$$f(x)=\begin{cases}\dfrac{\operatorname{sen}(2x)}{x} & \text{se } x<0\\[2mm] a(x^5+1)+bx^2 & \text{se } x\geq 0\end{cases}$$

Trovare tutti i valori $a, b \in \mathbb{R}$ t.c.

fatte!

a) f è continua in $\mathbb{R}$ $\iff$ $a=2,\ b\in\mathbb{R}$.

b) f def^{te} crescente per $x\to+\infty$

c) f ha un flesso in $x_0=1$

d) f ammette max. relativo in $x_0=1$.

Intorno a $x_0=1$ $f(x)=a(x^5+1)+bx^2$

$$f'(x)=5ax^4+2bx$$

$$f'(1)=0 \iff 5a+2b=0 \qquad \boxed{b=-\frac{5}{2}a}$$

$$f''(x)=20ax^3+2b$$

Se $f''(1)<0$, allora $x_0=1$ è pto di max locale stretto.

$$f''(1)<0 \iff 20a+2b=20a-5a=15a<0 \iff a<0.$$

se $f''(1)>0 \iff a>0$, allora $x_0=1$ è pto di min. locale stretto

$\Rightarrow$ non va bene.

se $f'(1)=f''(1)=0 \iff a=b=0 \iff f(x)=0$ ok! $x>0$

$$\boxed{\text{Sol}^{ni}: \quad b=-\frac{5}{2}a, \quad a\leq 0}$$

e) f è derivabile in $x_0=0$.

f deve essere continua $\iff a=2$.

$$f'_+(0)=\left(2(x^5+1)+bx^2\right)'\Big|_{x=0}=10x^4+2bx\Big|_{x=0}=0$$

$$f'_-(0) \cancel{\overset{?}{=}} \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \left(\frac{\sin(2x)}{x}\right)' =$$

perché f è continua in $x=0$.

$$= \lim_{x\to 0^-} \frac{2\cos(2x)\,x - \sin(2x)}{x^2} = \left(\frac{0}{0}\right) \cancel{\overset{?}{\underset{H}{=}}}$$

$$= \lim_{x\to 0^-} \frac{-4\sin(2x)\cancel{x} + \cancel{2\cos(2x)} - \cancel{2\cos(2x)}}{2\cancel{x}} = 0$$

sempre uguale a $f'_+(0)$.

Quindi la risposta a e) è la stessa di a).

$$\boxed{f \text{ derivabile in } x_0=0 \iff a=2,\ b\in\mathbb{R}}$$

$$f(x) = \log(ax+2) - b|x|$$

Trovare a,b t.c. f convessa nel suo dominio.

$$\text{dom } f = \{x\in\mathbb{R} : ax+2>0\} = \begin{cases} \left(-\frac{2}{a}, +\infty\right) & \text{se } a>0 \\ \mathbb{R} & \text{se } a=0 \\ \left(-\infty, -\frac{2}{a}\right) & \text{se } a<0 \end{cases}$$

C'è un problema di derivabilità in $x=0$, mentre f è sempre continua.

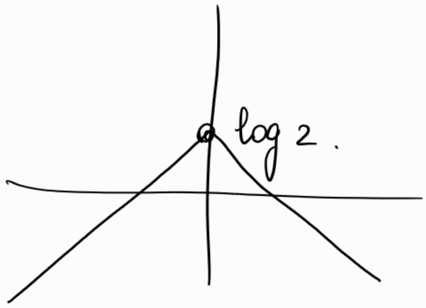
Per $x\neq 0$, $f'(x) = \frac{a}{ax+2} - b\ \text{sign}\, x$

$$f''(x) = -\frac{a^2}{(ax+2)^2} \leq 0 \qquad \forall x\neq 0.$$

Affinché f sia convessa l'unica possibilità è $a=0$.

$$\Rightarrow f(x) = \log 2 - b|x|$$

se $b>0$ $\quad f(x)$ →

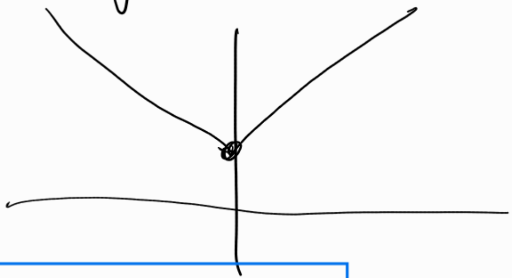


non è convessa.

se $b=0$ $\quad f(x) \equiv \log 2$ $\quad$ sì, è convessa.

se $b<0$

sì, è convessa.



Risposta è $\boxed{a=0,\ b\leq 0}$

Verificare con la def^{ne} che $\lim_{x\to 8} \sqrt[3]{x} = 2$.

$\forall \varepsilon>0$, cerco $\delta>0$ t.c. $\cancel{0<}|x-8|<\delta \Rightarrow |\sqrt[3]{x}-2|<\varepsilon$

1° modo) $\quad |\sqrt[3]{x}-2|<\varepsilon \iff 2-\varepsilon < \sqrt[3]{x} < 2+\varepsilon \iff$

$$\iff (2-\varepsilon)^3 < x < (2+\varepsilon)^3 = 8+\underbrace{12\varepsilon+6\varepsilon^2+\varepsilon^3}_{>0}$$

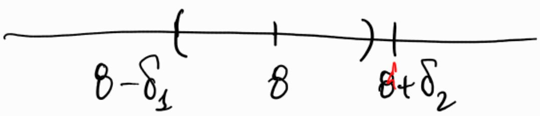
t^3 è strett. crescente in tutto $\mathbb{R}$

$$(2-\varepsilon)^3 = 8\underbrace{-12\varepsilon+6\varepsilon^2-\varepsilon^3}_{<0}$$

$$|\sqrt[3]{x}-2|<\varepsilon \iff \underbrace{-12\varepsilon+6\varepsilon^2-\varepsilon^3}_{-\delta_1} < x-8 < \underbrace{12\varepsilon+6\varepsilon^2+\varepsilon^3}_{\delta_2}$$

$$\Leftarrow \quad |x-8|<\delta = \min\{\delta_1,\delta_2\} = \min\{12\varepsilon-6\varepsilon^2+\varepsilon^3,\ 12\varepsilon+6\varepsilon^2+\varepsilon^3\}$$

$$= \delta_1 = 12\varepsilon-6\varepsilon^2+\varepsilon^3$$



2° modo $\quad \left|\sqrt[3]{x}-2\right| < \varepsilon \iff$

$A^3-B^3=(A-B)(A^2+AB+B^2)$

$A=\sqrt[3]{x} \qquad B=2.$

$$\left|\frac{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}\right| < \varepsilon$$

$$\frac{|x-8|}{\underbrace{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}_{\vee \atop 0}} < \varepsilon.$$

$$\frac{|x-8|}{\underbrace{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}_{\geq \atop 4 \ \forall x\geq 0}} \underset{\text{se } |x-8|<8}{<} \frac{|x-8|}{4} < \varepsilon$$

$|x-8| < 4\varepsilon$

$$\boxed{\delta = \min\{8, 4\varepsilon\}}$$

Fornire una stima dell'errore commesso sostituendo $\sin\frac{1}{2}$ con

$$\frac{1}{2}-\frac{1}{6\cdot 2^3}+\frac{1}{120\cdot 2^5}$$

$$\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}+E_5(x) \qquad \boxed{x_0=0}$$

$$E_5(x) = E_6(x)$$

$x=\frac{1}{2}$

$$\sin\frac{1}{2} = \frac{1}{2}-\frac{1}{6\cdot 2^3}+\frac{1}{120\cdot 2^5}+E_6\left(\frac{1}{2}\right)$$

$$E_6\left(\frac{1}{2}\right) = \frac{f^{(7)}(c)}{7!}\underbrace{\frac{1}{2^7}}_{(x-x_0)^7} = \frac{-\cos c}{7!\cdot 2^7} \qquad 0<c<\frac{1}{2}$$

$$\frac{\sqrt{3}}{2} < \cos c < 1$$

$$-\frac{1}{6\cdot 10^5} < E_6\left(\frac{1}{2}\right) < -\frac{\sqrt{3}}{2}\,\frac{1}{6\cdot 10^5}$$

$$\int 1\cdot\sqrt{1+x^2}\,dx = x\sqrt{1+x^2} - \int \frac{(x^2+1)-1}{\sqrt{1+x^2}}\,dx =$$

$$f'(x) = 1 \Rightarrow f(x) = x$$

$$g(x) = \sqrt{1+x^2} \Rightarrow g'(x) = \frac{x}{\sqrt{1+x^2}}$$

$$= x\sqrt{1+x^2} - \int\sqrt{1+x^2}\,dx + \int\frac{dx}{\sqrt{1+x^2}}$$

$$\int\sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\int\frac{dx}{\sqrt{1+x^2}} =$$

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\,\mathrm{settsinh}\,x + c$$

$$= \quad '' \quad + \frac{1}{2}\log\left(x+\sqrt{x^2+1}\right) + c.$$

$$\int 1\cdot \operatorname{arctg}\frac{1}{x}\,dx = x\operatorname{arctg}\frac{1}{x} + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx =$$

$$\left[\begin{array}{l} f'(x)=1 \Rightarrow f(x)=x \\ g(x)=\operatorname{arctg}\frac{1}{x} \Rightarrow g'(x)=\dfrac{1}{1+\frac{1}{x^2}}\cdot\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2+1} \end{array}\right]$$

$$= x\operatorname{arctg}\frac{1}{x} + \frac{1}{2}\log(x^2+1) + c.$$

$$\int_{-\pi/2}^{\pi/2} \cos^4 x\,dx \underset{f \text{ è pari}}{=} 2\int_0^{\pi/2}\cos^4 x\,dx =$$

$$\left[\cos^2 x = \frac{1+\cos(2x)}{2} \Rightarrow \cos^4 x = \frac{1}{4}\left(1+2\cos(2x)+\cos^2(2x)\right)\right]$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1 + 2\cos(2x) + \cos^2(2x)\right)dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \cancel{\sin(2x)\Big|_0^{\pi/2}} + \int_0^{\pi/2}\frac{1+\cos(4x)}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{4} + \cancel{\frac{1}{2}\int_0^{\pi/2}\cos(4x)\,dx}\right] = \frac{3}{8}\pi.$$

Oppure per parti:

$$2\int_0^{\pi/2}\cos^4 x\,dx = 2\int_0^{\pi/2}\cos^3 x\cos x\,dx =$$

$$\left[\begin{array}{l} f'(x)=\cos x \Rightarrow f(x)=\sin x \\ g(x)=\cos^3 x \Rightarrow g'(x) = -3\cos^2 x\sin x \end{array}\right]$$

$$= \cancel{2\sin x\cos^3 x\Big|_0^{\pi/2}} + 6\int_0^{\pi/2}\cos^2 x\,\underbrace{\sin^2 x}_{1-\cos^2 x}\,dx =$$

$$= 6\int_0^{\pi/2}\cos^2 x\,dx - 6\int_0^{\pi/2}\cos^4 x\,dx$$

$$\cancel{8}\,4\int_0^{\pi/2}\cos^4 x\,dx = \cancel{6}\,3\int_0^{\pi/2}\cos^2 x\,dx$$

$$\text{Int.} = 2\int_0^{\pi/2}\cos^4 x\,dx = \frac{3}{2}\underbrace{\int_0^{\pi/2}\cos^2 x\,dx}_{\pi/4}$$

$\int_0^{\pi/2}\cos^2 x\,dx$ si calcola o per parti oppure per bisezione.

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1) Calcolare $\int_0^1 x e^{x^2}dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}(e-1)$

2) Posto $I_n = \int_0^1 x^n e^{x^2}dx$, trovare una formula che esprima I_n in funzione di I_{n-2} $\quad n \geq 2$.

$$I_n = \int_0^1 x^n e^{x^2}dx = \int_0^1 x^{n-1}\, x e^{x^2}\,dx =$$

$$\left[\begin{array}{l} f'(x) = x e^{x^2} \Rightarrow f(x) = \frac{1}{2}e^{x^2} \\ g(x) = x^{n-1} \Rightarrow g'(x) = (n-1)\,x^{n-2} \end{array}\right]$$

$$= \underbrace{\frac{1}{2}x^{n-1}e^{x^2}\Big|_0^1}_{=\frac{e}{2}} - \frac{n-1}{2}\underbrace{\int_0^1 x^{n-2}e^{x^2}dx}_{I_{n-2}}$$

$$\Rightarrow \quad I_n = \frac{e}{2} - \frac{n-1}{2} I_{n-2} = \frac{1}{2}\left(e - (n-1) I_{n-2}\right)$$

3) Usare la formula trovata per trovare $I_5 = \int_0^1 x^5 e^{x^2} dx$

$$I_5 = \frac{1}{2}\left(e - 4 I_3\right) = \frac{1}{2}\left(e - 4\left[\frac{1}{2}\left(e - 2 I_1\right)\right]\right) =$$

$$= \frac{1}{2}\left(-e + 4 \underbrace{I_1}_{=\frac{1}{2}(e-1)}\right) =$$

$$= \frac{1}{2}\left(-e + 2(e-1)\right) = \frac{1}{2}(e-2)$$

Al variare di $\alpha, \beta > 0$, calcolare l'ordine di infinitesimo, per $x \to 0^+$, della funzione

$$f(x) = \sqrt{9 + x^2 + x^\alpha} - 3$$

$$= 3\left(\sqrt{1 + \underbrace{\frac{x^2 + x^\alpha}{9}}_{\to 0}} - 1\right) \sim 3 \, \frac{x^2 + x^\alpha}{18} = \frac{x^2 + x^\alpha}{6}$$

$$(1+t)^{1/2} - 1 \sim \frac{t}{2} \qquad t \to 0$$

$$f(x) \sim \begin{cases} \dfrac{x^\alpha}{6} & \text{se } \alpha < 2 \\ \dfrac{x^2}{3} & \alpha = 2 \\ \dfrac{x^2}{6} & \text{se } \alpha > 2 \end{cases} \qquad x \to 0^+$$

$$g(x) = \sqrt{9 + x^2 + x^\alpha} - 3 - \beta x^2$$

$$\sqrt{9+x^2+x^\alpha} = 3\sqrt{1+\frac{x^2+x^\alpha}{9}} = (*)$$

$$\frac{x^2+x^\alpha}{9} \to 0$$

$$\sqrt{1+t} = 1+\frac{t}{2}-\frac{t^2}{8}+o(t^2) \qquad t\to 0.$$

$$(*) = 3\left(1+\frac{x^2+x^\alpha}{18}-\frac{(x^2+x^\alpha)^2}{8\cdot 81}+o\left((x^2+x^\alpha)^2\right)\right)$$

$$g(x) = 3\left(\cancel{1}+\frac{x^2+x^\alpha}{18}-\frac{(x^2+x^\alpha)^2}{648}+o\left((x^2+x^\alpha)^2\right)\right)-\beta x^2$$

1° caso $0<\alpha<2$. $\Rightarrow g(x) \sim \frac{x^\alpha}{6}$ inf^{mo} di ordine α.

2° caso $\alpha=2$

$$g(x) = 3\left(\frac{2x^2}{18}-\frac{(2x^2)^2}{648}+o(x^4)\right)-\beta x^2$$

$$= \frac{x^2}{3}-\beta x^2+o(x^2) \sim \left(\frac{1}{3}-\beta\right)x^2 \text{ se } \beta\neq\frac{1}{3}.$$

$\alpha=2, \beta=\frac{1}{3}$.

$$g(x) = -\frac{12}{648}x^4+o(x^4)$$ infinitesimo di ordine 4.

$\alpha>2$.

<u>da concludere</u>