ESERCIZIO 1 ①

1. Riscriviamo lo spinore in coordinate sferiche

$$\psi = N \begin{pmatrix} -re^{i\varphi}\sin\theta \\ \sqrt{2}\, r\cos\theta \\ re^{i\varphi}\sin\theta \end{pmatrix} e^{-r/2a_0}$$

Dato che $re^{-r/2a_0} = C R_{21}(r)$ $\quad C = (2a_0)^{3/2}\sqrt{3}\,a_0$

otteniamo

$$\psi = \underbrace{NC\sqrt{\frac{8\pi}{3}}}_{A} \begin{pmatrix} Y_1^1 \\ Y_1^0 \\ Y_1^{-1} \end{pmatrix} R_{21}(r)$$

che possiamo riscrivere come

$$|\psi\rangle = A\left[\,|2\,1\,1\rangle|1\,1\rangle_s + |2\,1\,0\rangle|1\,0\rangle_s + |2\,1\,-1\rangle|1\,-1\rangle_s\right]$$

dove $|1\,S_z\rangle_s$ sono i ket dello spin e

$\langle\vec{r}\,|\,n\,\ell\,m\rangle = R_{n\ell}(r)\,Y_\ell^m(\theta,\varphi)$ è la parte spaziale

Imponendo
$\langle\psi|\psi\rangle = 1 \implies 3|A|^2 = 1 \quad |A| = \frac{1}{\sqrt{3}} \implies A = \frac{1}{\sqrt{3}}$ (scelta di fase)

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left[\,|2\,1\,1\rangle|1\,1\rangle_s + |2\,1\,0\rangle|1\,0\rangle_s + |2\,1\,-1\rangle|1\,-1\rangle_s\right]$$

Per rispondere alla domanda è necessario passare alla base in cui $J_1^2 J_z$ sono diagonali.

Notazione: dato che $\langle\vec{r}\,|\,n\,\ell\,m\rangle = R_{n\ell}(r)\,Y_\ell^m(\theta,\varphi)$

riscriviamo $|n\,\ell\,m\rangle = \underset{\substack{\uparrow \\ \text{parte radiale}}}{|n\,\ell\rangle_r}\ \underset{\substack{\uparrow \\ \text{parte angolare}}}{|\ell\,m\rangle}$

$$|\psi\rangle = \frac{1}{\sqrt{3}}|2\,1\rangle_r \left[|1\,1\rangle|1\,1\rangle_S + |1\,0\rangle|1\,0\rangle_S + |1\,-1\rangle|1\,-1\rangle_S \right]$$

= (tabelle CG 1×1) =

$$= \frac{1}{\sqrt{3}}|2\,1\rangle_r \left[\overset{l\ \ s\ \ J\ \ J_z}{|1\,1\ 1\ \ 2\ \ 2\rangle_J} + \sqrt{\frac{2}{3}}\,|1\,1\,1\,2\,0\rangle_J \right.$$

$$\left. - \sqrt{\frac{1}{3}}\,|1\,1\,1\,0\,0\rangle_J + |1\,1\,1\ \ 2\,-2\rangle_J \right]$$

Quindi

$$\text{prob}\,(J^2 = 6\hbar^2) = \frac{1}{3}\left(1 + \frac{2}{3} + 1\right) = \frac{8}{9}$$

$$\text{prob}\,(J^2 = 0) = \frac{1}{9}$$

$$\text{prob}\,(J_z = 2\hbar) = \frac{1}{3}$$

$$\text{prob}\,(J_z = 0) = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$$

$$\text{prob}\,(J_z = -\hbar) = \frac{1}{3}$$

2. Si tratta di una combinazione di stati con $n=2$.
Quindi $\text{prob}\left(E = -\frac{E_0}{4}\right) = 1$ [E_0 = energia stato fond.]

3. $\bar{L}\cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$ commuta con H.
Quindi il valor medio non dipende da t.

$$\langle\psi|\bar{L}\cdot\bar{S}|\psi\rangle = \frac{1}{2}\langle\psi|J^2|\psi\rangle - \frac{1}{2}(2\hbar^2 + 2\hbar^2)$$

$$= \frac{1}{2}\langle\psi|J^2|\psi\rangle - 2\hbar^2$$

$$= \frac{1}{2}\left[\frac{8}{9}\cdot 6\hbar^2 + \frac{1}{9}\cdot 0\right] - 2\hbar^2$$

$$= \frac{8}{3}\hbar^2 - 2\hbar^2 = \frac{2}{3}\hbar^2$$

4. Si tratta di un autostato di L^2 con autovalore $2\hbar^2$ (stato con $L=1$). Non si ottiene mai lo stato fondamentale

Potenziale

Per $-L \leq x \leq 0$ abbiamo

$$-\frac{\hbar^2}{2m}\psi'' - V_0\psi = -|E|\psi$$

($E<0$ per stati legati)

$$\psi'' + \frac{2m}{\hbar^2}(V_0-|E|)\psi = 0$$

Se $\lambda = \sqrt{\frac{2m}{\hbar^2}(V_0-|E|)}$ la soluzione che soddisfa $\psi(0)=0$ è $A\sin\lambda x$

Per $x \leq -L$ abbiamo $-\frac{\hbar^2}{2m}\psi'' = -|E|\psi$ se $k = \sqrt{\frac{2m|E|}{\hbar^2}}$

la soluzione è

$$\psi = Be^{kx} + Ce^{-kx}$$

↑ diverge per $x \to -\infty$

Quindi

$$\psi = \begin{cases} B\sin\lambda x & -L \leq x \leq 0 \\ Ae^{kx} & x \leq -L \\ 0 & x > 0 \end{cases}$$

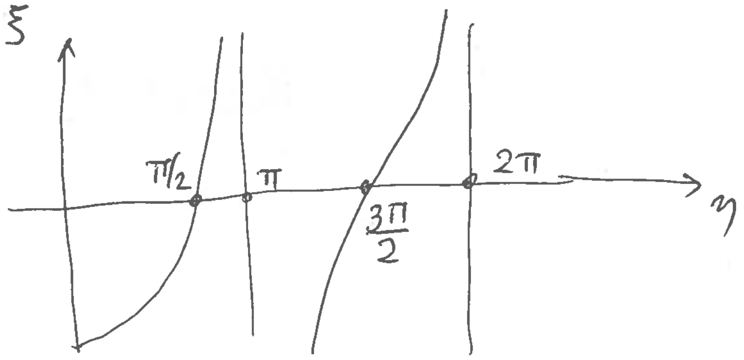
Raccordo in $x=-L$

$$\begin{cases} -B\sin\lambda L = Ae^{-kL} \\ B\lambda\cos\lambda L = kAe^{-kL} \end{cases} \Rightarrow -\frac{1}{\lambda}\tan\lambda L = \frac{1}{k}$$

$$\boxed{k = -\lambda\cot\lambda L}$$

Per trovare le soluzioni definiamo $\xi = kL$, $\eta = \lambda L$

$$\begin{cases} \xi^2 + \eta^2 = L^2(k^2+\lambda^2) = \frac{2mV_0}{\hbar^2}L^2 = R^2 \\ \xi = -\eta\cot\eta \end{cases} \qquad \xi, \eta > 0$$



Non ci sono stati legati per $R<\frac{\pi}{2}$

1 stato legato per $\frac{\pi}{2} \le R < \frac{3\pi}{2} = \frac{\pi}{2}+\pi$

⋮

n stati legati per $(n-1)\pi + \frac{\pi}{2} \le R < n\pi + \frac{\pi}{2}$

2. Se vi è un solo stato legato

$$\frac{\pi}{2} \le R < \frac{3\pi}{2} \qquad \left(\frac{\pi}{2}\right)^2 \le \frac{2mV_0}{\hbar^2}L^2 < \left(\frac{3\pi}{2}\right)^2$$

$$\frac{\hbar^2}{2mL^2}\left(\frac{\pi}{2}\right)^2 \le V_0 < \frac{\hbar^2}{2mL^2}\left(\frac{3\pi}{2}\right)^2 \qquad (*)$$

Se $|E| = \frac{V_0}{2}$ $\qquad k = \lambda = \sqrt{\frac{mV_0}{\hbar^2}}$

Quindi $k = -\lambda \cot \lambda L \Rightarrow \cot \lambda L = -1$

$$\lambda L = \left(\frac{3\pi}{4} + k\pi\right) \qquad k \ge 0$$

$$V_0 = \frac{\hbar^2}{mL^2}\left(\frac{3\pi}{4} + k\pi\right)^2$$

Solo la soluzione con $k=0$ soddisfa (*).

Quindi

$$V_0 = \frac{9\pi^2}{16}\frac{\hbar^2}{mL^2}$$

3. Per $k = \lambda = \frac{3\pi}{4}\frac{1}{L}$ abbiamo

$$-B\sin\lambda L = Ae^{-kL} \Rightarrow -B\sin\left(\frac{3\pi}{4}\right) = Ae^{-kL}$$

$$-B\frac{\sqrt{2}}{2} = Ae^{-kL}$$

Quindi

$$\psi = \begin{cases} B \sin kx & -L \leq x \leq 0 \\ A e^{kx} = -B \frac{\sqrt{2}}{2} e^{k(x+L)} & x < -L \end{cases}$$

Ora

$$\langle\psi|\psi\rangle = 1 \Rightarrow |B|^2 \int_{-L}^{0} dx \sin^2 kx + \frac{|B|^2}{2} \int_{-\infty}^{-L} dx\, e^{2k(x+L)}$$

$$= |B|^2 \frac{1}{2} \int_{-L}^{0} dx\, (1 - \cos 2kx) + \frac{|B|^2}{2} \frac{1}{2K} \left[e^{2k(x+L)} \right]_{-\infty}^{-L}$$

$$= \frac{|B|^2}{2} \left(L - \frac{1}{2k} \sin 2kL \right) + \frac{|B|^2}{2} \frac{1}{2k}$$

$$\left[\sin 2kL = \sin\left(\frac{3\pi}{2}\right) = -1 \right)$$

$$\langle\psi|\psi\rangle = |B|^2 \left(\frac{L}{2} + \frac{1}{4k} + \frac{1}{4k} \right) = |B|^2 \left(\frac{L}{2} + \frac{2L}{3\pi} \right)$$

$$= |B|^2 L \frac{3\pi + 4}{6\pi} \qquad |B|^2 = \sqrt{\frac{1}{L} \frac{6\pi}{3\pi+4}}$$

$$\text{prob}(x \leq -L) = \int_{-\infty}^{-L} dx \frac{|B|^2}{2} e^{2k(x+L)} = \frac{|B|^2}{4k}$$

$$= \frac{1}{L} \frac{6\pi}{3\pi+4} \cdot \frac{L}{3\pi} = \frac{2}{3\pi+4}$$

4. $\Delta E = v \int_{-\infty}^{+\infty} |\psi(x)|^2 x\, dx$

Calcoliamo separatamente gli integrali in $[-L, 0]$ e $[-\infty, -L]$

$$(a) = \int_{-L}^{0} |B|^2 \sin^2 kx \, x \, dx = \frac{|B|^2}{2} \int_{-L}^{0} x(1-\cos 2kx)\, dx$$

$$= \frac{|B|^2}{2}\left(-\frac{L^2}{2}\right) - \frac{|B|^2}{4}\frac{\partial}{\partial k}\int_{-L}^{0} dx \sin 2kx$$

$$= -\frac{|B|^2}{4} L^2 - \frac{|B|^2}{4}\frac{\partial}{\partial k}\left[-\frac{1}{2k}\cos 2kx\right]_{-L}^{0}$$

$$= -\frac{|B|^2}{4} L^2 - \frac{|B|^2}{4}\frac{\partial}{\partial k}\left[-\frac{1}{2k} + \frac{1}{2k}\cos 2kL\right]$$

$$= -\frac{|B|^2 L^2}{4} - \frac{|B|^2}{4}\left[\frac{1}{2k^2} - \frac{1}{2k^2}\cos 2kL - \frac{1}{2k}\sin 2kL \cdot 2L\right]$$

Se $kL = \frac{3\pi}{4}$ $\cos 2kL = 0$ $\sin 2kL = -1$. Quindi

$$(a) = -\frac{|B|^2}{4}\left[L^2 + \frac{1}{2k^2} + \frac{L}{k}\right]$$

$$= -\frac{|B|^2}{4} L^2 \left[1 + \frac{8}{9\pi^2} + \frac{4}{3\pi}\right]$$

$$= -\frac{|B|^2 L^2}{36\pi^2}\left(9\pi^2 + 8 + 12\pi\right)$$

$$(b) = \frac{|B|^2}{2}\int_{-\infty}^{-L} dx \, x \, e^{2k(x+L)}$$

$$= \frac{|B|^2}{4}\left\{\frac{\partial}{\partial k}\int_{-\infty}^{-L} dx \, e^{2k(x+L)} - 2L\int_{-\infty}^{-L} dx \, e^{2k(x+L)}\right\}$$

$$= \frac{|B|^2}{4}\left\{\frac{\partial}{\partial k}\left[\frac{1}{2k} e^{2k(x+L)}\right]_{-\infty}^{-L} - 2L\left[\frac{1}{2k} e^{2k(x+L)}\right]_{-\infty}^{-L}\right\}$$

$$= \frac{|B|^2}{4}\left\{\frac{\partial}{\partial k}\left(\frac{1}{2k}\right) - \frac{L}{k}\right\}$$

$$= -\frac{|B|^2}{4}\left\{\frac{1}{2k^2} + \frac{L}{k}\right\} = -\frac{|B|^2}{4}\left(\frac{8}{9\pi^2} + \frac{4}{3\pi}\right) L^2$$

$$= -\frac{|B|^2}{9\pi^2}\left(2 + 3\pi\right) L^2$$

Quindi

$$\langle\psi|x|\psi\rangle = -\frac{|B|^2L^2}{36\pi^2}\left(9\pi^2+12\pi+8\right)$$

$$-\frac{|B|^2L^2}{9\pi^2}\left(3\pi+2\right)$$

$$= -\frac{|B|^2L^2}{36\pi^2}\left(9\pi^2+12\pi+8+8+12\pi\right)$$

$$= -\frac{B|^2L^2}{36\pi^2}\left(9\pi^2+24\pi+16\right)$$

$$= -\frac{|B|^2L^2}{36\pi^2}\left(3\pi+4\right)^2$$

$$= -\frac{1}{36\pi^2}\,\frac{6\pi}{3\pi+4}\left(3\pi+4\right)^2 L$$

$$= -\frac{1}{6\pi}\left(3\pi+4\right)L$$

$$\Delta E = -\frac{U}{6\pi}\left(3\pi+4\right)L$$