$$\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x} = \left(\frac{0}{0}\right) = \frac{e}{2}.$$

$$e-(1+x)^{1/x} = e - e^{\frac{1}{x}\log(1+x)} \qquad x \to 0.$$

$$= e\left[1 - e^{\boxed{\frac{1}{x}\log(1+x)-1}}\right] \sim \underbrace{e\left(1-\frac{1}{x}\log(1+x)\right)} =$$

$\downarrow$ $0$

$\left[ e^t - 1 \sim t \quad \text{per } t \to 0 \right.$

infatti

$\left. \log(1+x) \sim x \to \frac{\log(1+x)}{x} \to 1 \right]$

$$\left[ \log(1+x) = x - \frac{x^2}{2} + o(x^2) \qquad \text{per } x \to 0 \right]$$

$$= e\left(1 - \frac{1}{x}\left(x - \frac{x^2}{2} + o(x^2)\right)\right) =$$

$$= e\left(\cancel{1} - \cancel{1} + \frac{x}{2} + o(x)\right) \sim \frac{e}{2}x$$

---

Risolvere

$$\operatorname{Im} z \left( \frac{|z|^2 + z^2}{2} - z - i \operatorname{Im} z (\operatorname{Re} z - 1) - 12 \right) - \frac{3}{2}(z + \bar{z}) - 2 = 0$$

$$z = x + iy \qquad x, y \in \mathbb{R}.$$

$$y\left( \frac{1}{2}\left(x^2 + \cancel{y^2} + x^2 - \cancel{y^2} + 2ixy\right) - x - \cancel{iy} - iy\left(x - \cancel{1}\right) - 12 \right) +$$

$$- \frac{3}{2}\left(x + \cancel{iy} + x - \cancel{iy}\right) - 2 = 0$$

$$y\left( x^2 + \cancel{ixy} - x - \cancel{ixy} - 12 \right) - 3x - 2 = 0$$

$y \quad (x^2-x-12) = 3x+2$

$$\boxed{y = \frac{3x+2}{x^2-x-12}}$$

$x^2-x-12 \neq 0.$

$x^2-x-12=0 \qquad x=4, \quad x=-3.$

per $x=4$ viene $\quad 0 = 14 \quad$ NO

$x=-3 \quad 0=-7 \quad$ NO.

Le soluzioni sono tutti i numeri complessi della forma

$$z = x + i\,f(x) \qquad \text{dove} \qquad f(x) = \frac{3x+2}{x^2-x-12} \qquad x \neq -3, 4.$$

---

Trovare tutti i numeri $z \in \mathbb{C}$ t.c.
$\frac{2+z}{2-z}$ sia immaginario puro, e disegnarli sul piano compl.

$z = x+iy \qquad x,y \in \mathbb{R}. \qquad (x,y) \neq (2,0)$

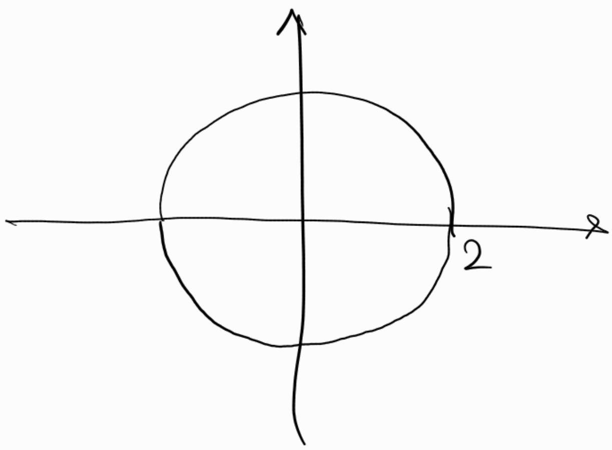
$$\frac{2+z}{2-z} = \frac{2+x+iy}{2-x-iy} \cdot \frac{2-x+iy}{2-x+iy} = \frac{(2+iy)^2-x^2}{(2-x)^2-(iy)^2} =$$

$$= \frac{4-y^2+4iy-x^2}{4+x^2-4x+y^2} = \underbrace{\frac{1}{4+x^2+y^2-4x}}_{\in\mathbb{R}} \left(4-x^2-y^2+4iy\right)$$

è immaginario puro se e solo se

$$4-x^2-y^2=0$$

è la circonferenza di centro l'origine e raggio 2



Calcolare l'ordine di infinito/infinitesimo, per $x \to 0^+$, delle seguenti funzioni

$$f(x) = x - \underbrace{\log\left(1 + \underset{\downarrow\, 0}{tg(x^3)}\right)}_{\sim\, tg(x^3) \sim x^3} = \underbrace{x\left(1 - \frac{\log\left(1+tg(x^3)\right)}{x}\right)}_{\downarrow\, 1} \sim x$$

$$\sim \frac{x^3}{x} = x^2 \to 0$$

infinitesimo di ordine 1 per $x \to 0^+$

$$g(x) = \frac{1}{x} - \frac{1}{\log\left(1+tg(x^3)\right)} = \frac{1}{x^3}\underbrace{\left(\underset{\downarrow\, 0}{x^2} - \underset{\downarrow\, 1}{\frac{x^3}{\log\left(1+tg(x^3)\right)}}\right)}_{\downarrow\, -1} \sim$$

$$\frac{1}{\log\left(1+tg(x^3)\right)} \sim \frac{1}{x^3}$$

$$\sim -\frac{1}{x^3}$$ infinito di ordine 3 per $x \to 0^+$

In alternativa:

$$g(x) = \frac{\log\left(1+tg(x^3)\right) - x}{x \log\left(1+tg(x^3)\right)} \sim -\frac{x}{x^4} = -\frac{1}{x^3}$$

$\log\left(1+tg(x^3)\right) - x \sim -x$ (precedente funzione)

$\log\left(1+tg(x^3)\right) \sim x^3$

$$h(x) = x^3 - \log\left(1 + tg(x^3)\right)$$

$$\log\left(1+tg(x^3)\right) -$$

$$tg\, t = t + \frac{t^3}{3} + o(t^4) \qquad t \to 0$$

$$tg\, x^3 = x^3 + \frac{x^9}{3} + o(x^{12}) = x^3 + o(x^6)$$

$$\log\left(1+\operatorname{tg}(x^3)\right) = \log\Big(1+\underbrace{x^3+o(x^6)}_{=\,t\to 0}\Big) =$$

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2) \qquad t\to 0$$

$$= x^3 + o(x^6) - \frac{1}{2}\underbrace{\left(x^3+o(x^6)\right)^2}_{=\,x^6+o(x^6)} + \underbrace{o\left(\left(x^3+o(x^6)\right)^2\right)}_{=\,o(x^6)} =$$

$$= x^3 - \frac{x^6}{2} + o(x^6)$$

$$h(x) = \cancel{x^3} - \left(\cancel{x^3} - \frac{x^6}{2} + o(x^6)\right) = \frac{x^6}{2} + o(x^6) \sim \frac{x^6}{2}$$

infinitesimo di ordine 6.

Errore tipico:

$$h(x) = x^3 - \log\left(1+\operatorname{tg}(x^3)\right)$$

$$\log\left(1+\operatorname{tg}(x^3)\right) \sim \operatorname{tg}(x^3) = x^3 + \frac{x^9}{3} + o(x^9)$$

errato perché nel 1° passaggio ho perso traccia di ogni termine che sia $o(x^3)$

$$x^3 - \underbrace{\log\left(1+\operatorname{tg}(x^3)\right)}$$

$$\log\left(1+\operatorname{tg}(x^3)\right) = \operatorname{tg}(x^3) - \frac{\left(\operatorname{tg}(x^3)\right)^2}{2} + o\left(\left(\operatorname{tg}(x^3)\right)^2\right)$$

$$\lim_{x\to+\infty} x^2 \underbrace{\log\left(1+\frac{1}{\sqrt[3]{x^2}}\right)}_{\sim \frac{1}{\sqrt[3]{x^2}}} \underbrace{\left[\operatorname{arctg}\sqrt[3]{x+1} - \operatorname{arctg}\sqrt[3]{x-1}\right]}_{\sim \frac{2}{3x^{4/3}}} = \frac{2}{3}$$

Vediamo la parentesi quadra.

Ricordo che $\boxed{\operatorname{arctg} t + \operatorname{arctg}\frac{1}{t} = \frac{\pi}{2}}$ $\forall t>0$

$$\operatorname{arctg} t = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{t} \qquad \forall t>0$$

$$\left[\operatorname{arctg}\sqrt[3]{x+1} - \operatorname{arctg}\sqrt[3]{x-1}\right] = \cancel{\frac{\pi}{2}} - \operatorname{arctg}\frac{1}{\sqrt[3]{x+1}} - \cancel{\frac{\pi}{2}} + \operatorname{arctg}\frac{1}{\sqrt[3]{x-1}} =$$

$$= \operatorname{arctg}\frac{1}{\sqrt[3]{x-1}} - \operatorname{arctg}\frac{1}{\sqrt[3]{x+1}} \qquad x-1=t\to+\infty$$

$$= \operatorname{arctg}\frac{1}{\sqrt[3]{t}} - \operatorname{arctg}\frac{1}{\sqrt[3]{t+2}} \qquad t\to+\infty$$

$$\operatorname{arctg} s = s - \frac{s^3}{3} + o(s^4) \qquad s\to 0$$

$$\operatorname{arctg}\frac{1}{\sqrt[3]{t}} = \frac{1}{\sqrt[3]{t}} - \frac{1}{3t} + o\left(\frac{1}{t^{4/3}}\right)$$

$$\operatorname{arctg}\frac{1}{\sqrt[3]{t+2}} = \frac{1}{\sqrt[3]{t+2}} - \frac{1}{3(t+2)} + \underbrace{o\left(\frac{1}{(t+2)^{4/3}}\right)}_{= o\left(\frac{1}{t^{4/3}}\right)}$$

$$\operatorname{arctg}\frac{1}{\sqrt[3]{t}} - \operatorname{arctg}\frac{1}{\sqrt[3]{t+2}} = \frac{1}{t^{1/3}} - \frac{1}{3t} + o\left(\frac{1}{t^{4/3}}\right) +$$

$$-\frac{1}{\sqrt[3]{t+2}}+\frac{1}{3(t+2)}=$$

$$=\frac{1}{t^{1/3}}-\cancel{\frac{1}{3t}}\underbrace{-\frac{1}{t^{1/3}\sqrt[3]{1+\frac{2}{t}}}}+\frac{1}{3t\left(1+\frac{2}{t}\right)}+o\left(\frac{1}{t^{4/3}}\right)\Big)$$

$$\frac{1}{3t}\left(1+\frac{2}{t}\right)^{?}=\cancel{\frac{1}{3t}}+o\left(\frac{1}{t^{4/3}}\right)$$

$$\frac{1}{3t}\cdot\frac{1}{1+\frac{2}{t}}\qquad \frac{2}{t}\to 0$$

$$\frac{1}{1+s}=1-s+s^2+o(s^2)$$

$$\frac{1}{3t}\left(1+o\left(\frac{1}{t^{1/3}}\right)\right)$$

$$-\frac{1}{t^{1/3}}\left(1+\frac{2}{t}\right)^{-1/3}=-\frac{1}{t^{1/3}}\left(1-\frac{2}{3t}\right)+o\left(\frac{1}{t^{4/3}}\right)$$

$$(1+s)^{-1/3}=1-\frac{s}{3}+\binom{-1/3}{2}s^2+o(s^2)$$

$$[\,\cdots\,]=\cancel{\frac{1}{t^{1/3}}}-\cancel{\frac{1}{t^{1/3}}}+\frac{2}{3t^{4/3}}+o\left(\frac{1}{t^{4/3}}\right)\sim\frac{2}{3t^{4/3}}=$$

$$=\frac{2}{3(x-1)^{4/3}}\sim\frac{2}{3x^{4/3}}$$

Sia
$$f(x)=\begin{cases} \dfrac{\sin(2x)}{x} & \text{se } x<0 \\ a(x^5+1)+bx^2 & \text{se } x\geq 0. \end{cases}$$

Trovare tutte le coppie $(a,b)\in\mathbb{R}^2$ che rendano vera ciascuna delle seguenti condizioni.

a) $f$ è continua in $\mathbb{R}$.

L'unico problema di continuità è in $x_0=0$, dove è continua da destra. $f$ è continua in $x_0=0$ se e solo se

$$\lim_{x\to 0^-} f(x) = \underbrace{\lim_{x\to 0^-} \frac{\operatorname{sen}(2x)}{x}}_{=2} = f(0) = a$$

Risposta: $a=2$, $b\in\mathbb{R}$.

b) $f$ definitivamente crescente per $x\to+\infty$

Per $x>0$ si ha $f(x)=a(x^5+1)+bx^2$.

$$f'(x)=5ax^4+2bx \xrightarrow[x\to+\infty]{} \begin{cases} +\infty & \text{se } a>0 \\ +\infty & a=0,\ b>0 \\ \equiv 0 & a=0,\ b=0 \\ -\infty & a=0,\ b<0 \\ -\infty & a<0 \end{cases}$$

se $(a>0,\ b\in\mathbb{R}) \vee (a=0,\ b>0) \Rightarrow \lim_{x\to+\infty} f'(x)=+\infty$

$\Rightarrow f'(x)$ def$^{te}$ $>0$ per $x\to+\infty$ $\Rightarrow f$ def$^{te}$ crescente (strettamente) per $x\to+\infty$. OK.

se $f'(x)\to-\infty \Rightarrow f$ def$^{te}$ strett. decrescente per $x\to+\infty$ NO

se $a=b=0 \Rightarrow f\equiv 0$ OK.

Risposta b) $(a>0, b\in\mathbb{R}) \vee (a=0, b\geq 0)$

c) $f$ ammette un flesso in $x_0=1$.

In un intorno di $x_0=1$ $\quad f(x)=a(x^5+1)+bx^2$.

$$f'(x)=5ax^4+2bx$$
$$f''(x)=20ax^3+2b$$

deve essere $f''(1)=0 \iff 20a+2b=0$

$$\boxed{b=-10a}$$

$$f'''(x)=60ax^2 \qquad f'''(1)=60a$$

$\Rightarrow$ se $a\neq 0$, $b=-10a \Rightarrow x_0=1$ pto di flesso.

Se $a=0$, $b=0$ $\quad f\equiv 0$ ha flessi in tutti i pti.

Risposta c) $\qquad b=-10a \quad \forall a\in\mathbb{R}$

d) $f$ ammette max relativo in $x_0=1$

e) $f$ è derivabile in $x_0=0$.

(da fare ...)