20/10/2025

ESERCIZIO 11. *Calcolare i seguenti limiti*

$$\lim_{x\longrightarrow 0}\frac{5x_1x_2}{\sqrt{x_1^2+3x_2^2}} \qquad \lim_{x\longrightarrow 0}\frac{\pi\ln(1+x_1^2)x_2}{x_1x_2} \qquad \lim_{x\longrightarrow 0}\frac{x_1^4}{x_2(x_1^2+x_2^2)} \qquad \lim_{x\longrightarrow 0}\frac{\sqrt{x_1x_2}}{1+2x_1^2+3x_2^2}$$

- $\displaystyle\lim_{x\to 0}\frac{5x_1x_2}{\sqrt{x_1^2+3x_2^2}} = 0$ infatti:

$$0 \leq \left|\frac{5x_1x_2}{\sqrt{x_1^2+3x_2^2}}\right| \underset{|ab|\leq\frac{1}{2}(a^2+b^2)}{\leq} \frac{5}{2}\,\frac{x_1^2+x_2^2}{\sqrt{x_1^2+3x_2^2}} \begin{cases} \leq \frac{5}{2}\cdot\frac{x_1^2+3x_2^2}{\sqrt{x_1^2+3x_2^2}} = \frac{5}{2}\sqrt{x_1^2+3x_2^2} \\ \leq \frac{5}{2}\cdot\frac{x_1^2+x_2^2}{\sqrt{x_1^2+x_2^2}} = \frac{5}{2}\sqrt{x_1^2+x_2^2} \end{cases} \xrightarrow[x\to 0]{} 0$$

- $\displaystyle\lim_{x\to 0}\frac{\pi\ln(1+x_1^2)\,x_2}{x_1x_2} = \lim_{x\to 0}\frac{\pi\ln(1+x_1^2)}{x_1} = \lim_{x\to 0}\frac{\pi x_1^2}{x_1} = 0$

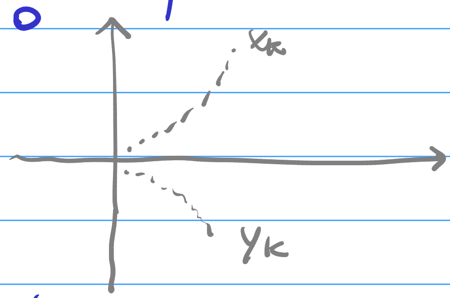
- $\displaystyle\lim_{x\to 0}\frac{x_1^4}{x_2(x_1^2+x_2^2)} \nexists$ perché:

$$f(x) = \frac{x_1^4}{x_2(x_1^2+x_2^2)} \underset{\text{C.P.}}{=} \frac{\rho^4\cos^4(\theta)}{\rho^3\operatorname{sen}(\theta)} = \rho\,\frac{\cos^4(\theta)}{\operatorname{sen}(\theta)} = \rho_k\,\frac{\cos^4(\rho_k)}{\pm\operatorname{sen}(\rho_k)} = \pm 1$$

$[x_1 = \rho\cos(\theta),\ x_2 = \rho\operatorname{sen}(\theta)]$

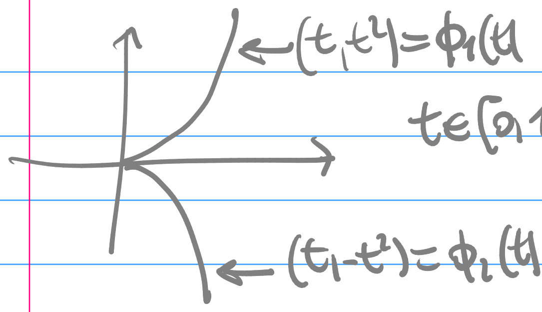
$x_k = (\rho_k\cos(\rho_k), \rho_k\operatorname{sen}(\rho_k))$ $\quad \rho_k \longrightarrow 0$

$y_k = (\rho_k\cos(\rho_k), \rho_k\operatorname{sen}(-\rho_k))$



$\phi(t) = (t, mt)$ $\quad t\in[0,1]$, $m\neq 0$

$$f(\phi(t)) = \frac{t^4}{mt(t^2+m^2t^2)} = \frac{t^4}{m(1+m^2)t^3} = \frac{t}{m(1+m^2)}$$

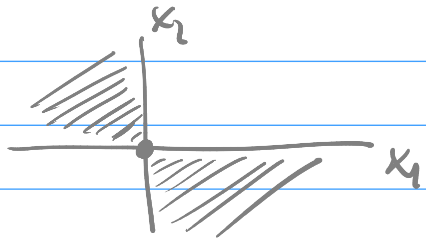


$t\in[0,1)$

$$f(\phi_1(t))=\frac{t^4}{t^2(t^2+t^4)}=\frac{1}{1+t^2}$$

$$f(\phi_2(t))=\frac{t^4}{-t^2(t^2+t^4)}=-\frac{1}{1+t^2}$$

- $$\lim_{x\to 0}\frac{\sqrt{x_1x_2}^{\,2}}{1+2x_1^2+3x_2^2}=0$$



visto che $\quad x_1x_2\to 0$
$\qquad\qquad\qquad 2x_1^2+3x_2^2\to 0$

OSS. $p:\underline{\underline{\mathbb{R}^n}}\to\mathbb{R}$ polinomio allora p è una funzione continua

$$a\,x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\qquad \alpha_1,\alpha_2,\dots,\alpha_n\geq 0$$

$$x\to\bar{x}\ \Rightarrow\ \rho=\left[(x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2+\dots+(x_n-\bar{x}_n)^2\right]^{1/2}\to 0$$

$$\Rightarrow\ \forall i=1,\dots,n\quad |x_i-\bar{x}_i|\to 0\ \Rightarrow\ x_i\to\bar{x}_i$$

per continuità (in $\mathbb{R}$) $\quad x_i^{\alpha_i}\longrightarrow \bar{x}_i^{\alpha_i}$

$$\Rightarrow\quad a\,x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}\longrightarrow a\,\bar{x}_1^{\alpha_1}\cdot\ldots\cdot\bar{x}_n^{\alpha_n}\qquad\blacksquare$$

Esercizio 4. *Si consideri lo spazio vettoriale* $C^0[0,\pi]$ *e le norme* $\|\cdot\|_i$, *con* $i = 1, 2, +\infty$, *definite in modo che, per* $f \in C^0[0,\pi]$, *valga*

$$\|f\|_1 = \int_0^\pi |f(s)|ds \qquad \|f\|_2 = \left[\int_0^\pi |f(s)|^2 ds\right]^{1/2} \qquad \|f\|_\infty = \max_{s\in[0,\pi]} |f(s)|$$

i. si mostri che esistono due costanti $C_1, C_2 > 0$ *tali che*

$$\|f\|_1 \le C_1\|f\|_\infty \qquad e \qquad \|f\|_2 \le C_2\|f\|_\infty \qquad \text{per ogni } f \in C^0[0,\pi]$$

ii. esistono delle costanti* $K_1, K_2 > 0$ *in modo che*

$$\|f\|_\infty \le K_i\|f\|_i \qquad \text{per ogni } f \in C^0[0,\pi] \quad e \quad i = 1, 2?$$

iii. esiste* $C_0 > 0$ *tale che* $\|f\|_1 \le C_0\|f\|_2$.

1. $$\|f\|_1 = \int_0^\pi |f(s)|ds \le \int_0^\pi \Big[\underbrace{\max_{[0,\pi]} |f(s)|}_{\|f\|_\infty}\Big] ds = \|f\|_\infty \int_0^\pi ds = \pi \|f\|_\infty$$

$C_1 = \pi$

$$\|f\|_2 = \left[\int_0^\pi |f(s)|^2 ds\right]^{1/2} \le \left[\int_0^\pi \|f\|_\infty^2 ds\right]^{1/2} = \|f\|_\infty \left[\int_0^\pi ds\right]^{1/2}$$

$$= \sqrt{\pi}\, \|f\|_\infty \qquad C_2 = \sqrt{\pi}$$

$$f_k(x) = \left(\frac{x}{\pi}\right)^k \xrightarrow[\text{ptmente}]{} f_\infty(x) = \begin{cases} 0 & x \in [0,\pi) \\ 1 & x = \pi \end{cases}$$

$$\|f_k\|_1 = \int_0^\pi |f_k(x)|dx = \int_0^\pi \left(\frac{x}{\pi}\right)^k dx = \left[\frac{\pi}{k+1}\left(\frac{x}{\pi}\right)^{k+1}\right]_0^\pi = \frac{\pi}{k+1} \xrightarrow[k\to+\infty]{} 0$$

se $\exists K_1 > 0$ avrei che $\boxed{\underbrace{\|f_k\|_\infty}_{1} \le K_1 \underbrace{\|f_k\|_1}_{\frac{\pi}{k+1} \to 0}}$ assurdo!

$\Downarrow$

$\nexists K_1$ come chiesto in ii.

iii. $\|f\|_1 = \int_0^\pi |f(s)|\,ds$

$$\|f\|_2^2 = \int_0^\pi [f(s)]^2\,ds =: (f|f)_2$$

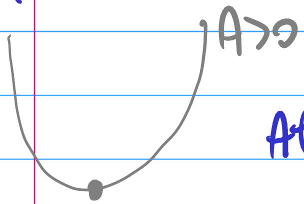
$$\||f|+t\cdot 1\|_2^2 = \int_0^\pi (|f|+t)^2\,ds = \int_0^\pi [f^2(s)+2t|f(s)|+t^2]\,ds$$

$t\in\mathbb{R}$

$$= \int_0^\pi f^2(s)\,ds + 2t\int_0^\pi |f(s)|\,ds + t^2 \underbrace{\int_0^\pi ds}_{=\pi>0} = H(t)$$

$$\min_{\mathbb{R}}(H) = H\left(-\frac{2\int_0^\pi |f(s)|\,ds}{2\pi}\right) = \int_0^\pi |f(s)|^2\,ds - \frac{2}{\pi}\left[\int_0^\pi |f(s)|\,ds\right]^2$$

$$+\pi\cdot\frac{1}{\pi^2}\left[\int_0^\pi |f(s)|\,ds\right]^2 = \int_0^\pi |f(s)|^2\,ds - \frac{1}{\pi}\left[\int_0^\pi |f(s)|\,ds\right]^2 \geq 0$$



$$\left[\; At^2+Bt+C=\phi(t) \Rightarrow \phi'(t) = 2At+B = 0 \text{ se } t = -\frac{B}{2A} \;\right]$$

$$\Rightarrow \int_0^\pi |f(s)|^2\,ds \geq \frac{1}{\pi}\left[\int_0^\pi |f(s)|\,ds\right]^2$$

$$\|f\|_2^2 \geq \frac{1}{\pi}\|f\|_1^2 \Rightarrow \|f\|_1 \leq \sqrt{\pi}\,\|f\|_2$$

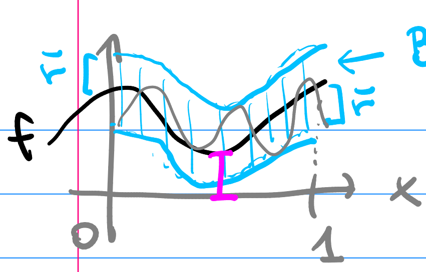
ESERCIZIO 2. *Si provi che l'insieme $A = \left\{f\in C^0[0,1] : f(x)>0 \text{ per ogni } x\in[0,1]\right\}$ è aperto in $\left(C^0[0,1], \|\cdot\|_\infty\right)$, poi si scriva l'espressione di A^c.*

se A è aperto allora $\forall f\in A$ $\exists\, r=r(f)>0$ tale che $B(f,r)\subseteq A$

$$B(f,r) = \{g\in C^0[0,1] : \|g-f\|_\infty < r\} \subseteq A = \{h\in C^0[0,1] : h(x)>0\}$$

osservo che $f\in A\subseteq C^0[0,1] \overset{W}{\Rightarrow} \exists\, x_0\in[0,1]$ tale che $f(x_0) = \min_{[0,1]}(f) = m > 0$

$\uparrow$
$f\in A$



$B(f,\bar r) \Rightarrow$ se $g\in B(f,\bar r)$ allora $\|f-g\|_\infty < \bar r$

$\Rightarrow |f(x)-g(x)| \le \|f-g\|_\infty < \bar r$

$\bar r := \frac{1}{2}m = \frac{1}{2}f(x_0)$

$$g\in B(f,\bar r) \Rightarrow \quad g(x) = g(x)-f(x)+f(x) \underset{\uparrow}{>} -\bar r + f(x) \ge -\bar r + m = \frac{1}{2}m > 0$$

$$\left[\ |g(x)-f(x)| < \bar r \quad \forall x\in[0,1] \Rightarrow -\bar r < g(x)-f(x) < \bar r\ \right]$$

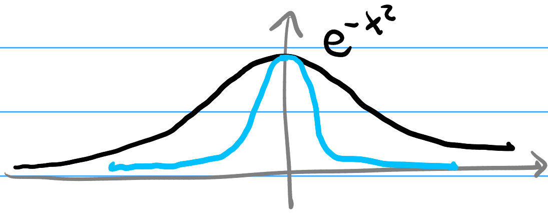
$$\Rightarrow g(x)>0 \quad \forall x\in[0,1] \Rightarrow g\in A$$

$$A = \{f\in C^0[0,1] : f(x)>0 \quad \forall x\in[0,1]\}$$

$$A^c = \{g\in C^0[0,1] : \exists p\in[0,1] : g(p)\le 0\}$$
$$= C^0[0,1]\setminus A$$

Esercizio 5. *Nello spazio $C^0[-1,1]$ si provi che, per $k=1,2,3,\dots$, si ha*
i. le funzioni $f_k(x)=e^{-kx^2}$ costituiscono una successione di Cauchy rispetto a $\|\cdot\|_2$,
ii. le funzioni $f_k(x)=e^{-kx^2}$ non sono una successione di Cauchy rispetto a $\|\cdot\|_\infty$,
iii. le funzioni $f_k(x)=e^{-|x|/k}$ costituiscono una successione di Cauchy rispetto a $\|\cdot\|_\infty$.

1. $$f_k(x) = e^{-kx^2} \longrightarrow \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases} = f_\infty(x)$$



$$\|f_k - f_{k+p}\|_2^2 = \int_{-1}^{1} \left[e^{-kx^2} - e^{-(k+p)x^2}\right]^2 dx = \int_{-1}^{1} e^{-2kx^2}\left[1-\cancel{e^{-px^2}}\right]^2 dx$$

$$\le \int_{-1}^{1} e^{-2kx^2}dx \underset{\substack{\uparrow \\ \sqrt{k}x=u}}{=} \int_{-\sqrt{k}}^{\sqrt{k}} \frac{1}{\sqrt{k}} e^{-2u^2}du \le \frac{1}{\sqrt{k}}\int_{\mathbb{R}} e^{-2u^2}du = \frac{C_G}{\sqrt{k}}$$

$$\Rightarrow \|f_k - f_{k+p}\|_2 \le \left[\frac{C_G}{\sqrt{k}}\right]^{1/2} \underset{\substack{\uparrow \\ ?}}{\le} \varepsilon \Rightarrow \frac{C_G^2}{k} \le \varepsilon^4, \quad k \ge \frac{C_G^2}{\varepsilon^4}$$

II. se $\{f_k\}$ fosse una successione di Cauchy in $C^0[-1,1]$
allora $f_k \to \bar{f} \in C^0[-1,1]$ per completezza

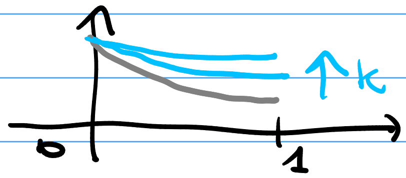
$$0 \le |f_k(x) - \bar{f}(x)| \le \max_{x \in [-1,1]} |f_k(x) - \bar{f}(x)| = \|f_k - \bar{f}\|_\infty \xrightarrow[k \to +\infty]{} 0$$

$\Rightarrow f_k(x) \to \bar{f}(x) \quad \forall x \in [-1,1]$ (convergenza in norma $\Rightarrow$ convergenza puntuale)

$\Rightarrow \boxed{\bar{f}(x) = f_\infty(x) \notin C^0[-1,1]}$ assurdo!

$\Rightarrow \{f_k\}$ non è di Cauchy

III. $\{f_k(x) = e^{|x|/k}\} \subseteq C^0[-1,1]$ è di Cauchy



$$f_k(x) = e^{-|x|/k} \xrightarrow[k \to +\infty]{} 1 = f_\infty(x) \quad \forall x \in [-1,1]$$

$\|f_k - f_{k+p}\|_\infty$

$$\|f_k - f_\infty\|_\infty = \max_{x \in [-1,1]} |1 - e^{-|x|/k}| = [1 - e^{-1/k}] \xrightarrow[k \to +\infty]{} 0$$

quindi $\{f_k\}$ è convergente, cioè di Cauchy!

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$f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ (aperto) f è differenziabile in $p \in A$ se $\exists\, W \in \mathbb{R}^n$ tale che

$$\lim_{u \to 0} \frac{f(p+u) - [f(p) + W \cdot u]}{\|u\|_2} = 0$$

$$\left[\, f(p+u) - [f(p) + W \cdot u] = o(\|u\|_2) \,\right]$$

TEOR. DIFFERENZIALE TOTALE

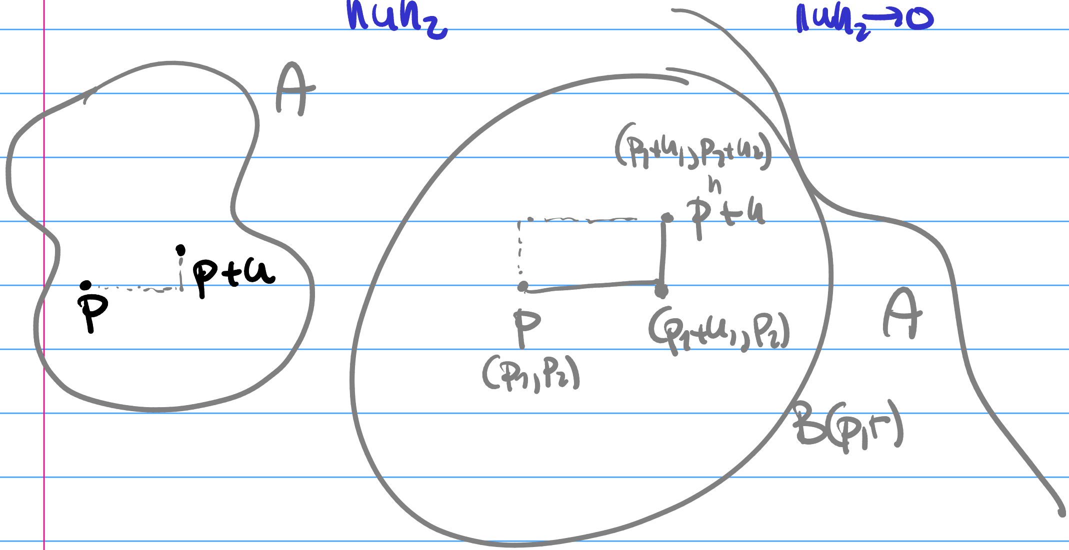
$f: \underset{\text{aperto}}{A} \subseteq \mathbb{R}^n \to \mathbb{R}$ $\qquad p \in A$

se $\underset{\forall i=1,\dots,n}{\exists\, \partial_i f(B(p,r))}$ e sono continue in $p \Rightarrow f$ è differenziabile in p.

$$\left[\, f \in C^1(A) \Rightarrow f \text{ è differenziabile in } A \,\right]$$

DIMOSTRAZIONE

$$\circledast \quad \frac{f(p+u) - [f(p) + \nabla f(p)\cdot u]}{\|u\|_2} \xrightarrow[\|u\|_2 \to 0]{} 0$$



$$\begin{aligned}
f(p+u) - f(p) &= f(p+u) \pm f(p+u_1e_1) - f(p) \\
&= [f(p+u) - f(p+u_1e_1)] + [f(p+u_1e_1) - f(p)] \\
&= [f(p_1+u_1, p_2+u_2) - f(p_1+u_1, p_2)] + [f(p_1+u_1, p_2) - f(p_1,p_2)] \\
&= u_2\, \partial_2 f(p_1+u_1, \xi) + u_1\, \partial_1 f(\eta, p_2)
\end{aligned}$$

con ξ tra p_2 e p_2+u_2 e η tra p_1 e p_1+u_1

$$\circledast = \frac{1}{\|u\|_2}\left[u_2\, \partial_2 f(p_1+u_1, \xi) + u_1\, \partial_1 f(\eta, p_2) - (\partial_1 f(p_1,p_2), \partial_2 f(p_1,p_2))\cdot(u_1,u_2)\right]$$

$$= \frac{1}{\|u\|_2}\left[u_1\left[\partial_1 f(\eta, p_2) - \partial_1 f(p_1,p_2)\right] + u_2\left[\partial_2 f(p_1+u_1, \xi) - \partial_2 f(p_1,p_2)\right]\right]$$

$$0 \leq \frac{|u_1|\cdot|\partial_1 f(\eta, p_2) - \partial_1 f(p_1,p_2)| + |u_2|\cdot|\partial_2 f(p_1+u_1,\xi) - \partial_2 f(p_1,p_2)|}{\|u\|_2}$$

$$\leq |\partial_1 f(\underset{\downarrow p_1}{\eta}, p_2) - \partial_1 f(p_1,p_2)| + |\partial_2 f(p_1+u_1,\underset{\downarrow p_2}{\xi}) - \partial_2 f(p_1,p_2)| \xrightarrow[u\to 0]{} 0$$

$$\frac{|u_i|}{\|u\|_2} = \frac{|u_i|}{\sqrt{u_1^2+u_2^2}} \leq \frac{|u_i|}{\sqrt{u_i^2}} = \frac{|u_i|}{|u_i|} = 1 \quad \text{per } i=1,2 \qquad \blacksquare$$

DEF. $f : \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}$ $\quad \partial_{ij} f(p) := \partial_j(\partial_i f(x))\big|_{x=p}$ $\quad i,j = 1,\dots,n$

TEOR. SCHWARZ $f : \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}$ se $f \in C^2(A) \Rightarrow \partial_{ij} f(x) = \partial_{ji} f(x)$ $\quad \forall i,j = 1,\dots,n$

DEF. $Hf(x) = (\partial_{ij} f(x))_{i,j=1,\dots,n}$ si dice matrice hessiana di f in x

OSS. se $f \in C^2(A) \Rightarrow (Hf) = (Hf)^T$ in A

ESERCIZIO 15. *Stabilire se le seguenti funzioni sono continue, derivabili, differenziabili o di classe* C^1 *in* ~~$\mathbb{R}^2$~~ $\mathbb{R}^3$

$$f(x,y,z) = x^2+y^2+z^3 \qquad g(x_1,x_2,x_3) = \begin{cases} \dfrac{x_1^3 - x_2^3}{x_1^2+x_2^2+2x_3^2} & x \neq O \\ 0 & x = O \end{cases} \qquad h(x_1,x_2,x_3) = x_1^2+x_2^2-x_3^2$$

Calcolare le equazioni cartesiane del piano tangente al grafico di f *in* $(1,1,2)$, *del piano tangente al grafico di* g *in* $(1,0,1)$ *e del piano tangente al grafico di* h *in* $(0,0,0)$.

$$f(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^3$$

$$\partial_1 f(x) = 2x_1 \qquad \partial_2 f(x) = 2x_2 \qquad \partial_3 f(x) = 3x_3^2$$

$\Rightarrow$ poiché $\partial_i f$ è un polinomio $\forall i = 1,2,3 \Rightarrow f \in C^1(\mathbb{R}^3)$

$\Rightarrow$ f è differenziabile in $\mathbb{R}^3$

$\partial_{11} f(x) = \partial_1(2x_1) = 2 \quad \partial_{12} f(x) = \partial_{13} f(x) = 0$

$\partial_{22} f(x) = \partial_2(2x_2) = 2 \quad \partial_{33} f(x) = \partial_3(3x_3^2) = 6x_3$

$\partial_{23} f(x) = 0$

$$Hf(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6x_3 \end{pmatrix}$$

eq. iperpiano Tangente: $x_4 = f(p) + \nabla f(p) \cdot (x-p) =: \pi(x)$

$= 10 + (2,2,12) \cdot (x_1 - 1, x_2 - 1, x_3 - 2)$

$p = (1,1,2)$

$f(1,1,2) = 1^2 + 1^2 + 2^3 = 10 \qquad = 2x_1 + 2x_2 + 12x_3 - 18$

$\nabla f(1,1,2) = (2,2,12)$

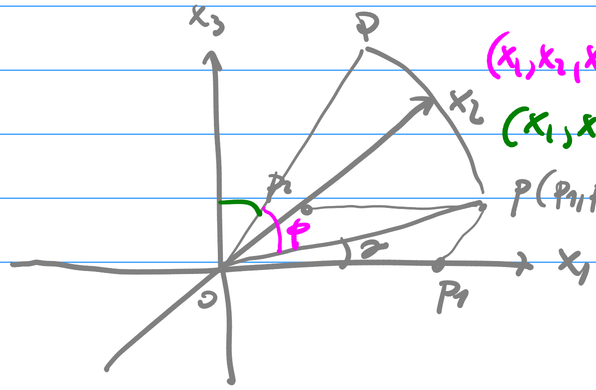
$(x-p) = (x_1 - 1, x_2 - 1, x_3 - 2)$

$\{2x_1 + 2x_2 + 12x_3 - x_4 = 18\} \subseteq \mathbb{R}^4$

$$g(x) = \begin{cases} \dfrac{x_1^3 - x_2^3}{x_1^2 + x_2^2 + 2x_3^2} & x \neq (0,0,0) \\ 0 & x = (0,0,0) \end{cases}$$

osservo che $g(x)$ è un rapporto di polinomi (funzione razionale) e che il denominatore è nullo solo x $x=0$, quindi $g \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ quindi derivabile, differenziabile e con hessiano simmetrico $\forall x \neq 0$.

¿ $\lim_{x \to 0} g(x) = \underset{g(0)}{0}$? $\quad g(x) =$



$(x_1, x_2, x_3) = r(\cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), \sin(\varphi))$

$(x_1, x_2, x_3) = r(\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi))$

$p(p_1, p_2) — (r, \theta) \qquad (r\cos(\theta), r\sin(\theta), 0)$

$(r, \theta, \varphi) \in [0, +\infty) \times [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$

$(r, \theta, \varphi) \in [0, +\infty) \times [0, 2\pi) \times [0, \pi]$

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$$0 \leq |g(x)| \overset{\text{c.s.}}{=} |g(r,\theta,\phi)| = \left| \frac{r^3\cos^3(\theta)\cos^3(\phi) - r^3\operatorname{sen}^3(\theta)\cos^3(\phi)}{r^2\cos^2(\theta)\cos^2(\phi) + r^2\operatorname{sen}^2(\theta)\cos^2(\phi) + 2r^2\operatorname{sen}^2(\phi)} \right|$$

$$\leq r \cdot \frac{|\cos^3(\phi)| \cdot (|\cos^3(\theta)| + |\operatorname{sen}^3(\theta)|)}{1+\operatorname{sen}^2(\phi)} \leq r\,\frac{2}{1} = 2r \xrightarrow[r\to 0]{} 0 = g(0)$$

g è continua in $\mathbb{R}^3$

$$\partial_1 g(0) = \lim_{k\to 0} \frac{g(k,0,0) - g(0,0,0)}{k} = \lim_{k\to 0} \frac{k^3}{k\cdot k^2} = 1$$

$$\partial_2 g(0) = \lim_{k\to 0} \frac{g(0,k,0) - g(0,0,0)}{k} = \lim_{k\to 0} \frac{-k^3}{k\cdot k^2} = -1$$

$$\partial_3 g(0) = \lim_{k\to 0} \frac{g(0,0,k) - g(0,0,0)}{k} = 0$$

$$\nabla g(0) = (1,-1,0)$$

$$\lim_{x\to 0} \frac{g(x) - [g(0) + \nabla g(0)\cdot x]}{\|x\|_2} = 0$$

$$\frac{g(x) - [g(0) + \nabla g(0)\cdot x]}{\|x\|_2} = \frac{1}{\|x\|_2}\left[\frac{x_1^3 - x_2^3}{x_1^2+x_2^2+2x_3^2} - (1,-1,0)\cdot(x_1,x_2,x_3)\right]$$

$$= \frac{1}{\|x\|_2}\left[\frac{\cancel{x_1^3} - \cancel{x_2^3} - (\cancel{x_1^3} + x_1x_2^2 + 2x_1x_3^2 - x_2x_1^2 - \cancel{x_2^3} - 2x_2x_3^2)}{x_1^2+x_2^2+2x_3^2}\right]$$

$$= \frac{1}{\|x\|_2}\left[\frac{x_1^2x_2 + 2x_2x_3^2 - x_1x_2^2 - 2x_1x_3^2}{x_1^2+x_2^2+2x_3^2}\right]$$

$$\overset{\text{c.s.}}{=} \frac{\cancel{r^3}[\cos^2(\theta)\cos^3(\phi)\operatorname{sen}(\theta) + 2\operatorname{sen}(\theta)\cos(\phi)\operatorname{sen}^2(\phi) - \dots]}{\cancel{r^3}(1+\operatorname{sen}^2(\phi))} \;\not\xrightarrow[r\to 0^+]{}\; 0$$

$\Rightarrow$ non è differenziabile in $x_0 = 0$!

TEOR. $f: \mathbb{R}^n \to \mathbb{R}$, sono equivalenti le seguenti affermazioni

i. f è continua,

ii. $\{x \in \mathbb{R}^n : f(x) > a\}$ è un aperto, $\forall a \in \mathbb{R}$,

iii. $\{x \in \mathbb{R}^n : f(x) \geq a\}$ è un chiuso, $\forall a \in \mathbb{R}$.

DEF. $f: \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \to \mathbb{R}^k$, $p \in A$ è differenziabile in p se

$$\exists\, M \in \mathcal{M}_{k,n} : \lim_{w \to 0} \frac{\| f(p+w) - [f(p) + M \cdot w] \|_2}{\|w\|_2} = 0$$

$$\left[f(x) = (f_1(x), f_2(x), \ldots, f_k(x)),\ \forall x \in A \right]$$

$$M = Jf(p) = (\partial_j f_i(p))_{i,j} \qquad i = 1, \ldots, k,\ j = 1, \ldots, n$$

matrice Jacobiana di f

TEOR. $f: \underset{\text{aperto}}{A \subseteq \mathbb{R}^n} \to \mathbb{R}^k$ differenziabile

$g: \underset{\text{aperto}}{B \subseteq \mathbb{R}^k} \to \mathbb{R}^p$ differenziabile

$f(A) \subseteq B$ allora $h(x) := g(f(x))$ è differenziabile

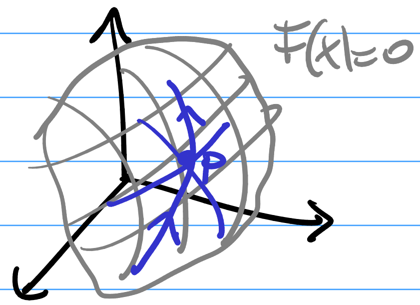
$$Jh(x_0) = Jg(f(x_0))\, Jf(x_0)$$

cioè $$(Jh(x_0))_{ij} = \partial_j h_i(x_0) = \sum_{s=1}^{k} \partial_s g_i(f(x_0))\, \partial_j f_s(x_0)$$

$$\left[Jh \in \mathcal{M}_{p,n},\ Jg \in \mathcal{M}_{p,k},\ Jf \in \mathcal{M}_{k,n} \right]$$

OSS. $F:\mathbb{R}^n \to \mathbb{R}$ di classe C^1

$V=\{x\in\mathbb{R}^n : F(x)=0\} \neq \emptyset$



$\phi:[-1,1]\to\mathbb{R}^n$ Tale che

$F(\phi(t))=0$ $\quad (\gamma = \mathrm{Im}(\phi)\subseteq V)$

e $\phi(0)=p$

$\phi:\mathbb{R}\to\mathbb{R}^n \quad J\phi(t)=\begin{pmatrix}\phi_1'(t)\\ \vdots \\ \phi_n'(t)\end{pmatrix}=\phi'(t)$

$JF(x)=\nabla F(x)=(\partial_1 f(x),\partial_2 f(x),\dots,\partial_n f(x))$

$\frac{d}{dt}(F(\phi(t)))=JF(\phi(t))\cdot J\phi(t)=\nabla F(\phi(t))\cdot\phi'(t)=0$

se scelgo $t=0 \Rightarrow \nabla F(\underbrace{\phi(0)}_{=p})\cdot \underbrace{\phi'(0)}_{=w}=0$

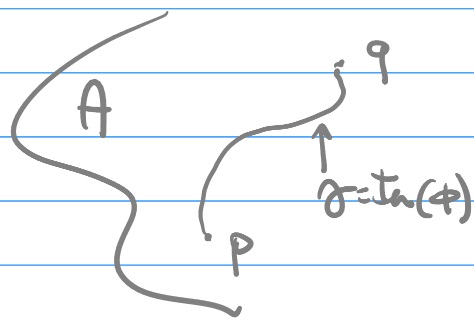
$w\in T_p(V) \leftarrow$ spazio Tangente a V nel punto $p\in V$

OSS. $f:\underset{\text{aperto}}{A}\subseteq\mathbb{R}^n\to\mathbb{R}$ $\quad A$ connesso

$f\in C^1(A)$ e $\nabla f(x)=0 \quad \forall x\in A \Rightarrow f$ è costante

se $\exists\, p,q\in A$ Tali che $f(p)\neq f(q)$, considero $\phi:[0,1]\to A$

$\phi(0)=p$ e $\phi(1)=q$



$0\neq f(p)-f(q)=f(\phi(0))-f(\phi(1))$

$=(0-1)\cdot\left[\frac{d}{dt}f(\phi(t))\right]_{t=\xi}$

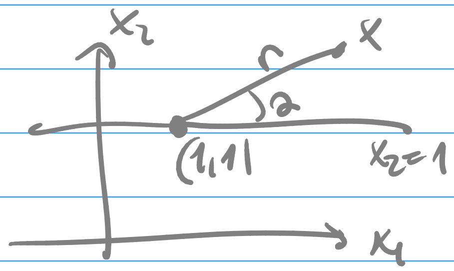
$=-\nabla f(\phi(\xi))\cdot\phi'(\xi)=0$?

ESERCIZIO 14. *Data*

$$f(x,y)=\begin{cases}\dfrac{(x-y)^3}{[(x-1)^2+(y-1)^2]^a} & (x,y)\neq(1,1)\\ 0 & (x,y)=(1,1)\end{cases}$$

studiare la continuità, la derivabilità e la differenziabilità di f al variare del parametro $a\in\mathbb{R}$.

$$\begin{cases} x_1 = 1 + r\cos(\vartheta) \\ x_2 = 1 + r\,\text{sen}(\vartheta)\end{cases} \qquad \vartheta\in[0,2\pi] \quad r\geq 0$$



$$f(r,\vartheta)=\begin{cases}\dfrac{r^3(\cos(\vartheta)-\text{sen}(\vartheta))^3}{r^{2a}} \\ 0 & r=0\end{cases}$$

$$r^{3-2a}(\cos(\vartheta)-\text{sen}(\vartheta))^3 \xrightarrow[r\to 0^+]{?} 0 = f(1,1)$$

se $3-2a>0 \Rightarrow a<3/2$

f è continua in $\mathbb{R}^2$ se e solo se $a<3/2$

Derivabilità: scriviamo i rapporti incrementali nel punto $(1,1)$

$$\frac{1}{k}[f(1+k,1)-f(1,1)] = \frac{1}{k}\cdot\frac{k^3}{k^{2a}} = k^{2-2a} \longrightarrow \begin{cases} 0 & \text{se } a<1 \\ 1 & \text{se } a=1 \\ \nexists & \text{se } a>1\end{cases}$$

$$\frac{1}{k}[f(1,1+k)-f(1,1)] = \frac{1}{k}\cdot\left[\frac{-k^3}{k^{2a}}\right] = -k^{2-2a} \longrightarrow \begin{cases} 0 & \text{se } a<1 \\ -1 & \text{se } a=1 \\ \nexists & \text{se } a>1\end{cases}$$

quindi f è derivabile in $(1,1)$ se e solo se $a\leq 1$.

Differenziabilità: distinguiamo i casi $a=1$ e $a<1$

se $a=1$ abbiamo

$$\lim_{w\to 0}\frac{f(1+w_1,1+w_2)-[f(1,1)+\overbrace{(1,-1)}^{\nabla f(1,1)}(w_1,w_2)]}{[w_1^2+w_2^2]^{1/2}}$$

$$= \lim_{w \to 0} \frac{w_1 w_2 (w_1 - w_2)}{\|w\|_2^3} = \lim_{r \to 0} sen(\alpha)\cos(\alpha)(\cos(\alpha) - sen(\alpha)) \neq 0$$

quindi la funzione non è differenziabile in $(1,1)$ se $\alpha = 1$.

Se $\alpha < 1$ i calcoli precedenti diventano

$$\lim_{w \to 0} \frac{f(1+w_1, 1+w_2) - [f(1,1) + \overset{(0,0)}{\overset{\shortparallel}{\nabla f(1,1)}} \cdot (w_1, w_2)]}{\|w\|_2}$$

$$= \lim_{w \to 0} \frac{(w_1 - w_2)^3}{\|w\|_2^{1+2\alpha}} = \lim_{r \to 0} r^{2(1-\alpha)} [\cos(\alpha) - sen(\alpha)]^3 = 0$$

visto che $0 \leq |(\cos(\alpha) - sen(\alpha))^3| \leq [|\cos(\alpha)| + |sen(\alpha)|]^3 = 8 \quad \forall \alpha \in \mathbb{R}$

quindi f è differenziabile in $(1,1)$ se e solo se $\alpha < 1$.

Concludiamo osservando che $\forall x \neq (1,1)$ f è differenziabile perché rapporto di funzioni differenziabili (in quanto il denominatore non si annulla).