$$\lim_{x\to 0} \frac{\overbrace{tg\,x}^{\sim x} - \overbrace{\sin^2 x}^{\sim x^2}}{x} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{\cancel{x}\left(\underset{\to 1}{\frac{tg\,x}{x}} - \underset{\to 0}{\frac{\sin^2 x}{x}}\right)}{\cancel{x}} = 1$$

$$\lim_{x\to 0} \frac{\overbrace{2-2\cos x}^{\sim x^2}}{\underbrace{sen(x^2) - 3\sin x - x^2}_{\sim -3x}} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{x^2}{-3x} = 0$$

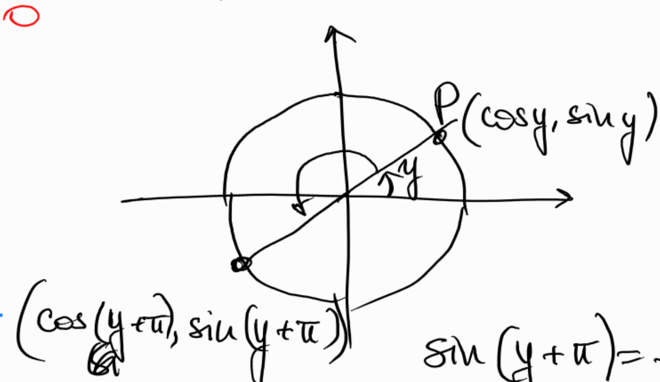
$$2 - 2\cos x = 2(1-\cos x) \sim 2\,\frac{x^2}{2} = x^2$$

$$\underbrace{sen(x^2)}_{\sim x^2} - \underbrace{3\sin x}_{\sim -3x} - x^2 = x\left(\underbrace{\boxed{\frac{sen(x^2)}{x}}}_{\to 0} \underbrace{\boxed{-\,3\,\frac{\sin x}{x}}}_{\to -3} - \underbrace{x}_{\to 0}\right) \sim -3x$$

$$\lim_{x\to \pi} \frac{\sin x}{x-\pi} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{\sin(\pi+y)}{y} = \lim_{y\to 0}\left(-\frac{\sin y}{y}\right) = -1$$

$x - \pi = y \to 0$

$x = \pi + y$



$$\sin(y+\pi) = -\sin y$$

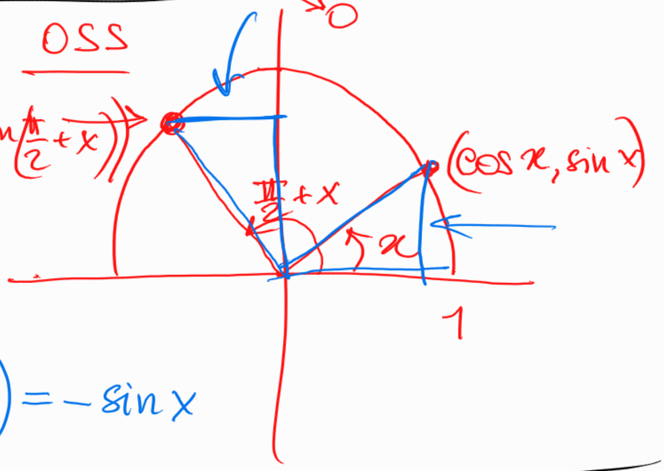
$$\lim_{n\to +\infty} \underbrace{\overbrace{(2n+5)}^{\sim 2n}}_{\to +\infty} \underbrace{\overbrace{\cos\left(\frac{\pi n^2 + 7}{2n^2+n}\right)}^{\sim \pi/4n}}_{\to \cos\frac{\pi}{2} = 0} = (+\infty \cdot 0) = \lim_{n\to +\infty} 2n\,\frac{\pi}{4n} = \frac{\pi}{2}$$

$$\cos\left(\frac{\pi n^2+7}{2n^2+n}\right) = \cos\left(\frac{\pi}{2} + \left(\frac{\pi n^2+7}{2n^2+n} - \frac{\pi}{2}\right)\right) = \cos\left(\frac{\pi}{2} + \frac{\cancel{2\pi n^2} + 14 - \cancel{2\pi n^2} - n\pi}{2(2n^2+n)}\right)$$

$$= \cos\left(\frac{\pi}{2} + \underbrace{\frac{(-n\pi+14)}{2(2n^2+n)}}_{\to 0}\right) = -\sin\left(\underbrace{\frac{-n\pi+14}{2(2n^2+n)}}_{\to 0}\right) \sim$$

$$\sim -\left(\frac{-n\pi+14}{2(2n^2+n)}\right) \sim \frac{n\pi}{4n^2} = \frac{\pi}{4n}$$

OSS



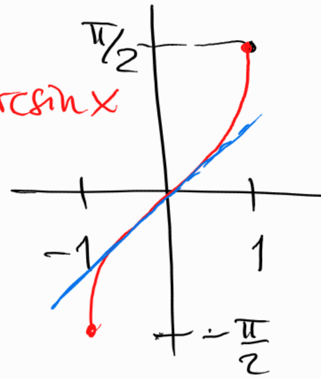
$$\cos\left(\frac{\pi}{2}+x\right) = -\sin x$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{y}{\sin y} = 1$$

$\arcsin x = y \to 0$

$x = \sin y$

$y = \arcsin x$



Formulazioni alternative

$\arcsin x \sim x$ per $x \to 0$

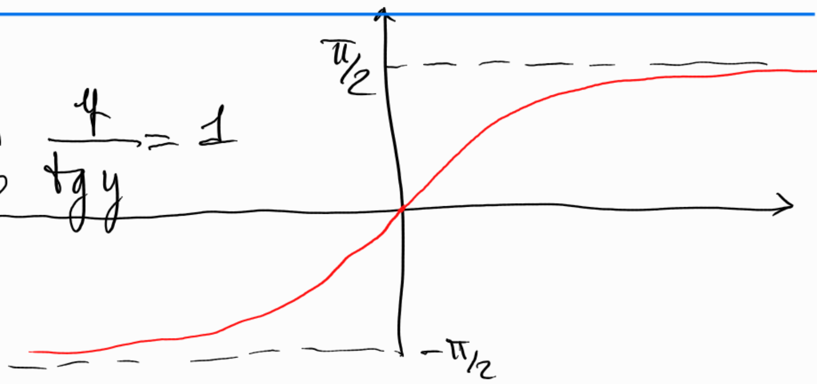
$\arcsin x = x(1 + o(1))$ "

$\arcsin x = x + o(x)$ "

$$\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{y}{\operatorname{tg} y} = 1$$

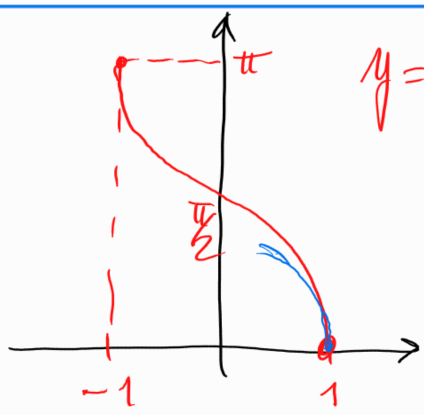
$y = \operatorname{arctg} x \to 0$

$x = \operatorname{tg} y$



$$\lim_{x\to 1^-} \frac{\arccos x}{\sqrt{1-x}} = \left(\frac{0}{0}\right) = (*)$$

$y = \arccos x$



$y = \arccos x \to 0^+$

$x = \cos y$

$$(*) = \lim_{y\to 0^+} \frac{y}{\sqrt{1-\cos y}} = \lim_{y\to 0^+} \sqrt{\underbrace{\frac{y^2}{1-\cos y}}_{\to 2}} = \sqrt{2}.$$

$$\lim_{x\to 1^-} \frac{\arccos x}{\sqrt{1-x}} = \sqrt{2}.$$

$$\arccos x \sim \sqrt{2(1-x)} \quad \text{per } x\to 1^-$$

$$\arccos x = \sqrt{2(1-x)}\,(1+o(1))$$

$$\arccos x = \sqrt{2(1-x)} + o\left(\sqrt{1-x}\right)$$

Limiti notevoli che coinvolgono e

$$\boxed{\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e}$$

già lo sappiamo, perché se $a_n \to +\infty$, allora $\lim_{n\to+\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$ e poi si applica il teorema ponte

$$\lim_{x\to +\infty}\left(1-\frac{1}{x}\right)^x = \lim_{x\to+\infty}\left[\left(1+\frac{1}{(-x)}\right)^{-x}\right]^{-1} =$$

$-x = y \to -\infty$

$$= \lim_{y\to-\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^{-1} = e^{-1} = \frac{1}{e}$$

$$\boxed{\lim_{x\to +\infty}\left(1+\frac{a}{x}\right)^x = e^a}$$

ovvio se $a=0$.

se $a\neq 0$

$$\left(1+\frac{a}{x}\right)^x = \Bigg[\overbrace{\left(1+\frac{1}{\underbrace{x/a}_{\to\pm\infty}}\right)^{x/a}}^{\to e}\Bigg]^a$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x+1}\right)^{x^2}=\left(1^{+\infty}\right)=\lim_{x\to+\infty}\Bigg[\underbrace{\left(1+\frac{1}{x+1}\right)^{x+1}}_{\downarrow\, e}\Bigg]^{\overset{\frac{x^2}{x+1}}{\downarrow +\infty}}=+\infty$$

$$\lim_{x\to 0^{\pm}}(1+x)^{1/x}=\left(1^{+\infty}\right)=\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y}=e$$

$$\frac{1}{x}=y\to+\infty$$

$$\boxed{\lim_{x\to 0}(1+x)^{1/x}=e}$$

Conseguenza immediata:

$$\lim_{x\to 0}\frac{1}{x}\log(1+x)=\lim_{x\to 0}\log\left((1+x)^{1/x}\right)=\log e=1.$$

$$\boxed{\lim_{x\to 0}\frac{\log(1+x)}{x}=1}$$

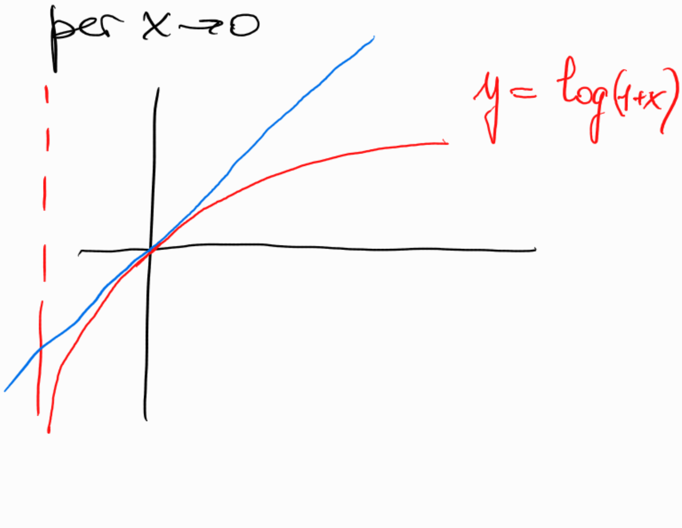
Attenzione: log naturale!

Si scrive anche così

$$\log(1+x)\sim x \qquad \text{per } x\to 0$$

$$\log(1+x)=x\,(1+o(1)) \qquad ''$$

$$\log(1+x)=x+o(x) \qquad \text{per } x\to 0$$



Attenzione: se la base del log è diversa cambia il risultato

$$\lim_{x\to 0} \frac{\log_b(1+x)}{x} = \lim_{x\to 0} \log_b\left((1+x)^{1/x}\right) = \log_b e = \frac{1}{\log b}$$

Rifacciamo

$$\lim_{x\to+\infty} \left(1+\frac{1}{x+1}\right)^{x^2} = \lim_{x\to+\infty} e^{x^2\log\left(1+\frac{1}{x+1}\right)} = +\infty$$

$$x^2 \underbrace{\log\left(1+\frac{1}{x+1}\right)}_{\sim} \sim \frac{x^2}{x} = x \to +\infty \qquad x\to+\infty$$

$$\frac{1}{x+1} \sim \frac{1}{x}$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \left(\frac{0}{0}\right) = \lim_{y\to 0} \frac{y}{\log(1+y)} = 1.$$

$$e^x - 1 = y \to 0$$

$$x = \log(1+y)$$

$$\boxed{\lim_{x\to 0} \frac{e^x-1}{x} = 1} \longleftarrow$$

In alternativa:

$$e^x - 1 \sim x \qquad \text{per } x\to 0$$

$$e^x - 1 = x\,(1+o(1)) \qquad ''$$

$$e^x = 1 + x + o(x) \qquad ''$$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \lim_{x\to 0} \boxed{\frac{e^{x\log a}-1}{x\log a}}_{\to 1} \log a = \quad \boxed{a>0}$$

$= \log a$

$$\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \log a}$$

$\boxed{\forall a > 0}$

OSS se $a=1$, il risultato è ovvio, perché $\frac{1^x - 1}{x} \equiv 0 \to 0 = \log 1$

Sia $\alpha \in \mathbb{R}$.

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \left(\frac{0}{0}\right) =$$

$$= \lim_{x\to 0} \frac{e^{\alpha \log(1+x)} - 1}{x} =$$

OSS $\alpha \log(1+x) \to 0$

$$= \lim_{x\to 0} \boxed{\frac{e^{\alpha \log(1+x)} - 1}{\alpha \log(1+x)}} \; \alpha \boxed{\frac{\log(1+x)}{x}} = \alpha$$

$\frac{\log(1+x)}{x} \to 1$

$\frac{e^{\alpha \log(1+x)} - 1}{\alpha \log(1+x)} = \frac{e^t - 1}{t}$ dove $t = \alpha \log(1+x) \to 0$

$\frac{e^t - 1}{t} \to 1$

$$\boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \qquad \forall \alpha \in \mathbb{R}.}$$

Alternativamente

$(1+x)^\alpha - 1 \sim \alpha x$ per $x \to 0$

$(1+x)^\alpha - 1 = \alpha x (1 + o(1))$ "

$(1+x)^\alpha = 1 + \alpha x + o(x)$ "

Per es. se $\alpha = 1/2$ $\qquad \sqrt{1+x} = 1 + \frac{x}{2} + o(x)$ per $x \to 0$

$\alpha = 2 \qquad (1+x)^2 = 1 + 2x + \underbrace{o(x)}_{= x^2} \qquad x \to 0$

$$\lim_{x \to +\infty} \left(\sqrt[3]{x^3 + 2x^2} - x \right) = (+\infty - \infty)$$

1° modo già noto $\qquad A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$$\sqrt[3]{x^3+2x^2} - x = \frac{\left(\sqrt[3]{x^3+2x^2} - x\right)\left((x^3+2x^2)^{2/3} + x\sqrt[3]{x^3+2x^2} + x^2\right)}{(x^3+2x^2)^{2/3} + x\sqrt[3]{x^3+2x^2} + x^2} =$$

$$= \frac{\cancel{x^3} + 2x^2 - \cancel{x^3}}{\underbrace{(x^3+2x^2)^{2/3}}_{\sim x^2} + \underbrace{x\sqrt[3]{x^3+2x^2}}_{\sim x^2} + x^2} = \frac{2x^2}{x^2(3 + o(1))} \to \frac{2}{3}$$

$x \to +\infty$

2° modo $\qquad \sqrt[3]{x^3+2x^2} - x = x\overbrace{\left(\sqrt[3]{1 + \frac{2}{x}} - 1\right)}^{\sim \frac{2}{3x}} \sim x\frac{2}{3x} = \frac{2}{3}$

$\frac{2}{x} \to 0$

$\sqrt[3]{1+t} - 1 \sim \frac{t}{3} \qquad t \to 0$

$\Rightarrow \sqrt[3]{1 + \frac{2}{x}} - 1 \sim \frac{2}{3x} \qquad$ per $x \to +\infty$