Oggi alle 14:00 in Aula 9 Tutoraggio con il Dott. G. Peluso.

Forme indeterminate: sono dei limiti che non si possono risolvere <u>automaticamente</u> con l'algebra (estesa) dei limiti, ma vanno risolti caso per caso.

$$+\infty-\infty\,,\quad 0\cdot(\pm\infty)\,,\quad \frac{\pm\infty}{\pm\infty}\,,\quad \frac{0}{0}$$

e poi

$$0^0 \qquad 1^{\pm\infty}\,,\quad (\pm\infty)^0$$

<u>Limiti di polinomi.</u>

$$P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_2 n^2 + a_1 n + a_0 = \sum_{j=0}^{k} a_j n^j$$

dove $k\in\mathbb{N}$, $a_0, a_1, \dots, a_k \in \mathbb{R}$, $a_k \neq 0$

$$\lim_{n\to+\infty} P_k(n) = \lim_{n\to+\infty} \underbrace{n^k}_{\to+\infty} \underbrace{\left(a_k + \underbrace{\frac{a_{k-1}}{n} + \frac{a_{k-2}}{n^2} + \dots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}}_{\to 0}\right)}_{\to a_k} =$$

$$= \begin{cases} +\infty & \text{se } a_k > 0 \\ -\infty & \text{se } a_k < 0 \end{cases}$$

$$\lim_{n\to+\infty} (3n^2 - \underbrace{2n^5}_{\uparrow} + n + 7) = -\infty$$

<u>Limiti di funzioni razionali (rapporti di polinomi) di n.</u>

$$\lim_{n\to+\infty} \frac{\overbrace{3n^2-2n+5}^{\to+\infty}}{\underbrace{-n^2+n+1}_{\to -\infty}} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to+\infty} \frac{\cancel{n^2}\overbrace{\left(3-\frac{2}{n}+\frac{5}{n^2}\right)}^{\to 3}}{\cancel{n^2}\underbrace{\left(-1+\frac{1}{n}+\frac{1}{n^2}\right)}_{\to -1}} = -3$$

$$\lim_{n\to+\infty} \frac{3n^2-2n+5}{-n^3+n+1} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to+\infty} \frac{\cancel{n^2}\overbrace{\left(3-\frac{2}{n}+\frac{5}{n^2}\right)}^{\to 3}}{\underbrace{n^{\cancel{3}}}_{\to+\infty}\underbrace{\left(-1+\frac{1}{n^2}+\frac{1}{n^3}\right)}_{\to -1}} = 0$$

$$\lim_{n\to+\infty} \frac{3n^3-2n+5}{-n^2+n+1} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to+\infty} \frac{n^{\cancel{3}}\,(3+\text{infinitesimo})}{\cancel{n^2}\,(-1+\text{infinitesimo})} = -\infty.$$

In generale:

$$P_k(n) = \sum_{j=0}^{k} a_j n^j \qquad Q_h(n) = \sum_{j=0}^{h} b_j n^j$$

$$a_0, \dots a_k,\ b_0, \dots b_h \in \mathbb{R}, \quad a_k, b_h \neq 0$$

$$\lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to+\infty} \frac{n^k}{n^h} \underbrace{\boxed{\frac{(a_k+\text{infinitesimo})}{(b_h+\text{infinitesimo})}}}_{\to \frac{a_k}{b_h}\neq 0} =$$

$$= \lim_{n\to+\infty} n^{k-h} \cdot \left(\frac{a_k}{b_h} + \text{infinitesimo}\right) =$$

$$= \begin{cases} \pm\infty & \text{se } k>h \quad \begin{cases} +\infty & \text{se } a_k \text{ e } b_h \text{ concordi di segno} \\ -\infty & \text{se } a_k \text{ e } b_h \text{ discordi} \end{cases} \\ \dfrac{a_k}{b_h} & \text{se } k=h \\ 0 & \text{se } k<h \quad \begin{cases} 0^+ & \text{se } a_k \text{ e } b_h \text{ concordi} \\ 0^- & \text{"} \quad \text{"} \quad \text{"} \quad \text{discordi} \end{cases} \end{cases}$$

In definitivo $\lim_{n\to+\infty} \frac{P_k(n)}{Q_h(n)} = \lim_{n\to+\infty} \frac{a_k n^k}{b_h n^h}$

Notazione: Se una successione a_n è infinitesima, cioè $\lim_{n\to+\infty} a_n = 0$, scriveremo $a_n = o(1)$ per $n\to+\infty$.

Es. $\frac{1}{n} = o(1)$ per $n\to+\infty$

$\frac{5}{n^2+3} = o(1)$ "

Ma (Attenzione alla notazione con l'uguale!) da questo non segue che $\frac{1}{n} = \frac{5}{n^2+3}$!!!

$$\lim_{n\to+\infty} \frac{3n^2+2n+5}{-n^2+n+1} = \lim_{n\to+\infty} \frac{\cancel{n^2}\,(3+o(1))}{\cancel{n^2}\,(-1+o(1))} = -3$$

Per indicare una successione che tende a 5, scriverò

$a_n = 5 + o(1)$

$$\lim_{n\to+\infty} \underbrace{\frac{n^\alpha - n}{n^\pi - n^3}}_{a_n} = \left(\frac{\infty}{\infty}\right) \qquad (\alpha\in\mathbb{R})$$

1° caso: $\alpha > 1$ $\quad \frac{n^\alpha - n}{n^\pi - n^3} = \frac{n^\alpha(1+o(1))}{n^\pi(1+o(1))} \longrightarrow \begin{cases} +\infty & \text{se } \alpha>\pi \\ 1 & \text{se } \alpha=\pi \\ 0 & \text{se } 1<\alpha<\pi \end{cases}$

(se serve, è 0^+)

2° caso $\alpha = 1$ $\Rightarrow a_n \equiv 0 \Longrightarrow \lim_{n\to+\infty} a_n = 0$.

3° caso $\alpha < 1$ $\Rightarrow \frac{n^\alpha - n}{n^\pi - n^3} = \frac{\cancel{n}\,(-1+o(1))}{n^{\cancel{\pi}\,\pi-1}(1+o(1))} \longrightarrow 0$ (se serve, è 0^-)

$$\lim_{n\to+\infty} \frac{2^n-5^n}{3^n+1} = \lim_{n\to+\infty} \frac{5^n\left(\frac{2^n}{5^n}-1\right)}{3^n(1+o(1))} =$$

$$\left(\frac{2}{5}\right)^n \to 0$$

$$= \lim_{n\to+\infty} \frac{5^n}{3^n}\underbrace{\boxed{\frac{(-1+o(1))}{(1+o(1))}}}_{\to -1} = \lim_{n\to+\infty} \underbrace{\left(\frac{5}{3}\right)^n}_{\to+\infty}\underbrace{(-1+o(1))}_{\to -1} = -\infty$$

$$\lim_{n\to+\infty}\left(\underbrace{\sqrt{n^3+n}}_{\to+\infty}\ \underbrace{-n}_{\to-\infty}\right) = (+\infty-\infty) = \lim_{n\to+\infty} \underbrace{n^{3/2}}_{\to+\infty}\underbrace{\left(\underbrace{\sqrt{1+\frac{1}{n^2}}}_{\to 1} - \underbrace{\frac{1}{\sqrt{n}}}_{\to 0}\right)}_{=1+o(1)} =$$

$$= +\infty$$

$$\lim_{n\to+\infty}\left(\sqrt{n^2+n}-2n\right) = (+\infty-\infty) =$$

$$= \lim_{n\to+\infty} n\underbrace{\left(\underbrace{\sqrt{1+\frac{1}{n}}}_{\to 1}-2\right)}_{\to -1} = -\infty$$

$$\lim_{n\to+\infty}\left(\sqrt{n^2+n}-n\right) = (+\infty-\infty) = \lim_{n\to+\infty} \underbrace{n}_{\to+\infty}\underbrace{\left(\sqrt{1+\frac{1}{n}}-1\right)}_{\to 0} = (+\infty\cdot 0)$$

??

In questo caso usiamo $A^2-B^2=(A-B)(A+B)$

$$\sqrt{n^2+n}-n = \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n} =$$

$$= \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \frac{\cancel{n}\ 1}{\cancel{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} \to \frac{1}{2}$$

$$\lim_{n\to+\infty}\Big(\underbrace{\sqrt[3]{n^3+n}}_{A}-\underbrace{n}_{B}\Big)=(+\infty-\infty)=$$

$$\left[A^3-B^3=(A-B)(A^2+AB+B^2)\right]$$

$$=\lim_{n\to+\infty}\frac{\left(\sqrt[3]{n^3+n}-n\right)\left((n^3+n)^{2/3}+n\sqrt[3]{n^3+n}+n^2\right)}{(n^3+n)^{2/3}+n\sqrt[3]{n^3+n}+n^2}=$$

$$=\lim_{n\to+\infty}\frac{\cancel{n^3}+n-\cancel{n^3}}{(n^3+n)^{2/3}+n\sqrt[3]{n^3+n}+n^2}=$$

$$=\lim_{n\to+\infty}\frac{\cancel{n}\ 1}{n^{\cancel{2}}\left(3+o(1)\right)}=0$$

$$\frac{(n^3+n)^{2/3}}{n^2}=\frac{\left(n^3\left(1+\frac{1}{n^2}\right)\right)^{2/3}}{n^2}=\frac{\cancel{n^2}\left(1+\frac{1}{n^2}\right)^{2/3}}{\cancel{n^2}}\to 1$$

$$\lim_{n\to+\infty}\Big[\underbrace{\log_{1/2}(n^3)}_{\downarrow\ -\infty}-\underbrace{\log_{1/2}(n+1)}_{\downarrow\ +\infty}\Big]=(-\infty+\infty)=$$

$$=\lim_{n\to+\infty}\log_{1/2}\Big(\underbrace{\frac{n^3}{n+1}}_{\downarrow\ +\infty}\Big)=-\infty$$

1) $\log_{1/2}A-\log_{1/2}B=\log_{1/2}\frac{A}{B}$

2) se $a_n\to+\infty\Rightarrow\log_{1/2}a_n\to-\infty$

$$\lim_{n\to+\infty}\Big(\underbrace{\frac{n^2-2n}{2n^2+1}}_{\downarrow\ 1/2}\Big)^{\overbrace{-n}^{\to\,-\infty}}=\left(\frac{1}{2}\right)^{-\infty}=+\infty$$

Vediamo in dettaglio,

$$\left(\frac{n^2-2n}{2n^2+1}\right)^{-n} = 2^{-n\log_2\left(\frac{n^2-2n}{2n^2+1}\right)} \to \left(2^{+\infty}\right) = +\infty$$

$$\underbrace{-n}_{\to -\infty} \; \underbrace{\log_2\left(\frac{n^2-2n}{2n^2+1}\right)}_{\to \log_2\left(\frac{1}{2}\right) = -1} \longrightarrow +\infty \qquad \frac{n^2-2n}{2n^2+1} \to 1/2$$

$$\underbrace{\left(\frac{n^2-2n}{n^2+1}\right)^{-n}}_{\shortparallel \;\; 2^{-n\log\left(\frac{n^2-2n}{n^2+1}\right)}} = \left(1^{-\infty}\right)$$

$$\underbrace{-n}_{\to -\infty} \; \underbrace{\boxed{\log\left(\frac{n^2-2n}{n^2+1}\right)}}_{\to \log 1 = 0} \longrightarrow \; ??$$

$$\lim_{n\to+\infty} n^{nx-3} \qquad x \in \mathbb{R}.$$

$$\boxed{x > 0} \Rightarrow nx-3 \xrightarrow{n\to+\infty} +\infty \Rightarrow \underbrace{n}_{\to+\infty}^{\overbrace{nx-3}^{\to+\infty}} \longrightarrow \left((+\infty)^{+\infty}\right) = +\infty$$

$$\boxed{x = 0} \Rightarrow n^{nx-3} = n^{-3} \longrightarrow 0$$

$$\boxed{x < 0} \Rightarrow nx-3 \to -\infty \Rightarrow n^{nx-3} \to \left((+\infty)^{-\infty}\right) = 0$$

Limiti di succ^{ni} monotòne.

Una succ^{ne} si dice crescente se $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$.
decrescente $\geq$

Una succ^{ne} monotòna è una succ^{ne} crescente oppure decrescente.

TEOREMA (limite di successioni monotòne)

Sia $\{a_n\}$ una succ^{ne} di numeri reali. Allora

1) se $\{a_n\}$ è crescente, $\exists$ esiste $\lim\limits_{n\to+\infty} a_n = \sup\limits_{n\in\mathbb{N}} a_n$

2) se $\{a_n\}$ è decrescente, $\exists \lim\limits_{n\to+\infty} a_n = \inf\limits_{n\in\mathbb{N}} a_n$

OSS. Permette, per es., di calcolare facilmente sup e inf di una succ^{ne}.

$$\sup \Bigg\{ \underbrace{\frac{5n^2-2}{n^2}}_{\substack{\shortparallel \\ 5-\frac{2}{n^2}}} \; ; \; n \in \mathbb{N}^+ \Bigg\} = \lim_{n\to+\infty} \frac{5n^2-2}{n^2} = 5$$

crescente.

Riferimento sul testo consigliato: §§ 2.8, 2.9.