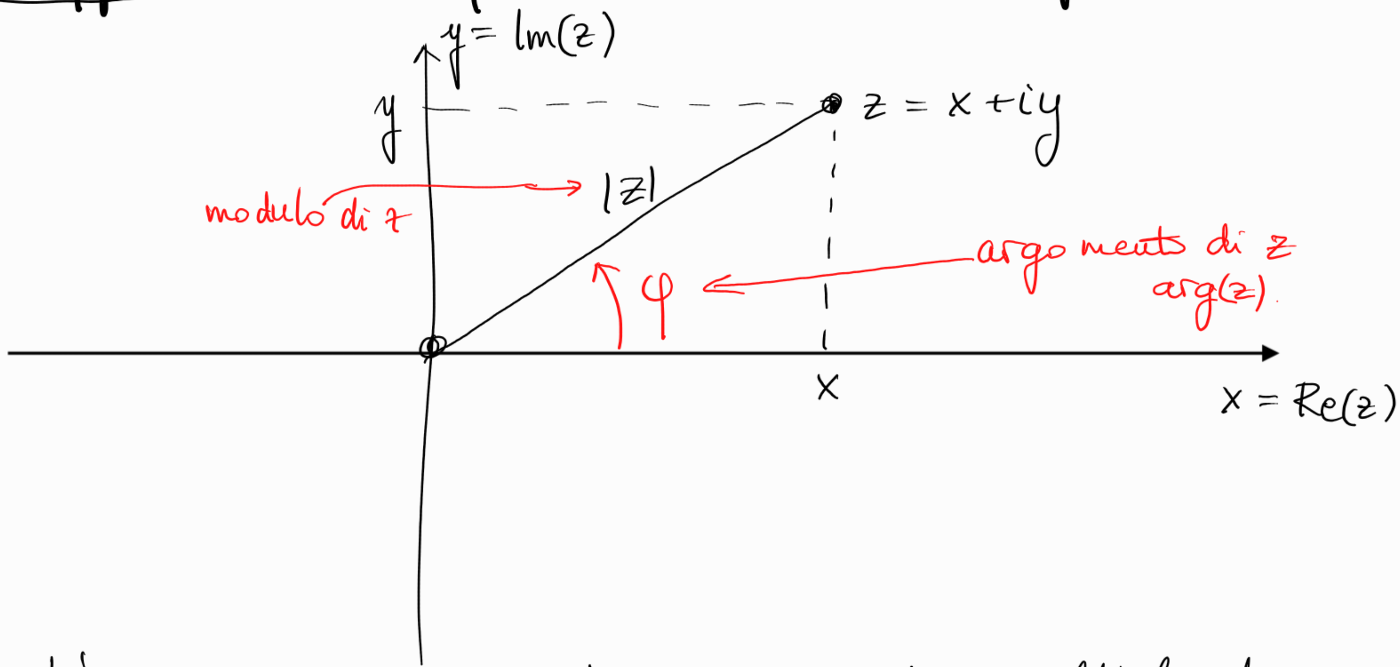
Rappresentazione polare dei numeri complessi



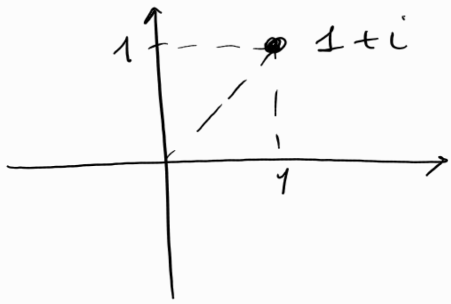
- L'argomento è determinato a meno di multipli di 2π.
- Se $z=0$ ($\Leftrightarrow |z|=0$), arg z è indeterminato.

$$\begin{cases} x = |z| \cos\varphi \\ y = |z| \sin\varphi \end{cases}$$

Dato $z = x+iy$, come si scrive in coord. polari?

$|z| = \sqrt{x^2+y^2}$, e φ è determinato da

$$\begin{cases} \cos\varphi = \dfrac{x}{|z|} \\ \sin\varphi = \dfrac{y}{|z|} \end{cases} \qquad \text{se } z \neq 0$$



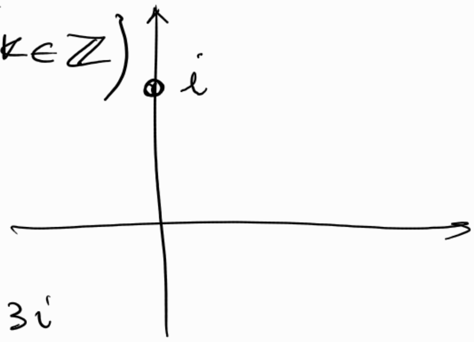
$$z = 1+i$$

$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\varphi = \frac{\pi}{4} + 2k\pi$$

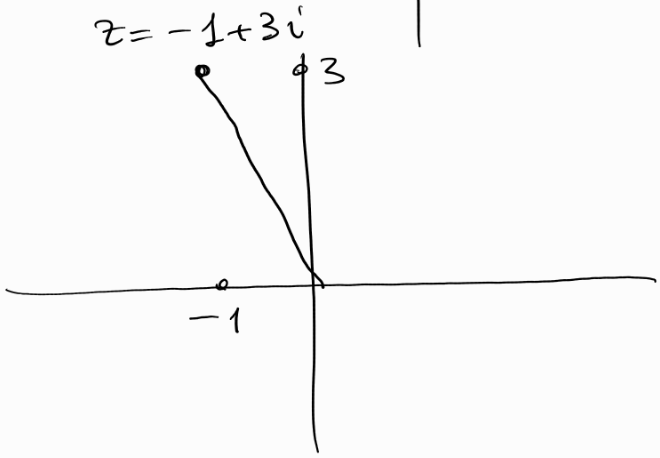
$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i \,\text{sen}\frac{\pi}{4}\right)$$

$z = i \qquad \Rightarrow |z| = 1, \; \varphi = \frac{\pi}{2} + 2k\pi \; (k \in \mathbb{Z})$



$$i = 1 \cdot \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$$

$z = -1 + 3i$



$|z| = \sqrt{1+9} = \sqrt{10}$

$$\begin{cases} \cos \varphi = \dfrac{-1}{\sqrt{10}} \\ \sin \varphi = \dfrac{3}{\sqrt{10}} \end{cases}$$

$$\varphi = \arccos\left(-\frac{1}{\sqrt{10}}\right) + 2k\pi = \pi - \arcsin\left(\frac{3}{\sqrt{10}}\right) + 2k\pi =$$

$$= \pi - \operatorname{arctg} 3 + 2k\pi$$

Proviamo a moltiplicare due numeri in notazione polare.

$z = |z| (\cos \varphi + i \sin \varphi)$

$w = |w| (\cos \theta + i \sin \theta)$

$$\Rightarrow zw = |z||w| \Big(\underbrace{\cos\varphi \cos\theta - \sin\varphi \sin\theta}_{\cos(\varphi+\theta)} + i \big(\underbrace{\cos\varphi \sin\theta + \sin\varphi \cos\theta}_{\sin(\varphi+\theta)} \big) \Big)$$

$$= |z||w| \left(\cos(\varphi+\theta) + i \sin(\varphi+\theta) \right)$$

TEOREMA se

1) Facendo il prodotto di due numeri complessi, i moduli si moltiplicano, gli argomenti si sommano.

$$z \cdot w = |z||w| \left(\cos(\varphi+\theta) + i \sin(\varphi+\theta) \right)$$

se $z = |z| (\cos\varphi + i \sin\varphi)$, $w = |w| (\cos\theta + i \sin\theta)$

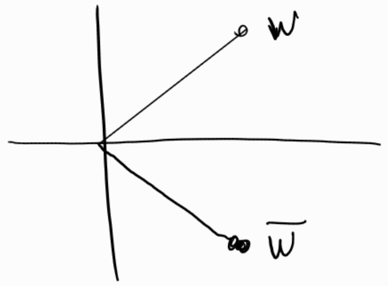
2) Facendo il rapporto di due numeri complessi, i moduli si dividono, gli argomenti si sottraggono.

$$\frac{z}{w} = \frac{|z|}{|w|}\Big(\cos(\varphi-\theta) + i\sin(\varphi-\theta)\Big).$$

Dim 2)

$$\frac{1}{w} = \frac{\bar{w}}{\underbrace{w\bar{w}}_{=|w|^2}} = \frac{\bar{w}}{|w|^2} =$$

[oss $\bar{w} = |w|\big(\cos(-\theta) + i\sin(-\theta)\big)$]



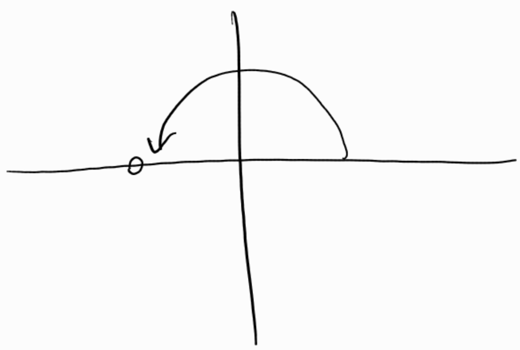
$$= \frac{|w|}{|w|^2}\big(\cos(-\theta) + i\sin(-\theta)\big) =$$

$$= \frac{1}{|w|}\Big(\cos(-\theta) + i\sin(-\theta)\Big)$$

$$\Rightarrow \frac{z}{w} = z\cdot\frac{1}{w} \overset{(1)}{=} \frac{|z|}{|w|}\Big(\cos(\varphi-\theta) + i\sin(\varphi-\theta)\Big) \qquad \square$$

Notazione esponenziale: se $\varphi \in \mathbb{R}$ poniamo

$$e^{i\varphi} := \cos\varphi + i\,\mathrm{sen}\,\varphi$$



$$\Rightarrow \quad e^{i0} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

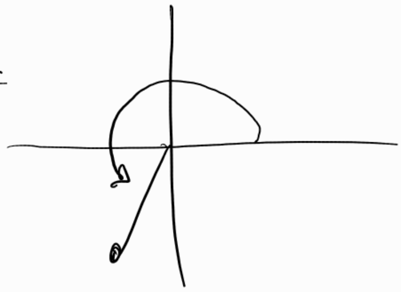
$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Formula di Eulero

$$e^{i\frac{4}{3}\pi} = \cos\left(\frac{4}{3}\pi\right) + i\left(\sin\frac{4}{3}\pi\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2} =$$

$$= -\frac{1+i\sqrt{3}}{2}$$



Quindi scriveremo

$$z = |z|(\cos\varphi + i\sin\varphi) = |z|\,e^{i\varphi}$$

$$zw = \left(|z|e^{i\varphi}\right)|w|e^{i\theta} = |z||w|\,e^{i(\varphi+\theta)}$$

$$\frac{z}{w} = \frac{|z|e^{i\varphi}}{|w|e^{i\theta}} = \frac{|z|}{|w|}\,e^{i(\varphi-\theta)}$$

Calcoliamo $z \cdot w$ e $\frac{z}{w}$ se $z = 1-i$, $w = 2+2i$

1) coord. cartesiane

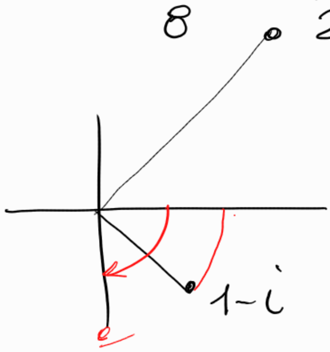
$$zw = (1-i)(2+2i) = 2+\cancel{2i}-\cancel{2i}+2 = 4$$

$$\frac{z}{w} = \frac{1-i}{2+2i} = \frac{1-i}{2+2i}\cdot\frac{2-2i}{2-2i} = \frac{2-2i-2i-2}{4+4} = \frac{-4i}{8} = -\frac{i}{2}$$

2) coord. polari

$$1-i = \sqrt{2}\,e^{-i\frac{\pi}{4}}$$

$$2+2i = 2\sqrt{2}\,e^{i\frac{\pi}{4}}$$



$$zw = \sqrt{2}\cdot 2\sqrt{2}\;\underbrace{e^{i\left(-\frac{\pi}{4}+\frac{\pi}{4}\right)}}_{e^{i0}=1} = 4$$

$$\frac{z}{w} = \frac{\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}}\;\underbrace{e^{i\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)}}_{e^{-i\frac{\pi}{2}}=-i} = -\frac{i}{2}$$

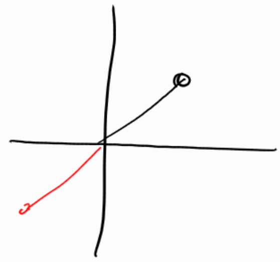
$(1+i)^4$

1) in coord. cartesiane

$$(1+i)^4 = 1^4 + 4\cdot 1^3\cdot i + 6\cdot 1^2\cdot i^2 + 4\cdot 1\cdot i^3 + i^4 =$$

$$= 1 + \cancel{4i} - 6 - \cancel{4i} + 1 = -4$$

2) in coord. polari: $\quad 1+i = \sqrt{2}\, e^{i\pi/4}$



$$(1+i)^4 = (\sqrt{2})^4\, e^{i\frac{\pi}{4}\cdot 4} = 4\, e^{i\pi} = -4$$

$$(1+i)^{21} = (\sqrt{2})^{21}\, \underbrace{e^{i\frac{\pi}{4}\cdot 21}}_{\shortparallel} = 1024\sqrt{2} = -1024\cancel{\sqrt{2}}\,\frac{(1+i)}{\cancel{\sqrt{2}}}$$

$$e^{i\frac{5}{4}\pi} = -\frac{(1+i)}{\sqrt{2}} \qquad = -1024(1+i)$$

Potenze di numeri complessi.

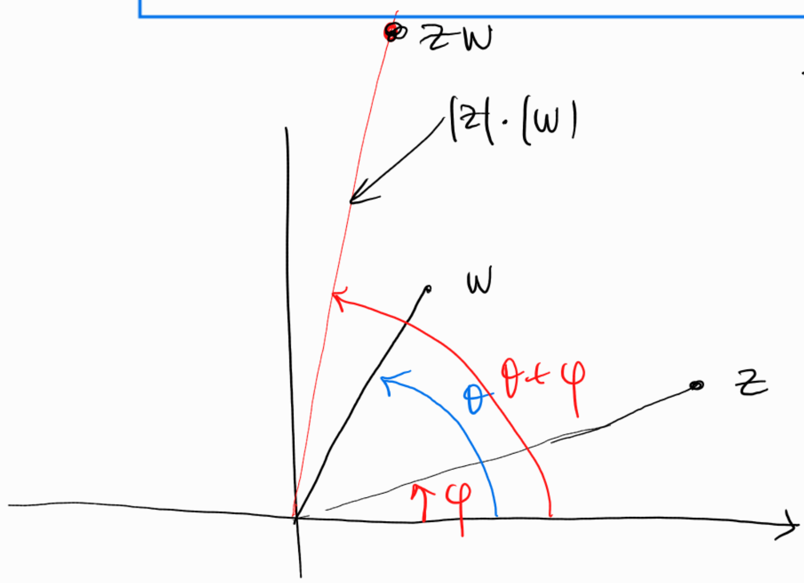
Se $\quad z = |z|\, e^{i\varphi}$, $\quad n \in \mathbb{N}$ $\quad\Rightarrow\quad z^n = |z|^n\, e^{in\varphi} =$

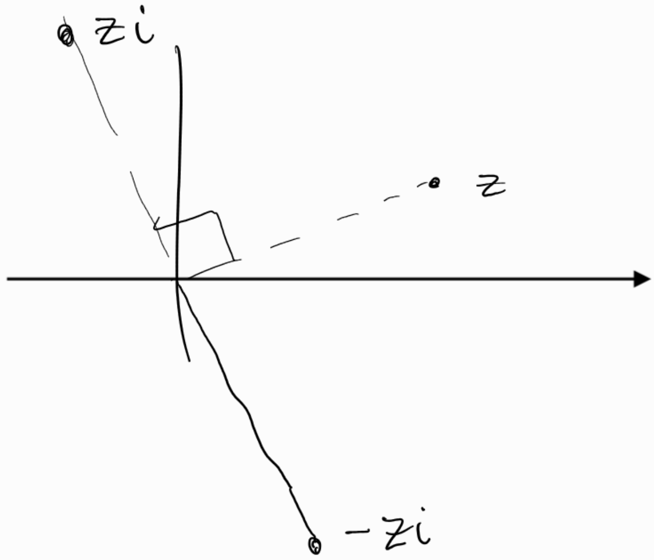
$$= |z|^n \left(\cos(n\varphi) + i\sin(n\varphi)\right)$$

se $|z| = 1$ si ottiene:

$$\boxed{(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi)}$$

formula di De Moivre





Radici n-esime di un numero complesso

Sia $z \in \mathbb{C}$, sia $n \in \mathbb{N}$, $n \geq 2$. Voglio risolvere l'eq^ne

$$w^n = z \qquad \text{in } w.$$

In altre parole, voglio trovare le radici n-esime di z.

Se $z=0$, $w=0$ è l'unica soluzione

se $z \neq 0$, scrivo $z = |z| e^{i\varphi}$

scrivo $w = |w| e^{i\theta}$

$w^n = z$ diventa $|w|^n e^{in\theta} = |z| e^{i\varphi}$

$$\Longrightarrow \begin{cases} |w|^n = |z| \\ n\theta = \varphi + 2k\pi \qquad (k \in \mathbb{Z}) \end{cases}$$

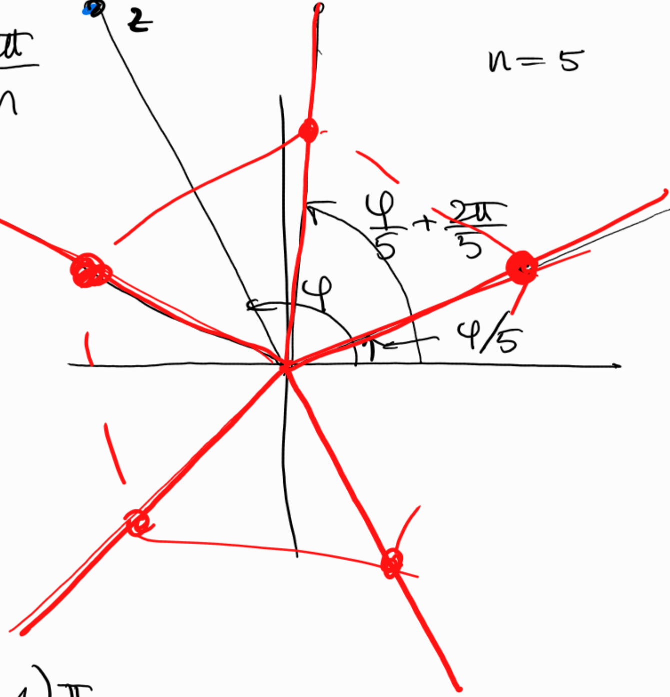
cioè

$$\begin{cases} |w| = \sqrt[n]{|z|} \\ \theta = \dfrac{\varphi + 2k\pi}{n} = \dfrac{\varphi}{n} + \dfrac{2k\pi}{n} \qquad k \in \mathbb{Z} \end{cases}$$

← radice n-esima nel senso dei reali

$k=0 \Rightarrow \theta_0 = \dfrac{\varphi}{n}$

$$k=1 \Rightarrow \theta_1 = \frac{\varphi}{n} + \frac{2\pi}{n}$$



$$k=2 \Rightarrow \theta_2 = \frac{\varphi}{n} + \frac{4\pi}{n}$$

$$k=3 \Rightarrow \theta_3 = \frac{\varphi}{n} + \frac{6\pi}{n}$$

$\vdots$

$$k=n-1 \Rightarrow \theta_{n-1} = \frac{\varphi}{n} + \frac{2(n-1)\pi}{n}$$

$$k=n \Rightarrow \theta_n = \frac{\varphi}{n} + 2\pi$$ fornisce lo stesso complesso di θ_0

$$k=n+1 \Rightarrow \theta_{n+1} = \theta_1 + 2\pi$$

Quindi abbiamo trovato che: se $z \neq 0$, le radici complesse n-esime di z sono n e sono date da

$$w_k = \sqrt[n]{|z|}\, e^{i\theta_k}, \text{ dove}$$

$$\theta_k = \frac{\varphi}{n} + \frac{2k\pi}{n}, \quad k=0,1,\dots,n-1$$

dove φ è l'arg. di z.

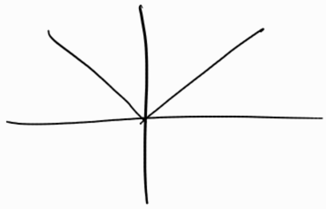
Queste n radici si trovano ai vertici di un poligono regolare di n lati centrato nell'origine e inscritto nella circ. di centro l'origine e raggio $\sqrt[n]{|z|}$.

Esempio Trovare le radici quarte di -4.

$$-4 = 4e^{i\pi}$$

Le radici quarte di -4 sono:

$$W_k = \underbrace{\sqrt[4]{4}}_{=\sqrt{2}}\, e^{i\theta_k} \quad \text{dove} \quad \theta_k = \frac{\pi}{4} + \frac{2k\pi}{4}$$



$$= \frac{\pi}{4} + \frac{k\pi}{2}$$

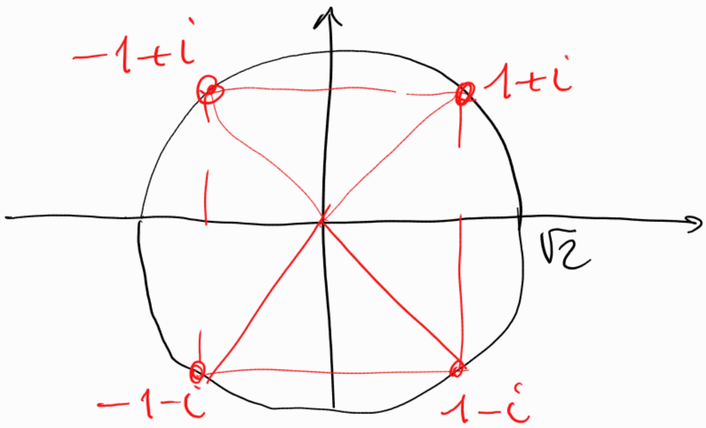
$k = 0, 1, 2, 3$.

$k=0 \Rightarrow W_0 = \sqrt{2}\, e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$

$k=1 \Rightarrow W_1 = \sqrt{2}\, e^{i\frac{3\pi}{4}} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1+i$

$k=2 \Rightarrow W_2 = \sqrt{2}\, e^{i\frac{5}{4}\pi} = \ldots\ldots = -1-i$

$k=3 \Rightarrow W_3 = \sqrt{2}\, e^{i\frac{7}{4}\pi} = \ldots\ldots = 1-i$



Radici quarte di 4:

$$4 = 4e^{i0}$$

Le radici quarte sono

$$W_k = \sqrt[4]{4}\, e^{i\theta_k} = \sqrt{2}\, e^{i\theta_k} \quad \text{dove}$$

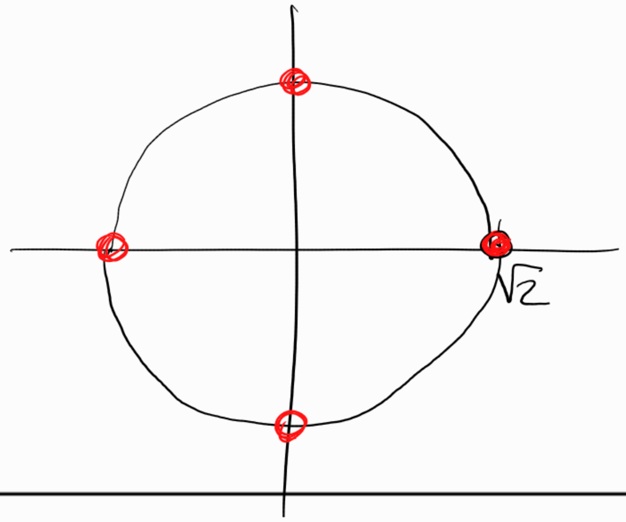
$$\theta_k = \frac{0}{4} + k\frac{2\pi}{4} = k\frac{\pi}{2} \qquad k = 0,1,2,3$$

$$w_0 = \sqrt{2}\, e^{i0} = \sqrt{2}$$

$$w_1 = \sqrt{2}\, e^{i\frac{\pi}{2}} = \sqrt{2}\, i$$

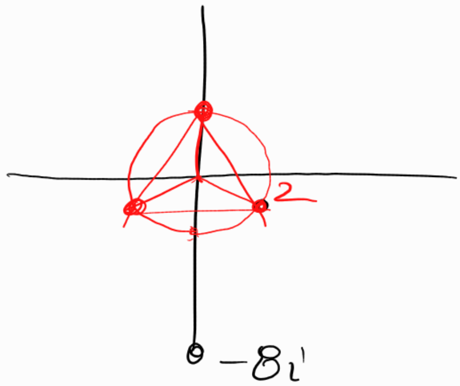
$$w_2 = \sqrt{2}\, e^{i\pi} = -\sqrt{2}$$

$$w_3 = \sqrt{2}\, e^{i\frac{3}{2}\pi} = -\sqrt{2}\, i$$



Calcolare le radici cubiche di $-8i$

$$-8i = 8\, e^{-\frac{\pi}{2} i}$$



Le radici cubiche sono tre:

$$W_k = \sqrt[3]{8}\, e^{i\theta_k} = 2e^{i\theta_k},$$

dove $\theta_k = -\frac{\pi}{6} + \frac{2k\pi}{3} \qquad k = 0,1,2.$

$$w_0 = 2\, e^{-i\frac{\pi}{6}} = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{3} - i$$

$$w_1 = 2e^{i\frac{\pi}{2}} = 2i$$

$$w_2 = 2e^{i\frac{7\pi}{6}} = 2\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\sqrt{3} - i$$

Radici quadrate di -4

$$-4 = 4e^{i\pi}$$

$$w_0 = \sqrt{4}\, e^{i\frac{\pi}{2}} = 2i$$

$$w_1 = \sqrt{4}\, e^{i\frac{3\pi}{2}} = -2i$$

__Esercizio.__ Risolvere l'eq^{ne} $|z|^3 z^3 = 64i$

Conviene usare coord. polari

$$z = |z|^3 e^{i\varphi}$$

l'eq^{ne} diventa

$$|z|^6 e^{i3\varphi} = 64i = 64 e^{i\frac{\pi}{2}}$$

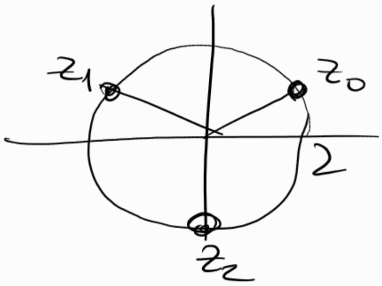
$$\Longrightarrow \quad |z|^6 = 64 \qquad \Longrightarrow \quad |z| = 2$$

$$3\varphi = \frac{\pi}{2} + 2k\pi$$

$$\varphi = \frac{\pi}{6} + \frac{2k\pi}{3} \qquad \cancel{k \in \mathbb{Z}}$$

$$k = 0, 1, 2$$

Tre sol^{ni}



$$z_0 = 2 e^{i\frac{\pi}{6}} = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) =$$

$$= \sqrt{3} + i$$

$$z_1 = 2 e^{i\frac{5}{6}\pi} = -\sqrt{3} + i$$

$$z_2 = 2 e^{i\frac{3}{2}\pi} = -2i$$