$$\lim_{x\to 0} \frac{e^{x^2}-1-\log(1+x+\operatorname{arctg} x)}{\sqrt{1+2x^4}-1}$$

Es. 11.46 di Marcellini-Sbordone

Questo esercizio si può fare anche senza usare il polinomio di Taylor

$$\sqrt{1+t} = 1+\frac{t}{2}+o(t) \qquad t\to 0 \qquad x^4=t\to 0$$

$$\sqrt{1+2x^4}-1 = 1+\frac{2x^4}{2}+o(x^4)-1 = x^4+o(x^4)\sim x^4.$$

$$e^t = 1+t+\frac{t^2}{2}+o(t^2) \quad t\to 0 \qquad x^2=t\to 0$$

$$e^{x^2} = 1+x^2+\frac{x^4}{2}+o(x^4) \qquad x\to 0$$

$$\operatorname{arctg} x = x-\frac{x^3}{3}+o(x^4)$$

$$\log(1+x+\operatorname{arctg} x) = \log\Big(1+\underbrace{2x-\frac{x^3}{3}+o(x^4)}_{t\to 0}\Big)$$

$$\log(1+t) = t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+o(t^4) \qquad t\to 0$$

$$\frac{1}{1+t} = 1-t+t^2-t^3+o(t^3) \qquad t\to 0$$

$$\log\Big(1+\underbrace{2x-\frac{x^3}{3}+o(x^4)}_{t}\Big) = 2x-\frac{x^3}{3}+o(x^4)-\frac{1}{2}\Big(4x^2-\frac{4}{3}x^4\Big)+$$

$$+\frac{1}{3}\big(8x^3\big)-\frac{1}{4}16x^4 =$$

$$= 2x-2x^2+\frac{7}{3}x^3-\frac{10}{3}x^4+o(x^4)$$

$$e^{x^2}-1-\log(\qquad) = x^2+\frac{x^4}{2}+o(x^4)-2x+2x^2-\frac{7}{3}x^3+\frac{10}{3}x^4$$

$$= -2x+o(x)$$

$$\lim \frac{-\!-}{-\!-} = \lim \frac{-2x}{x^4} \nexists \qquad \lim_{x\to 0^\pm} f = \mp\infty$$

$$\lim_{n\to+\infty} \frac{\left(2+2^{1/n}\right)^n + n^{\log n}}{\log n \left(3^n - \left(3 - \frac{1}{n\log n}\right)^n\right)}$$

$$3^n - \left(3 - \frac{1}{n\log n}\right)^n = 3^n \left(1 - \left(1 - \frac{1}{3n\log n}\right)^n\right) =$$

$$= 3^n \left(1 - e^{\underline{n \log\left(1 - \frac{1}{3n\log n}\right)}}\right) \sim 3^n \left(-n \log \underbrace{\left(1 - \frac{1}{3n\log n}\right)}_{\sim -\frac{1}{3n\log n}}\right) \sim$$

$$n \log\Big(1 - \underbrace{\frac{1}{3n\log n}}_{\downarrow \atop 0}\Big) \sim -\frac{\cancel{n}}{3\cancel{n}\log n} \to 0$$

$$e^t - 1 \sim t \qquad t \to 0.$$

$$\to \sim 3^n \left(-n\left(-\frac{1}{3n\log n}\right)\right) = 3^n \frac{1}{3\log n}$$

den $\sim 3^{n-1}$

Num: $\left(2+2^{1/n}\right)^n + n^{\log n} = e^{\overbrace{n\log(2+2^{1/n})}^{\nearrow \log 3}} + e^{\log^2 n} =$

$$e^{n\log(2+2^{1/n})}\Big(1 + \underbrace{e^{\log^2 n - n\log(2+2^{1/n})}}_{\searrow 0}\Big) \sim$$

$$\log^2 n - \underbrace{n\log\left(2+2^{1/n}\right)}_{\sim -n\log 3} \to -\infty$$

$$\sim e^{n\log\left(2+2^{1/n}\right)}$$

$$\lim_{n\to+\infty} \frac{e^{n\log\left(2+2^{1/n}\right)}}{\underbrace{3^{n-1}}_{= e^{(n-1)\log 3}}} = \lim_{n\to+\infty} e^{n\log\left(2+2^{1/n}\right)-(n-1)\log 3}$$

$$= 3\sqrt[3]{2}$$

$$a_n = \underbrace{n \log\left(2+2^{1/n}\right)}_{d} - (n-1)\log 3$$

$$n\log\left(2+2^{1/n}\right) = n\log\left(3+\left(2^{1/n}-1\right)\right) = n\log\left(3\cdot\left(1+\frac{2^{1/n}-1}{3}\right)\right) =$$

$$= n\log 3 + n\log\left(1+\frac{2^{1/n}-1}{3}\right)$$

$$a_n = n\log\left(1+\frac{2^{1/n}-1}{3}\right) + \log 3 \to \frac{\log 2}{3} + \log 3 = \log\left(2^{1/3}\cdot 3\right)$$

$$n\log\left(1+\underbrace{\frac{2^{1/n}-1}{3}}_{\to 0}\right) \sim n\,\frac{2^{1/n}-1}{3} = \frac{n}{3}\left(e^{\overbrace{\frac{\log 2}{n}}^{\to 0}}-1\right) \sim$$

$$\sim \frac{n}{3}\,\frac{\log 2}{n} \longrightarrow \frac{\log 2}{3}$$

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$$\lim_{x\to 0} (\cos x)^{\frac{1}{1-\cos x}} = \left(1^{\frac{1}{0^+}}\right) = \left(1^{+\infty}\right) \qquad \text{f.i.}$$

$$= \lim_{x\to 0} e^{\frac{1}{1-\cos x}\log(\cos x)} = e^{-1}.$$

$$\frac{\overbrace{\log(\cos x)}^{\sim -\frac{x^2}{2}}}{\underbrace{1-\cos x}_{\sim x^2/2}} \longrightarrow -1$$

$$1-\cos x = \cancel{1} - \left(\cancel{1} - \frac{x^2}{2} + o(x^2)\right)$$

$$\log(\cos x) = \log\big(1 + \underbrace{(\cos x - 1)}_{\to 0}\big) \sim \cos x - 1 \sim -\frac{x^2}{2}$$

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$$f(x) = \frac{\operatorname{tg} x}{\sqrt{3} - |\operatorname{tg} x|}$$

dominio $\quad x \neq \frac{\pi}{2} + k\pi$

$\operatorname{tg} x \neq \pm\sqrt{3}$

$\operatorname{tg} x \neq \sqrt{3} \iff x \neq \pm\frac{\pi}{3} + k\pi$

dominio $\quad x \neq \frac{\pi}{2} + k\pi, \ \pm\frac{\pi}{3} + k\pi$

$f$ è dispari ($f$. dispari fratto $f$. pari)

$f$ è periodica di periodo $\pi$.

$\Longrightarrow$ Basta studiarla in $[0, \frac{\pi}{2})$

Posso togliere il modulo.

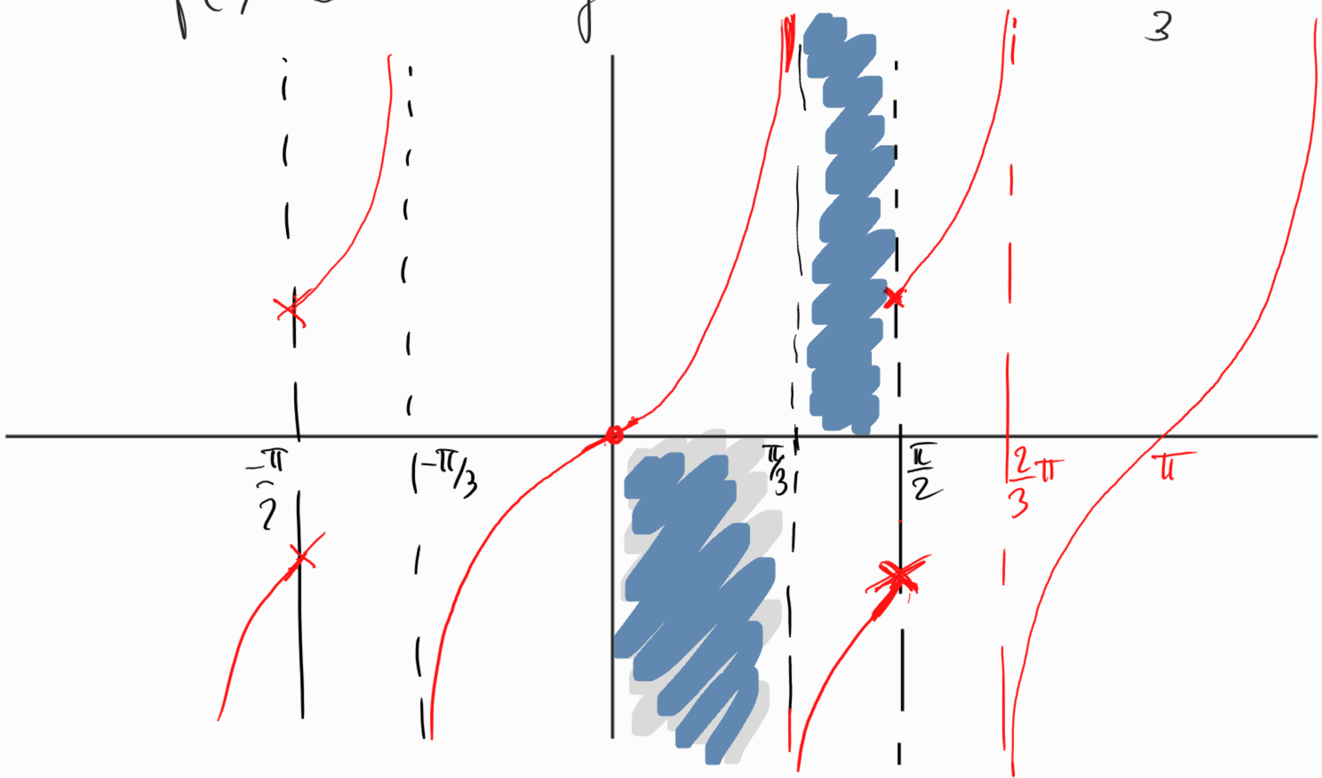
Studio $f(x) = \dfrac{\operatorname{tg} x}{\sqrt{3} - \operatorname{tg} x}$ in $[0, \frac{\pi}{2})$.

Segno.

$f(x) = 0 \iff x = 0$

$f(x) > 0 \iff 0 < \operatorname{tg} x < \sqrt{3} \iff 0 < x < \frac{\pi}{3}$

$f(x) < 0 \iff \operatorname{tg} x > \sqrt{3} \iff x > \frac{\pi}{3}$



Continuità e limiti    $f$ è continua nel suo dominio.

$$\lim_{x\to\frac{\pi}{3}^-} \frac{\overset{\to\sqrt{3}}{\operatorname{tg} x}}{\sqrt{3}-\underset{\downarrow\ \sqrt{3}^-}{\operatorname{tg} x}} = \frac{\sqrt{3}}{0^+} = +\infty$$

$$\lim_{x\to\frac{\pi}{3}^+} \frac{\operatorname{tg} x}{\sqrt{3}-\operatorname{tg} x} = \left(\frac{\sqrt{3}}{0^-}\right) = -\infty$$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\operatorname{tg} x}{\sqrt{3}-\operatorname{tg} x} = \lim_{t\to+\infty} \frac{t}{\sqrt{3}-t} = -1$$

$\operatorname{tg} x = t \to +\infty$

$x=\frac{\pi}{3}$ è asintoto verticale

Derivata prima    $f(x) = \dfrac{\operatorname{tg} x}{\sqrt{3}-\operatorname{tg} x}$

$$f'(x) = (1+\operatorname{tg}^2 x)\,\frac{\sqrt{3}-\cancel{\operatorname{tg} x}+\cancel{\operatorname{tg} x}}{(\sqrt{3}-\operatorname{tg} x)^2} = \frac{\sqrt{3}}{(\sqrt{3}-\operatorname{tg} x)^2}(1+\operatorname{tg}^2 x)$$

$$\forall x \in \left(0,\frac{\pi}{2}\right) \setminus \left(\frac{\pi}{3}\right)$$

In $x=0$ $\Rightarrow$ $f'_+(0) = \frac{\sqrt{3}}{3}$

Ma per simmetria $f'_-(0) = \frac{\sqrt{3}}{3}$

$\Rightarrow$ $f$ derivabile in $x=0$.

$f'(x)=0$ mai

$f'(x)>0 \quad \forall x \in \left(0,\frac{\pi}{2}\right)\setminus\left\{\frac{\pi}{3}\right\}$

$f$ è strett. crescente in $\left[0,\frac{\pi}{3}\right)$ e in $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} f'(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\sqrt{3}\,(1+tg^2x)}{(\sqrt{3}-tg\,x)^2} = \lim_{t\to+\infty}\frac{\sqrt{3}\,(1+t^2)}{(\sqrt{3}-t)^2}=\sqrt{3}$$

Derivata seconda

$$f''(x) = \left(\frac{\sqrt{3}\,(1+tg^2x)}{(tg\,x-\sqrt{3})^2}\right)' = \sqrt{3}\,(1+tg^2x)\,\frac{2tg\,x\,(tg\,x-\sqrt{3})^{\cancel{2}}-(1+tg^2x)\,2\,\cancel{(tg\,x-\sqrt{3})}}{(tg\,x-\sqrt{3})^{\cancel{4}\,3}}$$

$$=2\sqrt{3}\,\frac{(1+tg^2x)}{(tg\,x-\sqrt{3})^3}\left[\cancel{tg^2}x-\sqrt{3}\,tg\,x-1-\cancel{tg^2}x\right]=$$

$$=-\frac{2\sqrt{3}\,(1+tg^2x)\,(\sqrt{3}\,tg\,x+1)}{(tg\,x-\sqrt{3})^3} \qquad \forall x\in\left(0,\frac{\pi}{2}\right)\setminus\left\{\frac{\pi}{3}\right\}.$$

$$f''_+(0) = -\frac{2\sqrt{3}}{-3\sqrt{3}} = \frac{2}{3}$$

$$f''_-(0) = -\frac{2}{3}$$

$\Rightarrow$ $f$ non è derivabile 2 volte in $x=0$.

$f''(x)=0$ mai

$f''(x)>0 \iff 0<tg\,x<\sqrt{3} \iff x\in\left(0,\frac{\pi}{3}\right)$

$\phantom{f''(x)}<0 \iff tg\,x>\sqrt{3} \iff x\in\left(\frac{\pi}{3},\frac{\pi}{2}\right)$

$x=0$ è un punto di flesso.

$$f(x) = e^{-|x|}(x^2+2x+2) = \begin{cases} e^{-x}(x^2+2x+2) & x \geq 0 \\ e^{x}(x^2+2x+2) & x \leq 0. \end{cases}$$

Dominio: $\mathbb{R}$, non ci sono simmetrie apparenti

Segno: $f(x) > 0 \quad \forall x \in \mathbb{R}$

$f$ continua in $\mathbb{R}$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-x}(x^2+2x+2) = 0$$

$$\frac{\overbrace{x^2+2x+2}^{\sim x^2}}{e^x} \sim \frac{x^2}{e^x}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \underbrace{e^{x}}_{\downarrow\, 0}\underbrace{(x^2+2x+2)}_{\downarrow\, +\infty} = \lim_{t\to+\infty} e^{-t}(t^2-2t+2) = 0$$

$t = -x \to +\infty$

$y=0$ è asint. orizz. per $x \to \pm\infty$

Derivata $f$ sicuramente derivabile per $x \neq 0$.

$x>0 \quad f'(x) = \left(e^{-x}(x^2+2x+2)\right)' = e^{-x}(-x^2\cancel{-2x}\cancel{-2}\cancel{+2x}\cancel{+2}) =$

$= -x^2e^{-x}$

$x<0 \quad f'(x) = \left(e^{x}(x^2+2x+2)\right)' = e^{x}\left(x^2+2x+2+2x+2\right) =$

$= e^{x}(x^2+4x+4) = e^{x}(x+2)^2$

$\boxed{x=0} \quad f'_+(0) = -x^2e^{-x}\Big|_{x=0} = 0$

$f'_-(0) = e^{x}(x+2)^2\Big|_{x=0} = 4$

$x=0$ pto angoloso

$f'(x) = 0 \iff x = -2$

$f'(x) > 0 \iff x \in (-\infty, -2) \cup (-2, 0)$

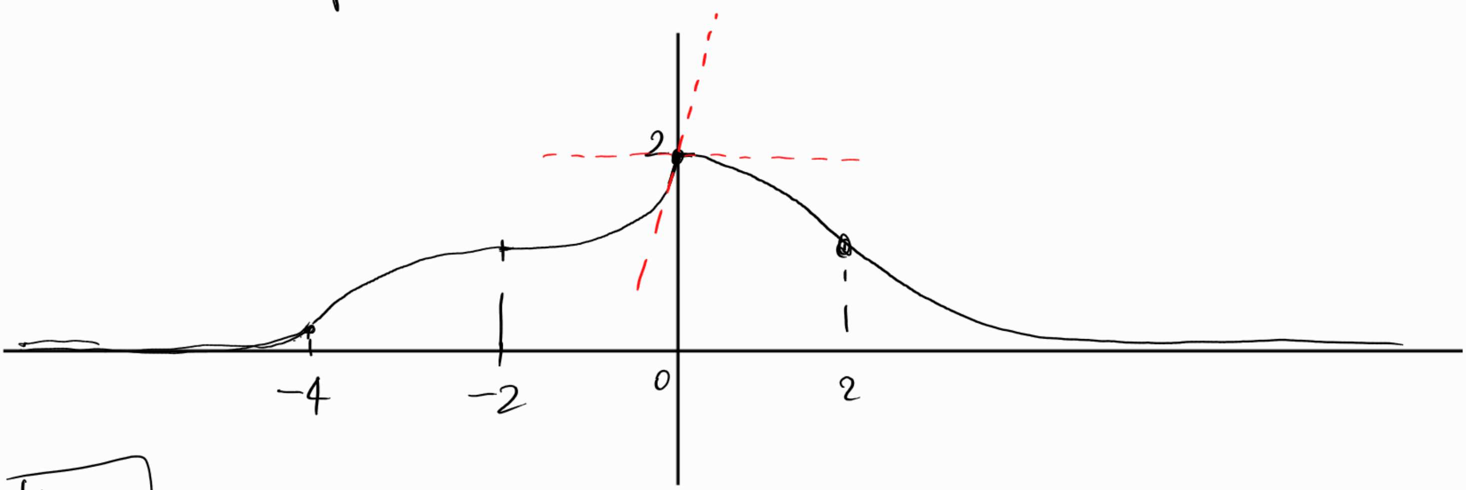
$f'(x) < 0 \iff x \in (0, +\infty)$

$f$ strett. crescente in $(-\infty, 0]$

$f$ strett. decrescente in $[0, +\infty)$

$x = -2$ è pto critico, ma non di estremo locale.

$x = 0$ è pto di massimo assoluto




$\boxed{x > 0}$

$f''(x) = \left(-x^2 e^{-x}\right)' = -e^{-x}\left(-x^2 + 2x\right) = e^{-x}\left(x^2 - 2x\right) = e^{-x} x(x-2)$

$f''(x) = 0 \iff x = 2$

$f''(x) > 0 \iff x > 2$

$f''(x) < 0 \iff 0 < x < 2$

$\boxed{x < 0}$

$f''(x) = \left(e^x (x+2)^2\right)' = e^x\left((x+2)^2 + 2(x+2)\right) = e^x (x+2)(x+4)$

$f''(x) = 0 \iff (x = -2) \vee (x = -4)$

$f''(x) > 0 \iff (x < -4) \vee (-2 < x < 0)$

$f''(x) < 0 \iff -4 < x < -2$

$f$ strett convessa in $(-\infty, -4]$, in $[-2, 0]$ e in $[2, +\infty)$.

$f$ " concava in $[-4, -2]$ e in $[0, 2]$.

$x = -4$, $x = -2$, $x = 2$ sono pti di flesso.