$$\lim_{x\to+\infty} \frac{\log\left(\frac{e^x+1}{e^{2x}+4}\right)}{x} = \left(\frac{-\infty}{+\infty}\right) = -1.$$

$$\frac{e^x+1}{e^{2x}+4} = \frac{e^x\left(1+\frac{1}{e^x}\right)}{e^{2x}\left(1+\frac{4}{e^{2x}}\right)} = \frac{1}{e^x}\left(\frac{1+\frac{1}{e^x}}{1+\frac{4}{e^{2x}}}\right) \sim \frac{1}{e^x} \to 0^+$$

($\frac{1}{e^x} \to 0$; $\left(1+\frac{1}{e^x}\right) \to 1$; $\left(1+\frac{4}{e^{2x}}\right) \to 1$; $\left(\frac{1+\frac{1}{e^x}}{1+\frac{4}{e^{2x}}}\right) \to 1$)

$$\log\left(\frac{e^x+1}{e^{2x}+4}\right) \to -\infty$$

$$\log\left(\frac{e^x+1}{e^{2x}+4}\right) = \log\left(\frac{1}{e^x}\left(\text{qualcosa che tende a } 1\right)\right) =$$

$$= \log\left(\frac{1}{e^x}\right) + \log\left(\text{qualcosa che tende a } 1\right)$$

$$= -x + \text{infinitesimo} =$$

$$= -x\left(1 + \frac{\text{infinitesimo}}{-x}\right) \sim -x$$

($\frac{\text{infinitesimo}}{-x} \to 0$, quindi $\left(1 + \frac{\text{infinitesimo}}{-x}\right) \to 1$)

$$\log\left(\frac{e^x+1}{e^{2x}+4}\right) = \log(e^x+1) - \log(e^{2x}+4) =$$

$$= \log\left(e^x\left(1+\frac{1}{e^x}\right)\right) - \log\left(e^{2x}\left(1+\frac{4}{e^{2x}}\right)\right)$$

$$= \log e^x + \log\left(1+\frac{1}{e^x}\right) - \log(e^{2x}) - \log\left(1+\frac{4}{e^{2x}}\right)$$

$$= x + \log\left(1+\frac{1}{e^x}\right) - 2x - \log\left(1+\frac{4}{e^{2x}}\right)$$

$$= -x + \log\left(1+\frac{1}{e^x}\right) - \log\left(1+\frac{4}{e^{2x}}\right)$$

$$\frac{\log\left(\frac{e^x+1}{e^{2x}+4}\right)}{x} = \frac{-x + \log\left(1+\frac{1}{e^x}\right) - \log\left(1+\frac{4}{e^{2x}}\right)}{x} =$$

$$= -1 + \frac{\log\left(1+\frac{1}{e^x}\right)}{x} - \frac{\log\left(1+\frac{4}{e^{2x}}\right)}{x} \to -1$$

$$\left(\frac{0}{+\infty}\right) = 0 \qquad \left(\frac{0}{+\infty}\right) = 0$$

$$\lim_{x\to 0^+} [\cos(2x)]^{\frac{1}{x^\alpha \sin x}} \qquad \alpha \in \mathbb{R}.$$

$$\cos(2x) \to 1$$

$$x^\alpha \sin x = x^{\alpha+1}\frac{\sin x}{x} \to \begin{cases} 0^+ & \text{se } \alpha > -1 \\ 1 & \alpha = -1 \\ +\infty & \alpha < -1 \end{cases}$$

$$\frac{\sin x}{x} \to 1$$

$$\frac{1}{x^\alpha \sin x} \to \begin{cases} +\infty & \text{se } \alpha > -1 \\ 1 & \text{se } \alpha = -1 \\ 0^+ & \text{se } \alpha < -1. \end{cases}$$

$\alpha < -1$ $\quad (\cos(2x))^{\frac{1}{x^\alpha \sin x}} \to (1^{0^+}) = 1$

$\alpha = -1$ $\quad (\cos(2x))^{\frac{1}{x^\alpha \sin x}} \to 1^1 = 1$

$\boxed{\alpha > -1}$ $\quad (\cos(2x))^{\frac{1}{x^\alpha \sin x}} \to (1^{+\infty})$ f. i.

$\boxed{\alpha > -1}$ $\quad (\cos(2x))^{\frac{1}{x^\alpha \sin x}} = e^{\frac{\log(\cos(2x))}{x^\alpha \sin x}}$

$$\frac{\log(\cos(2x))}{x^\alpha \sin x} = \frac{\log(\cos(2x))}{x^{\alpha+1}} \cdot \frac{x}{\sin x}$$

$$= \frac{\log(1+(\cos(2x)-1))}{\cos(2x)-1} \cdot \frac{(\cos(2x)-1)}{4x^2} \cdot \frac{4x^2}{x^{\alpha+1}} \cdot \frac{x}{\sin x}$$

(with $\cos(2x)-1 \to 0$; the four factors tend to: $\to 1$, $\to -1/2$, $= 4x^{1-\alpha}$, $\to 1$)

$\frac{\log(1+t)}{t} \to 1$ per $t \to 0$

$x^{1-\alpha} \to \begin{cases} 0 & \text{se } -1<\alpha<1 \\ 1 & \alpha = 1 \\ +\infty & \alpha > 1 \end{cases}$

$\frac{1-\cos t}{t^2} \to \frac{1}{2}$ per $t \to 0$

$\alpha > -1 \iff -\alpha < 1 \quad 1-\alpha < 2$

Riassumendo l'esponente

$$\frac{\log(\cos(2x))}{x^\alpha \sin x} \to \begin{cases} 0 & \text{se } -1<\alpha<1 \\ -2 & \text{se } \alpha = 1 \\ -\infty & \text{se } \alpha > 1. \end{cases}$$

$$e^{\frac{\log(\cos(2x))}{x^\alpha \sin x}} \to \begin{cases} 1 & \text{se } -1<\alpha<1 \\ e^{-2} & \text{se } \alpha = 1 \\ 0 & \text{se } \alpha > 1. \end{cases}$$

<u>Soluzione</u>: il limite vale $\begin{cases} 1 & \text{se } \alpha < 1 \\ e^{-2} & \alpha = 1 \\ 0 & \text{se } \alpha > 1 \end{cases}$