ESERCIZIO 1

Calcoliamo la matrice associata ad H nella base $|+;z\rangle \rightarrow (1,0)$, $|-;z\rangle \rightarrow (0,1)$

A questo fine dobbiamo calcolare $|\pm;x\rangle$

Si ha $S_x = \frac{\hbar}{2}\sigma_x$ con autoval. $\pm \hbar/2$

(1) Autoval $+\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{cases} b = a \\ a = b \end{cases} \qquad |+;x\rangle = (a,a)$$

La condizione di normalizzazione dà $a = \frac{1}{\sqrt{2}}$ (con scelta di fase)

$$|+;x\rangle = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

(2) Autoval $-\hbar/2$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{cases} b = -a \\ a = -b \end{cases} \qquad |-;x\rangle = (a,-a)$$

$a = \frac{1}{\sqrt{2}}$ per normalizzazione $\quad |-;x\rangle = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Quindi

$$\langle +;z|H|+;z\rangle = b\langle +;z\overset{1/\sqrt{2}}{|}+;x\rangle\langle -;x\overset{1/\sqrt{2}}{|}+;z\rangle - \frac{a}{4}$$
$$= \frac{b}{2} - \frac{a}{4}$$

$$\langle +|z|H|-;z\rangle = b\langle +;z|+;x\rangle\langle -;x|-;z\rangle + a$$
$$= a - \frac{b}{2}$$

$$\langle -,z|H|+;z\rangle = b\langle -;z|+;x\rangle\langle -;x|+;z\rangle = \frac{b}{2}$$

$$\langle -z|H|-;z\rangle = b\langle -;z|+;x\rangle\langle -;x|-;z\rangle = -\frac{b}{2}$$

$$H = \begin{pmatrix} \frac{b}{2} - \frac{a}{4} & a - \frac{b}{2} \\ \frac{b}{2} & -\frac{b}{2} \end{pmatrix}$$

(a) La hermiticità impone

$$\begin{cases} \left(\frac{b}{2} - \frac{a}{4}\right)^* = \left(\frac{b}{2} - \frac{a}{4}\right) \\ \left(-\frac{b}{2}\right)^* = \left(-\frac{b}{2}\right) \\ \left(a - \frac{b}{2}\right) = \left(\frac{b}{2}\right)^* \end{cases}$$

$\rightarrow$ a e b sono reali

$\rightarrow \quad a - \frac{b}{2} = \frac{b}{2} \qquad a = b \qquad \frac{a}{b} = 1$

(b) Se $a = b$

$$H = \begin{pmatrix} a/4 & a/2 \\ a/2 & -a/2 \end{pmatrix} = a \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = a\hat{H}$$

$$\det(\hat{H} - \lambda I) = \det \begin{pmatrix} 1/4 - \lambda & 1/2 \\ 1/2 & -\frac{1}{2} - \lambda \end{pmatrix} = \left(\frac{1}{4} - \lambda\right)\left(-\frac{1}{2} - \lambda\right) - \frac{1}{4}$$

$$= -\frac{1}{8} - \frac{\lambda}{4} + \frac{\lambda}{2} + \lambda^2 - \frac{1}{4} =$$

$$= \lambda^2 + \frac{\lambda}{4} - \frac{3}{8} = \frac{8\lambda^2 + 2\lambda - 3}{8}$$

$$(8\lambda^2 + 2\lambda - 3) = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1 + 24}}{8} = \frac{-1 \pm 5}{8} = \begin{cases} -3/4 \\ 1/2 \end{cases}$$

$$E_1 = -\frac{3a}{4} \qquad E_2 = \frac{a}{2}$$

(1) autovettore con energia $E_1 = -\frac{3a}{4}$

$$\begin{pmatrix} 1/4 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} \tilde{a} \\ b \end{pmatrix} = \left(-\frac{3}{4}\right) \begin{pmatrix} \tilde{a} \\ b \end{pmatrix} \rightarrow \begin{cases} \frac{\tilde{a}}{4} + \frac{b}{2} = -\frac{3}{4}\tilde{a} \\ \frac{\tilde{a}}{2} - \frac{b}{2} = -\frac{3b}{4} \end{cases}$$

$$\begin{cases} \frac{b}{2} = -\tilde{a} \\ \frac{\tilde{a}}{2} = -\frac{b}{4} \end{cases} \qquad b = -2\tilde{a} \qquad |v\rangle = (\tilde{a}, -2\tilde{a})$$

$$\langle v|v\rangle = |\tilde{a}|^2 + 4|\tilde{a}|^2 \Rightarrow |\tilde{a}| = \frac{1}{\sqrt{5}}$$

$$\left|-\frac{3}{4}a\right\rangle = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$$

(2) autovett. con $E_2 = +\frac{a}{2}$

$$\begin{pmatrix} 1/4 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} \tilde{a} \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{a} \\ b \end{pmatrix} \rightarrow \begin{cases} \frac{\tilde{a}}{4} + \frac{b}{2} = \frac{\tilde{a}}{2} \\ \frac{\tilde{a}}{2} - \frac{b}{2} = \frac{b}{2} \end{cases} \qquad \begin{matrix} \frac{b}{2} = \frac{\tilde{a}}{4} \\ \frac{\tilde{a}}{2} = b \end{matrix}$$

$$|v\rangle = \left(\tilde{a}, \frac{\tilde{a}}{2}\right) \qquad \langle v|v\rangle = |\tilde{a}|^2 + \frac{1}{4}|\tilde{a}|^2 = \frac{5}{4}|\tilde{a}|^2 \qquad |\tilde{a}| = \frac{2}{\sqrt{5}}$$

$$\left|\frac{a}{2}\right\rangle = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

c)

Decomponiamo lo stato in termini degli autostati di H

$$\left\langle +;z\middle|-\frac{3}{4}a\right\rangle = \frac{1}{\sqrt{5}} \qquad \left\langle +;z\middle|\frac{a}{2}\right\rangle = \frac{2}{\sqrt{5}}$$

$$|+;z\rangle = \frac{1}{\sqrt{5}}\left|-\frac{3}{4}a\right\rangle + \frac{2}{\sqrt{5}}\left|\frac{a}{2}\right\rangle$$

Valori ottenibili in una misura: $\frac{a}{2}$, $-\frac{3a}{4}$

$$\text{Prob}\left(E = -\frac{3a}{4}\right) = \frac{1}{5} \qquad \text{Prob}\left(E = \frac{a}{2}\right) = \frac{4}{5}$$

elemento (1,1) della matrice di H

$$\langle +;z|H|+;z\rangle = H_{11} = \frac{a}{4}$$

CHECK:

$$\langle +;z|H|+;z\rangle = \frac{1}{5}\left(-\frac{3a}{4}\right) + \frac{4}{5}\left(\frac{a}{2}\right) = -\frac{3a}{20} + \frac{2a}{5} = \frac{a}{20}(-3+8) = \frac{a}{4}$$

ok

(d)

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}} \exp\left(+ \frac{it}{\hbar}\frac{3}{4}a\right) \left|-\frac{3}{4}a\right\rangle$$

$$+ \frac{2}{\sqrt{5}} \exp\left(- \frac{it}{\hbar}\frac{a}{2}\right) \left|\frac{a}{2}\right)$$

Se $\omega = \frac{1}{\hbar}\left(\frac{a}{2} + \frac{3a}{4}\right) = \frac{5a}{4\hbar}$

$$|\psi(t)\rangle \overset{(\text{a meno di fase})}{=} \frac{1}{\sqrt{5}}\left|-\frac{3}{4}a\right\rangle + \frac{2}{\sqrt{5}} e^{-i\omega t}\left|\frac{a}{2}\right\rangle$$

e)

$$\text{Prob}\left(S_z = -\frac{\hbar}{2}\right) = \left|\langle -;z|\psi(t)\rangle\right|^2$$

Ora

$$\langle -;z|\psi(t)\rangle = \frac{1}{\sqrt{5}}\left\langle -;z\middle|-\frac{3}{4}a\right\rangle + \frac{2}{\sqrt{5}}\left\langle -z\middle|\frac{a}{2}\right\rangle e^{-i\omega t}$$

$$= \frac{1}{\sqrt{5}}\left(-\frac{2}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right) e^{-i\omega t}$$

$$= -\frac{2}{5} + \frac{2}{5} e^{-i\omega t} = -\frac{2}{5}\left(1 - e^{-i\omega t}\right)$$

$$\text{Prob}\left(S_z = -\frac{\hbar}{2}\right) = \frac{4}{25}\left|1 - e^{-i\omega t}\right|^2$$

$$= \frac{4}{25}(2 - 2\cos\omega t) = \frac{8}{25}(1 - \cos\omega t)$$

$$= \frac{8}{25}\left(1 - \cos\frac{5at}{4\hbar}\right)$$

①

a) Riscriviamo la funzione d'onda come

$$\psi = a\sqrt{4\pi}\ f_a(r)\, Y_0^0\, \chi_1 + b\, f_b(r)\, Y_1^0\, \chi_2 \qquad \left[Y_0^0 = \frac{1}{\sqrt{4\pi}}\right]$$

Per il principio di Pauli la funzione d'onda deve essere dispari sotto scambio $\vec{r}_\ell = \bar{r}_2 - \bar{r}_1 \to -\vec{r} = \bar{r}_1 - \bar{r}_2$

Dato che sotto scambio (parità) $Y_0^0 \to Y_0^0$, $Y_1^0 \to -Y_1^0$ la funzione di spin χ_1 deve essere dispari e χ_2 pari sotto scambio. Quindi χ_1 è autofunzione di S^2 con autovalore 0, χ_2 autofunzione di S^2 con autovalore $2\hbar^2$ $(S=1)$. Quindi

$$\psi = a\sqrt{4\pi}\ f_a(r)\, Y_0^0\, |\underset{S}{0}\ \underset{S_z}{0}\rangle + b\, f_b(r)\, Y_1^0\, |\underset{S}{1}\ \underset{S_z}{0}\rangle$$

$$\psi_a(r,\theta,\varphi) = f_a(r)\, Y_0^0$$
$$\psi_b(r,\theta,\varphi) = f_b(r)\, Y_1^0$$

sono normalizzate $\int d^3r\, |\psi_{a,b}|^2 = 1$

Quindi

$$\langle\psi|\psi\rangle = 1 \implies |a|^2 4\pi + |b|^2 = 1$$

$$\text{Prob}\,(S^2 = 2\hbar^2) = \frac{1}{3} \implies |b|^2 = \frac{1}{3} \quad \rightarrow \quad 4\pi|a|^2 = \frac{2}{3}$$

$$a = \frac{1}{\sqrt{6\pi}} \qquad b = \frac{1}{\sqrt{3}} \qquad (a, b \text{ reali positivi})$$

b) Calcoliamo lo spettro di H_0

Nel CM: $H_0 = \frac{1}{2\mu} p^2 - \frac{\alpha}{r}$ $\qquad \mu = \frac{m}{2}$

Quindi (spettro coulombiano)

$$E_n = -\frac{1}{2}\frac{a^2\mu}{\hbar^2}\frac{1}{n^2} = -\frac{\bar{E}\alpha}{4}\frac{1}{n^2}$$

I primi livelli sono

$$E_1 = -\frac{E_\alpha}{4} \qquad E_2 = -\frac{E_\alpha}{16} \qquad E_3 = -\frac{E_\alpha}{36}$$

Dato che una misura di E_α fornisce sempre un risultato minore di $-E_\alpha/20$, ψ deve essere combinazione degli stati Coulombiani con $n=1,2$.

$f_a Y_0^0$ ha $L^2=0$, $L_z=0$ e quindi deve essere combinazione degli stati con $l=0$ che sono due; il livello $n=1$ e lo stato $l=0$ del livello $n=2$. Quindi

$$f_a = A R_{10}(r) + B R_{20}(r)$$

$$\int |f_a|^2 r^2 dr = |A|^2 \int R_{10}^2 r^2 dr + |B|^2 \int R_{20}^2 r^2 dr = |A|^2 + |B|^2 = 1$$

$f_b Y_1^0$ ha $L^2 = 2\hbar$, $L_z = 0$. Vi è una sola possibilità

$$f_b = R_{21}(r) e^{i\alpha}$$

Quindi

$$\psi = \sqrt{\frac{2}{3}} \left[A R_{10}(r) + B R_{20}(r) \right] Y_0^0 \overset{S\,S_z}{|0\,0\rangle} + \sqrt{\frac{1}{3}} e^{i\alpha} R_{21} Y_1^0 |1\,0\rangle$$

$$H[R_{10} Y_0^0] = -\frac{E_\alpha}{4} \quad (n=1)$$

$$H[R_{20} Y_0^0] = -\frac{E_\alpha}{16} \quad (n=2)$$

$$H[R_{21} Y_1^0] = -\frac{E_\alpha}{16} \quad (n=2)$$

Quindi

$$\langle\psi|H_0|\psi\rangle = \frac{2}{3}|A|^2\left(-\frac{E_\alpha}{4}\right) + \frac{2}{3}|B|^2\left(-\frac{E_\alpha}{16}\right) + \frac{1}{3}\left(-\frac{E_\alpha}{16}\right)$$

$$= -E_\alpha \left[\frac{1}{6}|A|^2 + \frac{1}{24}|B|^2 + \frac{1}{48} \right]$$

Quindi

$$\langle\psi_0|H|\psi_0\rangle = -\frac{E_\alpha}{8} \Longrightarrow$$

$$\frac{|A|^2}{6} + \frac{|B|^2}{24} + \frac{1}{48} = \frac{1}{8}$$

$$8|A|^2 + 2|B|^2 + 1 = 6$$

$$8|A|^2 + 2|B| = 5 \qquad \text{Ora} \quad |B|^2 = 1 - |A|^2$$

$$6|A|^2 + 2 = 5 \qquad |A|^2 = \frac{1}{2} \qquad |A| = \frac{1}{\sqrt{2}} \qquad |B| = \frac{1}{\sqrt{2}}$$

Quindi $A = \frac{1}{\sqrt{2}} e^{i\beta} \qquad B = \frac{1}{\sqrt{2}} e^{i\gamma}$

Abbiamo

$$\psi = \frac{1}{\sqrt{3}}\left(e^{i\beta} R_{10} Y_0^0 + e^{i\gamma} R_{20} Y_0^0\right)|00\rangle + \frac{1}{\sqrt{3}} e^{i\alpha} R_{21} Y_1^0 |10\rangle$$

Vi sono tre fasi arbitrarie. Possiamo eliminarne UNA (e solo UNA). Poniamo $\beta = 0$

c)

$$\psi = \frac{1}{\sqrt{3}}\left(e^{-iE_1 t/\hbar} R_{10} Y_0^0 + e^{i\gamma} e^{-iE_2 t/\hbar} R_{20} Y_0^0\right)|00\rangle$$
$$+ \frac{1}{\sqrt{3}} e^{i\alpha} e^{-iE_2 t/\hbar} R_{21} Y_1^0 |10\rangle$$

Calcoliamo $\langle\psi|z^2|\psi\rangle = \langle\psi|r^2\cos^2\theta|\psi\rangle$

$$\langle\psi|z^2|\psi\rangle = \frac{1}{3}\int dr\, r^2 \left|e^{-iE_1 t/\hbar} R_{10} + e^{i\gamma} e^{-iE_2 t/\hbar} R_{20}\right|^2 r^2 \times$$
$$\int d\Omega\, |Y_0^0|^2 \cos^2\theta$$
$$+ \frac{1}{3}\int dr\, r^2 R_{21}^2\, r^2 \int d\Omega |Y_1^0|^2 \cos^2\theta$$

Ora

$$\int d\Omega\, |Y_0^0|^2 \cos^2\theta = \frac{1}{4\pi} \underset{\text{[int in }\phi]}{2\pi} \int_{-1}^{1} dx\, x^2 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3} \quad [x = \cos\theta]$$

$$\int d\Omega\, |Y_0^1|^2 \cos^2\theta = \frac{3}{4\pi}\int d\Omega \cos^4\theta = \frac{3}{4\pi} \underset{\text{[int in }\phi]}{2\pi} \int dx\, x^4 = \frac{3}{2}\cdot\frac{2}{5} = \frac{3}{5}$$

$$\langle\psi|z^2|\psi\rangle = \frac{1}{3}\int_0^\infty dr\, r^4 \left| R_{10} + e^{i\gamma} e^{i(E_1-E_2)t/\hbar} R_{20}\right|^2 \cdot \frac{1}{3}$$

$$+ \frac{1}{3}\int_0^\infty dr\, r^4 R_{21}^2 \cdot \frac{3}{5}$$

$$= \frac{1}{9}\int_0^\infty dr\, r^4 \left(R_{10}^2 + R_{20}^2 + 2\cos\left(\gamma + \frac{(E_1-E_2)t}{\hbar}\right) R_{10}R_{20}\right)$$

$$+ \frac{1}{5}\int_0^\infty dr\, r^4 R_{21}^2$$

$$= \frac{1}{9}a_0^2\left[3 + 42 + 2\left(-\frac{512\sqrt{2}}{243}\right)\cos\left(\gamma + \frac{E_1-E_2}{\hbar}t\right)\right]$$

$$+ \frac{1}{5}\cdot 30\, a_0^2$$

$$= 11\, a_0^2 - \frac{1024}{243\cdot 9}\sqrt{2}\, a_0^2 \cos\left(\gamma + \frac{E_1-E_2}{\hbar}t\right)$$

Ora definiamo $\omega_\alpha = \frac{E_2-E1}{\hbar} = -\frac{E_\alpha}{16\hbar} + \frac{E_\alpha}{4\hbar} = \frac{3E_\alpha}{16\hbar}$

Quindi

$$\cos\left(\gamma + \frac{E_1-E_2}{\hbar}t\right) = \cos(\omega_\alpha t - \gamma)$$

Quindi $T = \frac{2\pi}{\omega_\alpha} = \frac{32\pi\hbar}{3E_\alpha}$