+\sin x+\frac{3}{x}\right]dx = \left(\frac{(x-1)^4}{4}-\cos x+3\log x\right)\Bigg|_1^2 =$$

$$= \frac{1}{4} - \cos 2+\cos 1 + 3\log 2 - \cancel{3\log 1}$$

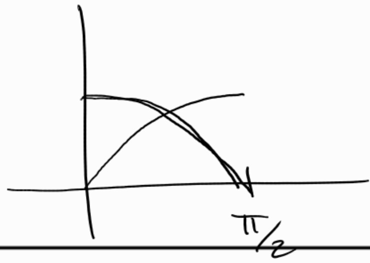
$$\int_0^1 \cos(5x-1)\,dx = \frac{\operatorname{sen}(5x-1)}{5}\Bigg|_0^1 = \frac{1}{5}\left[\operatorname{sen}4+\operatorname{sen}1\right]$$

$$\int_0^\pi \cos^2 x\,dx = \frac{1}{2}\int_0^\pi\left(1+\cos(2x)\right)dx = \frac{1}{2}\left(\pi+\cancel{\frac{\operatorname{sen}(2x)}{2}}\right)\Bigg|_0^\pi = \frac{\pi}{2}$$

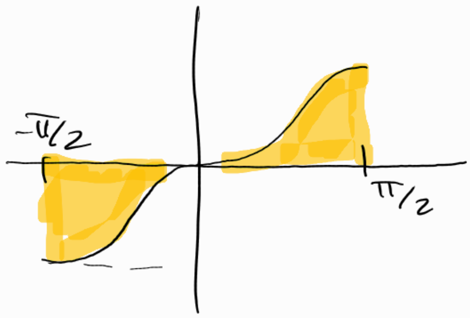
$$\cos(2x) = 2\cos^2 x-1 \implies \boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$$

$$\cos(2x) = 1-2\sin^2 x \implies \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\int_0^\pi \sin^2 x\,dx = \frac{1}{2}\int_0^\pi\left(1-\cancel{\cos(2x)}\right)dx = \frac{\pi}{2}$$



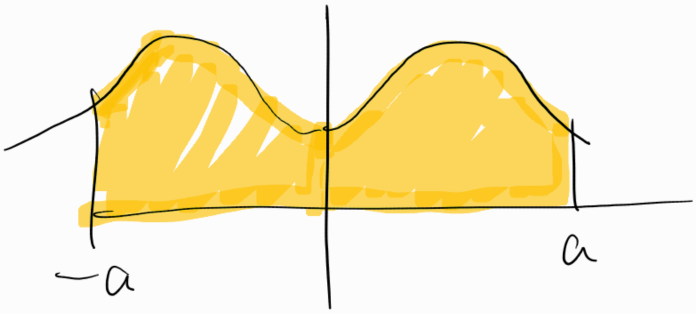
$$\int_{-\frac{\pi}{2}}^{\pi/2} \sin^5 x\,dx = 0$$



Più in generale

$$\int_{-a}^{a} f(x)\,dx = 0 \qquad \text{se } f \text{ è dispari} \quad \forall a>0$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \qquad \text{se } f \text{ è pari}$$



$$\int_{-\frac{\pi}{2}}^{\pi/2} \cos^3 x \, dx = 2\int_0^{\pi/2} \cos^3 x \, dx = 2\int_0^{\pi/2} \cos x \,(1-\sin^2 x)\, dx =$$

$$= 2\int_0^{\pi/2} (\cos x - \cos x \sin^2 x)\, dx =$$

$$= 2\left(\sin x - \frac{\sin^3 x}{3}\right)\Bigg|_0^{\pi/2} = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$$

Integrazione per parti

Nasce dalla formula per la derivata del prodotto.

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Supponiamo $f, g \in C^1([a,b])$, cioè derivabili in $[a,b]$ con derivata continua.

$$\Rightarrow \quad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \forall x \in [a,b]. \quad (*)$$

Integro tra a e b.

$$\underbrace{\int_a^b (f(x)g(x))'\,dx}_{= f(x)g(x)\big|_a^b} = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

Riscrivo

$$\boxed{\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx}$$

Formula di integrazione per parti per integrali definiti.

Se invece a partire da $(*)$ prendo gli integrali indefiniti, ottengo la formula di integrazione per parti per integrali indefiniti.

$$\boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx}$$

$$\int x e^x\,dx = xe^x - \int e^x \cdot 1\,dx = xe^x - e^x + c$$
$$= e^x(x-1) + c$$

$e^x = f'(x) \Rightarrow f(x) = e^x$

$x = g(x) \Rightarrow g'(x) = 1$

Se invece avessi scelto

$$f'(x) = x \Rightarrow f(x) = \frac{x^2}{2}$$

$$g(x) = e^x \Rightarrow g'(x) = e^x$$

avrei ottenuto

$$\int x e^x \, dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx$$

non conviene perché arrivo a un integrale più difficile.

Stesso esercizio ma per integrali definiti.

$$\int_0^1 x e^x \, dx$$

1° modo Usando l'integrale indefinito (già calcolato) ottengo

$$\int_0^1 x e^x \, dx = e^x(x-1)\Big|_0^1 = 0 + 1 = 1$$

2° modo

$$\int_0^1 x e^x \, dx = \underbrace{x e^x \Big|_0^1}_{= e} - \underbrace{\int_0^1 e^x \, dx}_{e^x \big|_0^1 = e - 1} =$$

$$f'(x) = e^x \Rightarrow f(x) = e^x$$

$$g(x) = x \Rightarrow g'(x) = 1$$

$$= e - (e-1) = 1$$

Il metodo si può iterare:

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx = x^2 e^x - 2e^x(x-1) + c$$

$$= e^x(x^2 - 2x + 2) + c$$

$$f'(x) = e^x \Rightarrow f(x) = e^x$$

$$g(x) = x^2 \Rightarrow g'(x) = 2x$$

$$\int_1^e \log x\,dx = \int_1^e 1\cdot \log x\,dx = x\log x\Big|_1^e - \int_1^e \underbrace{x\cdot\frac{1}{x}}_{1}\,dx$$

$$\begin{bmatrix} f'(x)=1 \Rightarrow f(x)=x \\ g(x)=\log x \Rightarrow g'(x)=\frac{1}{x}\end{bmatrix} \qquad = e-(e-1)=1$$

Integrale indefinito

$$\int \log x\,dx = x\log x - \int 1\,dx = x\log x - x + c =$$
$$= x(\log x - 1) + c$$

$$\int \log(x-3)\,dx = \int 1\cdot\log(x-3)\,dx =$$

$$f'(x)=1 \Rightarrow f(x)=x-3$$
$$g(x)=\log(x-3) \Rightarrow g'(x)=\frac{1}{x-3}$$

$$= (x-3)\log(x-3) - \int (x-3)\frac{1}{x-3}\,dx =$$

$$= (x-3)\log(x-3) - x + c$$

$$\int_0^\pi e^x\cos(3x)\,dx =$$

$$f'(x)=e^x \Rightarrow f(x)=e^x$$
$$g(x)=\cos(3x) \Rightarrow g'(x)=-3\sin(3x)$$

$$= e^x\cos(3x)\Big|_0^\pi - \int_0^\pi (-3\sin(3x))\,e^x\,dx =$$

$$= e^\pi(-1) - 1 + 3\int_0^\pi \sin(3x)\,e^x\,dx$$

$f'(x) = e^x \Rightarrow f(x) = e^x$

$g(x) = \sin(3x) \Rightarrow g'(x) = 3\cos(3x)$

$$= -e^{\pi} - 1 + 3\left[\underbrace{\cancel{e^x \sin(3x)\Big|_0^{\pi}}}_{\overset{\shortparallel}{0}} - 3\int_0^{\pi} e^x \cos(3x)\,dx\right]$$

$$\Rightarrow \int_0^{\pi} e^x \cos(3x)\,dx = -\left(e^{\pi}+1\right) - 9\int_0^{\pi} e^x \cos(3x)\,dx$$

$$\Rightarrow \cancel{10}\int_0^{\pi} e^x \cos(3x)\,dx = \frac{-\left(e^{\pi}+1\right)}{10}$$

$$\int \sin(3x)\cos x\,dx =$$

$f'(x) = \cos x \Rightarrow f(x) = \sin x$

$g(x) = \sin(3x) \Rightarrow g'(x) = 3\cos(3x)$

$$= \sin x \sin(3x) - 3\int \sin x \cos(3x)\,dx =$$

$f'(x) = \sin x \Rightarrow f(x) = -\cos x$

$g(x) = \cos(3x) \Rightarrow g'(x) = -3\sin(3x)$

lo porto a 1° membro

$$= \sin x \sin(3x) - 3\left[-\cos x \cos(3x) - 3\int \cos x \sin(3x)\,dx\right] =$$

$$= \sin x \sin(3x) + 3\cos x \cos(3x) + 9\int \cos x \sin(3x)\,dx$$

$$\Rightarrow (1-9)\int \cos x \sin(3x)\,dx = \sin x \sin(3x) + 3\cos x \cos(3x) + C$$

$$\Rightarrow \int \cos x \sin(3x)\,dx = -\frac{1}{8}\left(\sin x \sin(3x) + 3\cos x \cos(3x)\right) + C_1$$

Si potevano anche usare le formula di Werner

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right].$$

Avremmo trovato

$$\int \sin(3x)\cos x\,dx = \frac{1}{2}\int\left[\sin(4x)+\sin(2x)\right]dx$$

$$= \frac{1}{2}\left(-\frac{\cos(4x)}{4}-\frac{\cos(2x)}{2}\right)+c$$

Si può provare che le due funzioni trovate con i due metodi differiscono per una costante.

$$\int \cos^2 x\,dx$$

<u>1° modo</u> (già visto): formule di bisezione

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\int \cos^2 x\,dx = \frac{1}{2}\int\left(1+\cos(2x)\right)dx = \frac{1}{2}\left(x+\frac{\sin(2x)}{2}\right)+c$$

<u>2° modo</u>: per parti

$$\int \cos^2 x\,dx = \int \cos x\cos x\,dx =$$

$$f'(x)=\cos x \Rightarrow f(x)=\sin x$$

$$g(x)=\cos x \Rightarrow g'(x)=-\sin x$$

$$= \sin x\cos x + \int \underbrace{\sin^2 x\,dx}_{=\,1-\cos^2 x\,dx}$$

porto a 1° membro

$$= \sin x\cos x + x - \int \cos^2 x\,dx$$

$$\Rightarrow \cancel{2}\int \cos^2 x\, dx = \frac{\sin x \cos x + x}{2} + c.$$

e questo è la stessa funzione di prima.

$$\int \operatorname{arctg} x\, dx = \int 1 \cdot \operatorname{arctg} x\, dx =$$

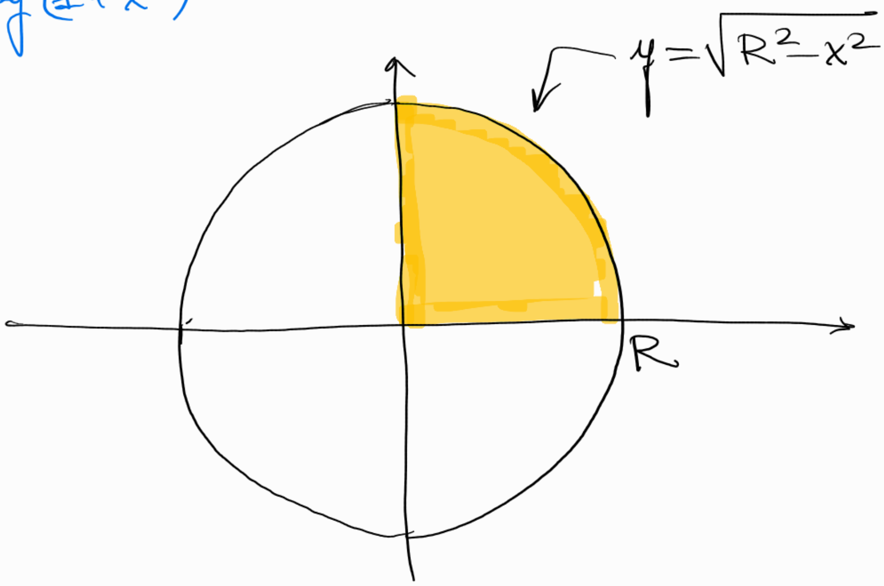
$$f'(x) = 1 \Rightarrow f(x) = x$$

$$g(x) = \operatorname{arctg} x \Rightarrow g'(x) = \frac{1}{1+x^2}$$

$$= x \operatorname{arctg} x - \frac{1}{2}\underbrace{\int \frac{2x}{1+x^2}\, dx}_{\log(1+x^2)} = x \operatorname{arctg} x - \frac{1}{2}\log(1+x^2) + c$$

Area del cerchio

$$x^2 + y^2 \leq R^2$$



$$\text{Area} = 4\int_0^R \sqrt{R^2-x^2}\, dx$$

Calcolo $\displaystyle\int_0^R \sqrt{R^2-x^2}\, dx = \int_0^R 1 \cdot \sqrt{R^2-x^2}\, dx =$

$$f'(x) = 1 \Rightarrow f(x) = x$$

$$g(x) = \sqrt{R^2-x^2} \Rightarrow g'(x) = \frac{-2x}{2\sqrt{R^2-x^2}} = -\frac{x}{\sqrt{R^2-x^2}}$$

$$= \cancel{x\sqrt{R^2-x^2}\Big|_0^R} + \int_0^R \frac{x^2}{\sqrt{R^2-x^2}}\, dx =$$

porto a 1° membro

$$= \int_0^R \frac{x^2 - R^2 + R^2}{\sqrt{R^2 - x^2}}\, dx = \boxed{- \int_0^R \sqrt{R^2 - x^2}\, dx} + \int_0^R \frac{R^2}{\sqrt{R^2 - x^2}}\, dx$$

$$\cancel{2} \int_0^R \sqrt{R^2 - x^2}\, dx = \frac{R^2}{2} \int_0^R \frac{dx}{\sqrt{R^2 - x^2}} =$$

$$= \frac{R^2}{2} \int_0^R \frac{dx}{R\sqrt{1 - \left(\frac{x}{R}\right)^2}} = \frac{R^2}{2} \arcsin\left(\frac{x}{R}\right)\Bigg|_0^R =$$

$$\left[\left(\arcsin\left(\frac{x}{R}\right)\right)' = \frac{1/R}{\sqrt{1 - \left(\frac{x}{R}\right)^2}} \right] = \frac{R^2}{2} \arcsin 1 = \frac{R^2}{2} \frac{\pi}{2}$$

$$\Rightarrow \text{Area del cerchio} = 4 \int_0^R \sqrt{R^2 - x^2}\, dx = 4 \frac{\pi R^2}{4} = \pi R^2$$