Sostituzioni speciali

3) $\int R(\sin x, \cos x)\, dx$

sost. $t = \operatorname{tg}\frac{x}{2}$, $dx = \frac{2\, dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$x \neq (2k+1)\pi \qquad k \in \mathbb{Z}$

Questa sost. trasforma l'integrale in una funzione razionale di t.

OSS Non sempre le sost. standard sono la via migliore

$\int \cos^7 x \, dx$

$t = \operatorname{tg}\frac{x}{2}$ non conviene

Usare le formule di iterazione integrando per parti

$I_n(x) = \int \cos^n x \, dx$

$$\int \cos^6 x \cdot \cos x \, dx =$$

$$= \int (1-\sin^2 x)^3 \cos x \, dx$$

$\sin x = t$

$\cos x\, dx = dt$

$$= \int (1-t^2)^3 \, dt$$

$$= \int (1 - 3t^2 + 3t^4 - t^6)\, dt =$$

$$= \left(t - t^3 + \frac{3}{5}t^5 - \frac{t^7}{7}\right)_{t=\sin x} + c$$

3') Se la funzione razionale dipende solo da $\cos^2 x$, $\sin^2 x$, $\sin x \cos x$. invece di $t = \operatorname{tg}\frac{x}{2}$ conviene

$$\boxed{t = \operatorname{tg} x} \quad \begin{cases} \cos^2 x = \dfrac{1}{1+\operatorname{tg}^2 x} = \dfrac{1}{1+t^2} \\ \sin^2 x = 1 - \cos^2 x = 1 - \dfrac{1}{1+t^2} = \dfrac{t^2}{1+t^2} \\ \sin x \cos x = \underbrace{\dfrac{\sin x}{\cos x}}_{= t} \underbrace{\cos^2 x}_{= \frac{1}{1+t^2}} = \dfrac{t}{1+t^2} \end{cases}$$

$$x = \operatorname{arctg} t + k\pi \Rightarrow \boxed{dx = \frac{dt}{1+t^2}} \qquad x \neq (2k+1)\frac{\pi}{2} \quad k \in \mathbb{Z}.$$

$$\int \frac{3\sin^2 x + 1}{2 + \cos^2 x}\, dx = (*)$$

a) se passiamo a $t = \operatorname{tg}\frac{x}{2}$ $\quad dx = \frac{2}{1+t^2} dt$

$$(*) = \int \frac{3 \cdot \frac{4t^2}{(1+t^2)^2} + 1}{2 + \frac{(1-t^2)^2}{(1+t^2)^2}} \cdot \frac{2}{1+t^2} dt =$$

$$= \int \frac{12t^2 + (1+t^2)^2}{2(1+t^2)^2 + (1-t^2)^2} \frac{2}{1+t^2} dt$$

b) se invece prendiamo $t = \operatorname{tg} x$.

$$\int \frac{3\sin^2 x + 1}{2 + \cos^2 x} dx = \int \frac{3 \frac{t^2}{1+t^2} + 1}{2 + \frac{1}{1+t^2}} \cdot \frac{dt}{1+t^2} =$$

$$= \int \frac{3t^2 + 1 + t^2}{2 + 2t^2 + 1} \cdot \frac{dt}{1+t^2} =$$

$$= \int \frac{4t^2+1}{(2t^2+3)(1+t^2)} \, dt =$$

$$= \int \left(\frac{A \cdot 4t + B}{2t^2+3} + \frac{C \cdot 2t + D}{1+t^2} \right) dt = (*)$$ bisogna trovare A, B, C, D

$$4t^2+1 = \cancel{4At(1+t^2)} + B(1+t^2) + \cancel{2Ct(2t^2+3)} + D(2t^2+3)$$

$\cancel{-2At}$

$$\begin{cases} \cancel{0 = 4A + 4C \Rightarrow C = -A} \\ 4 = B + 2D \\ \cancel{0 = 4A - 6A} \quad \cancel{A = 0} = C \\ 1 = B + 3D \end{cases}$$

$$\boxed{D = -3}$$

$$\boxed{B = 1 - 3D = 10}$$

$$(*) = \int \frac{10}{2t^2+3} dt - \int \frac{3}{1+t^2} dt$$

$$\int \frac{3}{1+t^2} dt = 3 \, \underbrace{\text{arctg}}\, t$$

$$\text{arctg tg } x = x \, (+k\pi)$$

$$\int \frac{10}{2t^2+3} dt = \frac{10}{3} \int \frac{dt}{\underbrace{\frac{2}{3}t^2 + 1}_{\left(\frac{\sqrt{2}}{\sqrt{3}} t\right)^2 + 1}} = \frac{10}{\cancel{3}_{\sqrt{3}}} \frac{\cancel{\sqrt{3}}}{\sqrt{2}} \, \text{arctg}\left(\frac{\sqrt{2}}{\sqrt{3}} t\right)$$

4) Integrali del tipo $\int R\left(x, \underbrace{\sqrt{1-x^2}}_{-1 \leq x \leq 1}\right) dx$

Varie sostituzioni possibili.

a) $x = \cos t$ $\quad t \in [0, \pi]$

$\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = |\sin t| = \sin t.$

$dx = -\sin t \, dt.$

L'integrale diventa $-\int R(\cos t, \sin t) \sin t \, dt$

un integrale del tipo 3)

b) $\sqrt{1-x^2} = \sqrt{(1-x)(1+x)} = \sqrt{\frac{1-x}{1+x} \cdot (1+x)^2}$

è del tipo 2).

Esempio $\int \sqrt{2-4x^2} \, dx =$ (oss questo si può anche fare per parti)

$= \sqrt{2} \int \sqrt{1-2x^2} \, dx =$ $\quad \sqrt{2}\, x = \cos t$ $\quad t \in [0, \pi]$ $\quad x = \frac{\cos t}{\sqrt{2}}.$

$\sqrt{1-2x^2} = \sqrt{1-\cos^2 t} = \sin t$

$dx = -\frac{\sin t}{\sqrt{2}} \, dt$

$= -\cancel{\sqrt{2}} \int \sin t \, \frac{\sin t}{\cancel{\sqrt{2}}} dt = -\int \sin^2 t \, dt$

e questo lo abbiamo già calcolato.

2° modo

$\sqrt{2} \int \sqrt{1-2x^2} \, dx =$

$\sqrt{2}\, x = s$

$dx = \frac{ds}{\sqrt{2}}$

$$= \int \sqrt{1-s^2}\, ds = \int \underbrace{\sqrt{\frac{1-s}{1+s}}}_{t} \cdot (1+s)\, ds = (*)$$

$$\sqrt{\frac{1-s}{1+s}} = t \iff \frac{1-s}{1+s} = t^2 \iff 1-s = t^2(1+s)$$

$$s(1+t^2) = 1-t^2 \qquad s = \frac{1-t^2}{1+t^2}$$

$$ds = \frac{-2t(1+t^2)-(1-t^2)2t}{(1+t^2)^2}\, dt =$$

$$= \frac{2t}{(1+t^2)^2}\left[-1-t^2-1+t^2\right] dt = -\frac{4t}{(1+t^2)^2}$$

$$(*) = \int t \left(1 + \frac{1-t^2}{1+t^2}\right)\left(\frac{-4t}{(1+t^2)^2}\right) dt =$$

$$= -8\int \frac{t^2}{(1+t^2)^3} = \int\left(\frac{A}{1+t^2} + \frac{B}{(1+t^2)^2} + \frac{C}{(1+t^2)^3}\right) dt$$

(Integrali già noti)

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5) $\int R\left(x, \sqrt{x^2+c}\right) dx$

Varie sostituzioni possibili:

a) $\sqrt{x^2+c} = x+t$, cioè $t = \sqrt{x^2+c} - x$

$$\cancel{x^2}+c = (x+t)^2 = \cancel{x^2}+t^2+2xt$$

$$\Rightarrow \quad x = \frac{c-t^2}{2t} \quad \Rightarrow \quad \sqrt{x^2+c} = x+t = \frac{c-t^2}{2t} + t =$$

$$= \frac{c+t^2}{2t}$$

$$dx = \frac{-2t^2-(c-t^2)}{2t^2}\,dt = \frac{-t^2-c}{2t^2}\,dt = -\frac{t^2+c}{2t^2}\,dt$$

$$\Rightarrow \int R\left(x, \sqrt{x^2+c}\right)dx = \int R\left(\frac{c-t^2}{2t}, \frac{c+t^2}{2t}\right)\left(-\frac{t^2+c}{2t^2}\right)dt$$

$$\int \frac{dx}{x+\sqrt{x^2+3}} =$$

$$\sqrt{x^2+3} = x+t$$

$$\cancel{x^2}+3 = \cancel{x^2}+t^2+2xt$$

$$x = \frac{3-t^2}{2t}$$

$$\sqrt{x^2+3} = x+t = \frac{3-t^2}{2t}+t = \frac{3+t^2}{2t}$$

$$dx = \frac{-2t^2-(3-t^2)}{2\;t^2}\,dt = -\frac{t^2+3}{2t^2}\,dt$$

$$= \int \frac{1}{\frac{3-t^2}{\cancel{2t}}+\frac{3+t^2}{\cancel{2t}}}\left(-\frac{t^2+3}{\cancel{2}t^{\cancel{2}}}\right)dt =$$

$$= -\int \frac{t^2+3}{6t}\,dt = -\frac{1}{6}\int\left(t+\frac{3}{t}\right)dt =$$

$$= -\frac{1}{6}\left(\frac{t^2}{2}+3\log|t|\right)+c \qquad t=\sqrt{x^2+3}-x$$

$$= -\frac{1}{6}\left(\frac{\left(\sqrt{x^2+3}-x\right)^2}{2}+3\log\left(\sqrt{x^2+3}-x\right)\right)+c$$

$$\sqrt{3-x^2}$$

Altro modo

$$\int \frac{dx}{x+\sqrt{x^2+3}} = \qquad \boxed{x = \sqrt{3}\,\sinh t}$$

$$\sqrt{x^2+3} = \sqrt{3\sinh^2 t+3} =$$

$$= \sqrt{3}\sqrt{\sinh^2 t + 1} = \sqrt{3}\cosh t$$

$$dx = \sqrt{3}\cosh t \, dt.$$

$$= \int \frac{\cancel{\sqrt{3}}\cosh t}{\cancel{\sqrt{3}}\sinh t + \cancel{\sqrt{3}}\cosh t}\, dt \qquad \sinh t = \frac{e^t - e^{-t}}{2} \qquad \cosh t = \frac{e^t + e^{-t}}{2}$$

$$= \int \frac{e^t + e^{-t}}{e^t - \cancel{e^{-t}} + e^t + \cancel{e^{-t}}}\, dt = \int \frac{e^t + e^{-t}}{2e^t}\, dt$$

$$= \frac{1}{2}\int \left(1 + e^{-2t}\right) dt = \frac{1}{2}\left(t - \frac{1}{2}e^{2t}\right) =$$

$$\left[\begin{aligned} t &= \operatorname{settsinh}\left(\frac{x}{\sqrt{3}}\right) = \log\left(\frac{x}{\sqrt{3}} + \sqrt{\frac{x^2}{3} + 1}\right) = \\ &= \log\left(\frac{x + \sqrt{x^2+3}}{\sqrt{3}}\right) = \log\left(x + \sqrt{x^2+3}\right) - \frac{1}{2}\log 3 \end{aligned}\right]$$

$$= \frac{1}{2}\log\left(\frac{x + \sqrt{x^2+3}}{\sqrt{3}}\right) - \frac{1}{4}\underbrace{e^{\log\left(\left(\frac{x+\sqrt{x^2+3}}{\sqrt{3}}\right)^2\right)}}_{\frac{(x+\sqrt{x^2+3})^2}{3}} + c$$

Calcoliamo l'integrale definito

$$\int_{-2}^{-1} \frac{\sqrt{x^2 - 1}}{x}\, dx$$

Calcoliamo l'$\int$ indefinito

1° modo $\qquad \sqrt{x^2 - 1} = x + t \implies \cancel{x^2} - 1 = \cancel{x^2} + t^2 + 2xt$

$$\boxed{x = -\frac{1+t^2}{2t}}$$

$$\sqrt{x^2-1} = x+t = -\frac{1+t^2}{2t}+t = \frac{-1-t^2+2t^2}{2t} = \frac{t^2-1}{2t}$$

$$dx = -\frac{1}{2}\,\frac{2t^2-(1+t^2)}{t^2}\,dt = -\frac{1}{2}\,\frac{t^2-1}{t^2}\,dt$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{t^2-1}{2\cancel{t}}\left(+\frac{\cancel{2t}}{1+t^2}\right)\left(+\frac{1}{\cancel{2}}\,\frac{(t^2-1)}{t^2}\right)dt =$$

$$= \frac{1}{2}\int \frac{(t^2-1)^2}{t^2(1+t^2)}\,dt = \text{funz. fratta}$$

(divisione + scomposizione in fratti semplici)

e poi ricordarsi che

$$t = \sqrt{x^2-1} - x$$

2° modo

$$\int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x}\,dx \qquad x = -\cosh t$$

prendiamo $t \geq 0$ (perché $x \leq -1$)

$$dx = -\sinh t\,dt$$

$$\sqrt{x^2-1} = \sqrt{\cosh^2 t - 1} = |\sinh t| = \sinh t.$$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sinh t}{-\cosh t}(-\sinh t)\,dt =$$

$$= \int \frac{\sinh^2 t}{\cosh t}\,dt = \int R(e^t)\,dt$$

3° modo

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{2}\int 2x\,\frac{\sqrt{x^2-1}}{x^2}\,dx =$$

$$x^2 = t \qquad 2x\,dx = dt$$

$$= \frac{1}{2}\int \frac{\sqrt{t-1}}{t}\,dt =$$

$$\sqrt{t-1} = s$$

$$t = s^2+1 \qquad dt = 2s\,ds$$

$$= \frac{1}{\cancel{2}} \int \frac{s}{s^2+1} \cdot \cancel{2} s \, ds = \int \frac{s^2+1-1}{s^2+1} ds = \int \left(1 - \frac{1}{s^2+1}\right) ds$$

$$= s - \operatorname{arctg} s + c. \qquad s = \sqrt{x^2-1}$$

$$= \sqrt{x^2-1} - \operatorname{arctg}\left(\sqrt{x^2-1}\right) + c.$$

$$\int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x} dx = \left(\sqrt{x^2-1} - \operatorname{arctg}\sqrt{x^2-1}\right)\Big|_{-2}^{-1} =$$

$$= -\sqrt{3} + \underbrace{\operatorname{arctg}\sqrt{3}}_{= \pi/3}$$

Se avessimo avuto

$$\int R\left(\sqrt{4x^2-1}, x\right) dx, \quad \text{varie opzioni}$$

$$\sqrt{4x^2-1} = 2x + t \Rightarrow \cancel{4x^2} - 1 = \cancel{4x^2} + t^2 + 4xt$$

oppure $\quad 2x = \pm \cosh t \quad$ a seconda del segno di x.

Attenzione: ci sono molte funzioni, anche abbastanza semplici, le cui primitive non si possono scrivere in termini di funzioni elementari. Esempi

$$\int e^{x^2} dx \quad \int e^{-x^2} dx, \quad \int \frac{\sin x}{x} dx, \quad \int \sin(x^2) dx, \quad \int \frac{e^x}{x} dx$$

Una primitiva di e^{x^2} è $\int_0^x e^{t^2} dt$ ma non si può esprimere come combinazione di funzioni elementari

Dopo aver mostrato che si tratta di un integrale secondo Riemann, calcolare

$$\int_{-32}^{32} \frac{1}{\sqrt[5]{x}} \operatorname{arctg}\left(\frac{\sqrt[5]{x}}{2}\right) dx$$

$$f(x) = \frac{1}{\sqrt[5]{x}} \operatorname{arctg}\left(\frac{\sqrt[5]{x}}{2}\right)$$ non è definita in $x=0$,

ma è estendibile con continuità in $x=0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{\sqrt[5]{x}} \underbrace{\operatorname{arctg}\left(\frac{\sqrt[5]{x}}{2}\right)}_{\sim \frac{\sqrt[5]{x}}{2}} = \frac{1}{2}.$$

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ 1/2 & x = 0 \end{cases}$$

tale estensione è integrabile secondo Riemann in $[-32,32]$ perché continua. Chiaramente f resta integrabile (con lo stesso integrale) se prendo un altro valore in $x=0$.

Calcoliamo l'integrale

$$\int_{-32}^{32} f(x)\,dx \underset{\substack{\uparrow \\ f \text{ pari}}}{=} 2\int_0^{32} f(x)\,dx = 2\underbrace{\int_0^{32} \frac{1}{\sqrt[5]{x}} \operatorname{arctg}\left(\frac{\sqrt[5]{x}}{2}\right) dx}_{} = (*)$$

sost. $t = \sqrt[5]{x} \Rightarrow x = t^5 \Rightarrow dx = 5t^4\,dt$

$x=0 \Rightarrow t=0$, $x=32 \Rightarrow t=2$.

$$= 2\int_0^2 \frac{1}{t} \operatorname{arctg}\left(\frac{t}{2}\right) 5\, t^4\,dt =$$

$$= 10\int_0^2 t^3 \operatorname{arctg}\left(\frac{t}{2}\right) dt =$$

per parti

$$f'(t) = t^3 \Rightarrow f(t) = \frac{t^4}{4}$$

$$g(t) = \operatorname{arctg}\left(\frac{t}{2}\right) \Rightarrow g'(t) = \frac{1}{1+\frac{t^2}{4}} \cdot \frac{1}{2} =$$

$$= \frac{4}{4+t^2} \cdot \frac{1}{2} = \frac{2}{4+t^2}$$

$$= 10 \left[\underbrace{\frac{t^4}{4} \operatorname{arctg}\left(\frac{t}{2}\right)\Bigg|_0^2}_{4 \operatorname{arctg} 1 = \pi} - \frac{1}{2}\int_0^2 \frac{t^4}{4+t^2}\, dt \right]$$

$$\int_0^2 \frac{t^4}{4+t^2}\, dt =$$

$$\begin{array}{l|l} t^4 & t^2+4 \\ \hline -t^4 \quad -4t^2 & t^2-4 \\ \quad\quad\quad -4t^2 & \\ \quad\quad\quad 4t^2+16 & \\ \quad\quad\quad\quad 16 & \end{array}$$

$$= \int_0^2 \left(t^2 - 4 + \frac{16}{4+t^2}\right) dt$$

$$= \frac{8}{3} - 8 + 16 \int_0^2 \frac{dt}{4+t^2}$$

$$= \quad '' \quad + \frac{16}{4} \int_0^2 \frac{dt}{1+\left(\frac{t}{2}\right)^2}$$

$$= \quad '' \quad + 4 \cdot 2 \operatorname{arctg}\left(\frac{t}{2}\right) =$$

$$= \frac{8}{3} - 8 + 8 \operatorname{arctg}\left(\frac{t}{2}\right)\Bigg|_0^2 = -\frac{16}{3} + 8 \cdot \frac{\pi}{4} = -\frac{16}{3} + 2\pi.$$

Risultato $= 10 \left[\cancel{\pi} + \frac{8}{3} - \cancel{\pi} \right] = \frac{80}{3}$

$$\int x^3 \log\left(x^8+5x^4+6\right) dx = (*)$$

sost. $x^4 = t \quad \Rightarrow \quad 4x^3 dx = dt \quad \Rightarrow \quad x^3 dx = \dfrac{dt}{4}$

$$= \frac{1}{4}\int 1\cdot \log\left(t^2+5t+6\right) dt =$$

sugg. scrivere

$\log\left(t^2+5t+6\right) = \log\left((t+2)(t+3)\right)$

$= \log|t+2| + \log|t+3|$

per parti

$f'(t) = 1 \Rightarrow f(t) = t$

$g(t) = \log\left(t^2+5t+6\right) \Rightarrow g'(t) = \dfrac{2t+5}{t^2+5t+6}$

$$= \frac{1}{4}\left[t \log\left(t^2+5t+6\right) - \int \frac{2t^2+5t}{t^2+5t+6}\, dt\right] =$$

$$\begin{array}{rrr|l} 2t^2 & +5t & & t^2+5t+6 \\ \cline{4-4} -2t^2 & -10t & -12 & 2 \\ \hline / & -5t & -12 & \end{array}$$

$$= \frac{1}{4}\left[t \log\left(t^2+5t+6\right) - \int \left(2 - \frac{5t+12}{t^2+5t+6}\right) dt\right]$$

$$= \frac{1}{4}\left[t \log\left(\_\_\_\right) - 2t + \int \frac{5t+12}{\underbrace{t^2+5t+6}_{(t+2)(t+3)}}\, dt\right]$$

$$\frac{5t+12}{(t+2)(t+3)} = \frac{A}{t+2} + \frac{B}{t+3} = \frac{2}{t+2} + \frac{3}{t+3}$$

$$5t+12 = A(t+3) + B(t+2)$$

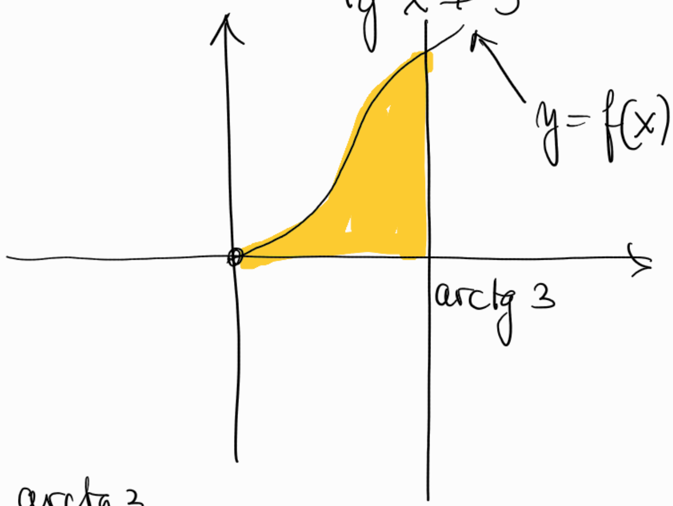
$$\begin{cases} 5 = A + B \\ 12 = 3A + 2B \end{cases} \qquad 10 = 2A + 2B$$

$$A = 2 \qquad B = 5 - A = 3$$

$$\int \frac{5t+12}{(t+2)(t+3)} = 2 \log|t+2| + 3 \log|t+3|$$

Calcolare l'area della regione delimitata dagli assi coordinati, dalla retta $x = \operatorname{arctg} 3$ e dal grafico di

$$f(x) = \frac{tg^2 x + tg^4 x}{tg^2 x + 9}$$



$$\text{Area} = \int_0^{\operatorname{arctg} 3} f(x)\, dx =$$

$$= \int_0^{\operatorname{arctg} 3} \frac{tg^2 x + tg^4 x}{tg^2 x + 9}\, dx = \int_0^{\operatorname{arctg} 3} \frac{tg^2 x \overbrace{(1 + tg^2 x)}^{(tg\, x)'} dx}{tg^2 x + 9} =$$

$$t = tg\, x \qquad (1 + tg^2 x)\, dx = dt.$$

$$x = 0 \Rightarrow t = 0$$

$$x = \operatorname{arctg} 3 \Rightarrow t = 3$$

$$= \int_0^3 \frac{t^2+9-9}{t^2+9}\,dt = \int_0^3 \left(1 - \frac{9}{t^2+9}\right) dt = 3 - 9\int_0^3 \frac{dt}{t^2+9}$$