Integrali di funzioni razionali

$$\int \frac{P_n(x)}{Q_m(x)}\,dx$$

- Possiamo supporre $n < m$ (altrimenti: divisione)
- Essenziale: saper scomporre $Q_m(x)$

$$\int \frac{1}{(x+1)\underbrace{(x^2+x+1)}_{\text{irriducibile.}}}\,dx = (*)$$

derivata del denom.

$$\frac{1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{B(2x+1)+C}{x^2+x+1}$$

si trovano A, B, C ..

$$(*) = A \boxed{\int \frac{dx}{x+1}} + B \underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{\log(x^2+x+1)} + C \boxed{\int \frac{dx}{x^2+x+1}}$$

$\int \frac{dx}{x+1} = \log|x+1|$

sost. $x^2+x+1 = t$
$(2x+1)\,dx = dt$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}} = \int \frac{dx}{\left(\frac{2x+1}{2}\right)^2+\frac{3}{4}} =$$

$$x^2+x+\frac{1}{4} = \left(x+\frac{1}{2}\right)^2$$

$$= \frac{4}{3}\int \frac{dx}{\left(\frac{2x+1}{2}\right)^2 \frac{4}{3}+1} = \frac{4}{3}\int \frac{dx}{\left(\frac{2x+1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\right)^2+1} =$$

$$= \frac{4}{3} \int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} = \frac{4}{3}\frac{\sqrt{3}}{2} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c_1$$

$$\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c_1$$

Resta un caso da considerare: radici complesse coniugate con molteplicità > 1.

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{(1+x^2) - x^2}{(1+x^2)^2}\,dx = \underbrace{\int \frac{dx}{1+x^2}}_{\operatorname{arctg} x} - \int \frac{x^2}{(1+x^2)^2}\,dx =$$

$$= \operatorname{arctg} x - \frac{1}{2}\int x \cdot \frac{2x}{(1+x^2)^2}\,dx = (*)$$

$$f'(x) = \frac{2x}{(1+x^2)^2} \Rightarrow f(x) = -\frac{1}{1+x^2}$$

$$g(x) = x \Rightarrow g'(x) = 1$$

$$\int \frac{2x}{(1+x^2)^2}\,dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{1+x^2}$$

$$1+x^2 = t$$
$$2x\,dx = dt$$

$$(*) = \operatorname{arctg} x - \frac{1}{2}\left[-\frac{x}{1+x^2} + \int \frac{dx}{1+x^2}\right] =$$

$$= \frac{1}{2}\left[\operatorname{arctg} x + \frac{x}{1+x^2}\right] + c$$

$$\boxed{\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left[\operatorname{arctg} x + \frac{x}{1+x^2}\right] + c.}$$

$$\int \frac{dx}{(1+x^2)^3} = \int \frac{(1+x^2)-x^2}{(1+x^2)^3}\,dx =$$

$$= \underbrace{\int \frac{dx}{(1+x^2)^2}}_{\text{già calcolato}} - \int \frac{x^2}{(1+x^2)^3}\,dx =$$

$$= \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\int x \left(\frac{2x}{(1+x^2)^3}\right) = (*)$$

Per parti:: $f'(x) = \frac{2x}{(1+x^2)^3} \Rightarrow f(x) = -\frac{1}{2(1+x^2)^2}$

$g(x) = x \Rightarrow g'(x) = 1$

$$(*) = \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\left[-\frac{x}{2(1+x^2)^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2}\right]$$

$$= \frac{3}{4}\underbrace{\int \frac{dx}{(1+x^2)^2}}_{\frac{1}{2}\left[\frac{x}{1+x^2} + \operatorname{arctg} x\right]} + \frac{1}{4}\frac{x}{(1+x^2)^2} =$$

$$=$$

$$\boxed{\int \frac{dx}{(1+x^2)^3} = \frac{3}{8}\frac{x}{1+x^2} + \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{8}\operatorname{arctg} x}$$

$$\int \frac{dx}{(1+x^2)^4} = ?$$

per casa

Si può trovare una formula iterativa per

$$I_n(x) = \int \frac{dx}{(1+x^2)^n}$$

$$\boxed{I_n(x) = \frac{2n-3}{2n-2} I_{n-1} + \frac{1}{2n-2} \frac{x}{(1+x^2)^{n-1}}}$$

Esercizio Riassuntivo

$$\int \frac{P_n(x)}{(5x-2)(x-1)^3 \underbrace{(x^2+2x+17)}_{\text{irriducibile}}{}^2} dx = (*) \quad [\text{supponiamo } n \leq 7]$$

$$\frac{P_n(x)}{\text{denomin.}} = \frac{A}{5x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} +$$

$$+ \frac{E(2x+2)+F}{x^2+2x+17} + \frac{G(2x+2)+H}{(x^2+2x+17)^2}$$

Una volta trovati $A, B, \dots H$,

$$(*) = A \underbrace{\int \frac{dx}{5x-2}}_{= \frac{1}{5}\log|5x-2|} + B \underbrace{\int \frac{dx}{x-1}}_{= \log|x-1|} + C \underbrace{\int \frac{dx}{(x-1)^2}}_{-\frac{1}{x-1}} + D \underbrace{\int \frac{dx}{(x-1)^3}}_{-\frac{1}{2(x-1)^2}} +$$

$$+ E \underbrace{\int \frac{2x+2}{x^2+2x+17} dx}_{\log(x^2+2x+17)} + F \int \frac{dx}{x^2+2x+17} + G \underbrace{\int \frac{2x+2}{(x^2+2x+17)^2} dx}_{-\frac{1}{x^2+2x+17}} + H \int \frac{dx}{(x^2+2x+17)^2}$$

$$\int \frac{dx}{x^2+2x+17} = \int \frac{dx}{(x^2+2x+1)+16} = \int \frac{dx}{(x+1)^2+16} = \frac{1}{16} \int \frac{dx}{\left(\frac{x+1}{4}\right)^2 + 1} =$$

$$= \frac{1}{16} \cdot 4 \quad \operatorname{arctg}\left(\frac{x+1}{4}\right) = \frac{1}{4} \operatorname{arctg}\left(\frac{x+1}{4}\right) + c_1.$$

$$\int \frac{dx}{(x^2+2x+17)^2} = \frac{1}{256} \int \frac{dx}{\left(\left(\frac{x+1}{4}\right)^2+1\right)^2} =$$

$\left[\text{sost} \quad \frac{x+1}{4} = t\right]$

$dx = 4dt$

$$= \frac{1}{128} \left[\operatorname{arctg}\left(\frac{x+1}{4}\right) + \frac{\frac{x+1}{4}}{1+\left(\frac{x+1}{4}\right)^2} \right] + c$$

$$x^2+2x+17 = (x+1)^2+16 = 16\left(\left(\frac{x+1}{4}\right)^2+1\right)$$

$$\boxed{\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\left[\operatorname{arctg} x + \frac{x}{1+x^2}\right] + c.}$$

$$\int \frac{dx}{x^4+16} =$$

Scomponiamo x^4+16

- nei complessi si trovano le radice quarte di -16.
- a occhio

$(x^2+4)^2$

$\quad$ "

$$x^4+16 = (x^4+16+8x^2) - 8x^2 =$$

$$= \left(x^2-2\sqrt{2}x+4\right)\left(x^2+2\sqrt{2}x+4\right).$$

$$= \int \frac{dx}{(x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4)} =$$

$$\left[\frac{1}{(x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4)} = \frac{A(2x-2\sqrt{2})+B}{x^2-2\sqrt{2}x+4} + \frac{C(2x+2\sqrt{2})+D}{x^2+2\sqrt{2}x+4} \right]$$

Si trovano A, B, C, D.

$$= A \boxed{\int \frac{2x-2\sqrt{2}}{x^2-2\sqrt{2}x+4}} dx + B\int \frac{dx}{x^2-2\sqrt{2}x+4} +$$

$\log(x^2-2\sqrt{2}x+4)$ viene un arcotangente

$$+ C \int \frac{2x+2\sqrt{2}}{x^2+2\sqrt{2}x+4} dx + D \int \frac{dx}{x^2+2\sqrt{2}x+4}$$

$\log(x^2+2\sqrt{2}x+4)$ altro arctg

Sostituzioni speciali

Nel seguito le funzioni $R(x)$ oppure $R(x,y)$ sono funzioni razionali fratte, cioè rapporti di polinomi nelle variabili indicate

1) $\int R(e^x)\, dx$

si pone $e^x = t \Rightarrow x = \log t \Rightarrow dx = \frac{dt}{t}$

$$\Rightarrow \int R(e^x) dx = \int \frac{R(t)}{t} dt$$

$$\int \frac{e^x-1}{e^{2x}+1} dx = \qquad e^x = t \qquad dx = \frac{dt}{t}$$

$$= \int \frac{t-1}{t^2+1} \frac{dt}{t} = \int \left(\frac{A}{t} + \frac{2Bt+C}{t^2+1}\right) dt = (*)$$

Troviamo A, B, C. $\quad \frac{t-1}{t(t^2+1)} = \frac{A}{t} + \frac{2Bt+C}{t^2+1}$

$$t-1 = A(t^2+1) + 2Bt^2 + Ct$$

$$\begin{cases} 0 = A + 2B \\ 1 = C \\ -1 = A \end{cases} \qquad B = -\frac{A}{2} = \frac{1}{2}$$

$$(*) = -\int \frac{dt}{t} + \frac{1}{2}\int \frac{2t}{1+t^2}\,dt + \int \frac{dt}{1+t^2}$$

$$= -\log|t| + \frac{1}{2}\log(1+t^2) + \operatorname{arctg} t + c_1 \qquad t = e^x$$

$$= -x + \frac{1}{2}\log(1+e^{2x}) + \operatorname{arctg} e^x + c_1$$

Attenzione: a volte convengono sostituzioni diverse

$$\int \frac{e^{2x}-1}{e^{2x}+1}\,dx = \int \frac{t-1}{t+1}\,\frac{dt}{2t}$$

si può fare come prima $(e^x = t)$

ma è meglio $e^{2x} = t$

$2x = \log t$

$x = \frac{dt}{2t}$

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2) Integrali della forma $\int R(x, \sqrt[n]{ax+b})\,dx = (*)$

$a, b \in \mathbb{R}$

$n \in \mathbb{N}$

Si pone $\sqrt[n]{ax+b} = t \Rightarrow ax+b = t^n$

$$x = \frac{t^n - b}{a} \Rightarrow dx = \frac{n}{a}t^{n-1}dt$$

$$(*) = \int R\left(\frac{t^n-b}{a}, t\right)\frac{n}{a}t^{n-1}dt$$

$$\int \frac{dx}{(x+6)\sqrt{x+2}} = \int \frac{2\cancel{t}\,dt}{(t^2-2+6)\cancel{t}}$$

$\sqrt{x+2} = t$

$x = t^2 - 2$

$dx = 2t\,dt$

$$= 2\int \frac{dt}{t^2+4} = \frac{2}{4}\int \frac{dt}{\left(\frac{t}{2}\right)^2+1} = \frac{1}{\cancel{2}}\cdot\cancel{2}\ \text{arctg}\left(\frac{t}{2}\right) + c$$

$$= \text{arctg}\left(\frac{\sqrt{x+2}}{2}\right) + c$$

OSS. Questo è un caso particolare della classe

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$$

$$\sqrt[n]{\frac{ax+b}{cx+d}} = t \Rightarrow \frac{ax+b}{cx+d} = t^n \Rightarrow ax+b = t^n(cx+d)$$

$$x = \text{---}$$

$$dx = \text{---}$$

$$\int \frac{1}{x}\sqrt{\frac{3+x}{x+1}}\, dx = \qquad \sqrt{\frac{3+x}{x+1}} = t$$

$$\frac{3+x}{x+1} = t^2$$

$$= \int \frac{\cancel{t^2-1}}{3-t^2}\, t \cdot \left(-\frac{4t}{(t^2-1)^{\cancel{2}}}\right) dt = \qquad 3+x = t^2(x+1)$$

$$x(t^2-1) = 3-t^2$$

$$= \int \frac{-4t^2}{(3-t^2)(t^2-1)}\, dt = \qquad \boxed{x = \frac{3-t^2}{t^2-1}}$$

$$= \int \left(\frac{A}{t-\sqrt{3}} + \frac{B}{t+\sqrt{3}} + \frac{C}{t-1} + \frac{D}{t+1}\right) dt \qquad dx = \frac{-2t(t^2-1)-(3-t^2)2t}{(t^2-1)^2}\, dt =$$

trovati A, B, C, D, l'integrale è calcolato $\qquad \frac{2t\left[-\cancel{t^2}+1-3+\cancel{t^2}\right]}{(t^2-1)^2}\, dt$

$$= -\frac{4t}{(t^2-1)^2}\, dt$$

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③ Integrali della forma $\int R(\sin x, \cos x)\, dx$

Una sost. tipica è $\boxed{t = \operatorname{tg}\frac{x}{2}}$ $\Longleftrightarrow$ $\frac{x}{2} = \operatorname{arctg} t + k\pi \quad (k \in \mathbb{Z})$

$x = 2\operatorname{arctg} t + 2k\pi$

$$\boxed{\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2},} \qquad \boxed{dx = \frac{2\,dt}{1+t^2}}$$

N.B. deve essere $x \neq (2k+1)\pi$

$$\int \frac{dx}{\sin x}, \qquad \int \frac{2\sin x - 1}{\sin x + \cos x}\, dx$$

$$\int \frac{dx}{\sin x} = (*) \qquad x \neq k\pi$$

1° modo $\qquad t = \operatorname{tg}\frac{x}{2}$

$$(*) = \int \frac{\cancel{1+t^2}}{\cancel{2}t} \cdot \frac{\cancel{2}\,dt}{\cancel{1+t^2}} = \int \frac{dt}{t} = \log|t| + c =$$

$$= \log\left|\operatorname{tg}\frac{x}{2}\right| + c.$$

2° modo

$$\int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x}\, dx = \int \frac{\sin x}{1-\cos^2 x}\, dx =$$

$\cos x = t \qquad -\sin x\, dx = dt$

$$= -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1} = \int \left(\frac{A}{t-1} + \frac{B}{t+1}\right) dt =$$

$$1 = A(t+1) + B(t-1)$$

$$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \qquad B = -A \qquad 2A = 1$$

$$A = \tfrac{1}{2} \qquad B = -\tfrac{1}{2}$$

$$= \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = \frac{1}{2} \left[\log |t-1| - \log |t+1| \right] + c$$

$$= \frac{1}{2} \log \left(\frac{1-\cos x}{1+\cos x} \right) + c$$

Controllare che i due risultati trovati differiscono per una costante.