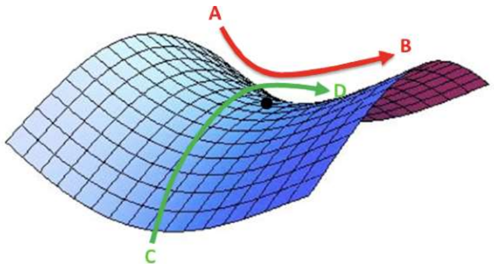
Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 11

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Motivation and Outline

The goal of this lesson is to introduce the so-called **Watson transformation**, a technique that exploits complex integration (in particular, the Residue Theorem in a sort of *reverse fashion*) to convert slowly converging series into rapidly converging ones.

In the second part of this lesson we will illustrate such a technique by following the **original paper** by G. N. Watson, appeared in 1918.

However, this will require the introduction of a considerable number of concepts about **spherical waves**, which will be the object of the first part of this lesson.

Scalar Spherical Wave Functions

Scalar Helmholtz Equation: Spherical Coordinates

Let ψ be a scalar wave function, i.e., a solution of the *scalar Helmoltz equation*

$$\nabla^2\psi + k^2\psi = 0$$

In spherical coordinates, this reads

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0}$$

Following the method of **separation of variables**, we seek solutions in the form

$$\psi = \mathrm{R}\left(r\right)\Theta\left(\theta\right)\Phi\left(\phi\right)$$

which we will call *spherical wave functions*.

Separation of Variables

Substituting the latter into the Helmholtz equation, dividing by ψ , and multiplying by $r^2 \sin^2 \theta$, we obtain

$$\frac{\sin^2 \theta}{\mathrm{R}} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{dR}}{\mathrm{d}r}\right) + \frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + k^2 r^2 \sin^2\theta = 0$$

The ϕ dependence is now separated out, and we let

$$\boxed{\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -\mu^2}$$

where μ is a constant.

Separation of Variables

Substituting this into the preceding equation and dividing by $\sin^2\theta$, we obtain

$$\frac{1}{\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{dR}}{\mathrm{d}r}\right)+\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{\mu^2}{\sin^2\theta}+k^2r^2=0$$

so that also the r and θ dependences are separated. In particular, we let

$$\boxed{\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)-\frac{\mu^2}{\sin^2\theta}=-\nu\left(\nu+1\right)}$$

(the properties of the Θ function depend upon whether or not ν is an integer $\nu=n$, which motivates the apparently strange choice of the separation constant.)

Separation of Variables

Finally, substituting this into the preceding equation we obtain

$$\frac{1}{\mathrm{R}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{dR}}{\mathrm{d}r}\right)-\nu\left(\nu+1\right)+k^2r^2=0$$

so that, collecting the above results, we have **three separated equations**

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{dR}}{\mathrm{d}r}\right)+\left[k^2r^2-\nu\left(\nu+1\right)\right]\mathrm{R}=0$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[\nu\left(\nu+1\right)-\frac{\mu^2}{\sin^2\theta}\right]\Theta=0$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+\mu^2\Phi=0$$

The Radial Dependence

The R equation is closely related to Bessel's equation and admits as solutions the so-called **spherical Bessel functions**:

$$b_\nu\left(kr\right) = \sqrt{\frac{\pi}{2kr}} B_{\nu+1/2}\left(kr\right)$$

where $B_{\nu+1/2}$ is any ordinary Bessel (cylindrical) function (J, Y, $H^{(1)}$, $H^{(2)}$) of order $\nu+1/2$.

The spherical Bessel functions are actually *simpler* than the ordinary Bessel functions…

Spherical Bessel Functions

For example, the zero-order functions are:

$$j_0\left(kr\right)=\frac{\sin\left(kr\right)}{kr}$$

$$h_0^{(1)}\left(kr\right)=\frac{e^{jkr}}{jkr}$$

$$y_0\left(kr\right)=-\frac{\cos\left(kr\right)}{kr}$$

$$h_0^{(2)}\left(kr\right)=-\frac{e^{-jkr}}{jkr}$$

The higher-order functions with $\nu = n$ integer are polynomials in $1/kr$ times $\sin(kr)$ and $\cos(kr)$.

The Angular θ Dependence

The θ equation is related to Legendre's equation and its solutions are called **associated Legendre functions**:

$$L_\nu^\mu\left(\cos\theta\right) = a_1 P_\nu^\mu\left(\cos\theta\right) + a_2 Q_\nu^\mu\left(\cos\theta\right), \qquad a_{1,2} \in \mathbb{C}$$

where the indices ν and μ are those related to the radial and azimuthal functions, respectively, and:

- $P_\nu^\mu\left(\cos\theta\right)$ are the associated Legendre functions of the **first kind**

(regular for $\theta \in \left[0, \pi\right]$ only when ν is an integer)

- $Q_\nu^\mu\left(\cos\theta\right)$ are the associated Legendre functions of the **second kind**

(singular for $\theta = 0, \pi$ for any value of ν, μ)

θ Dependence: Legendre Polynomials

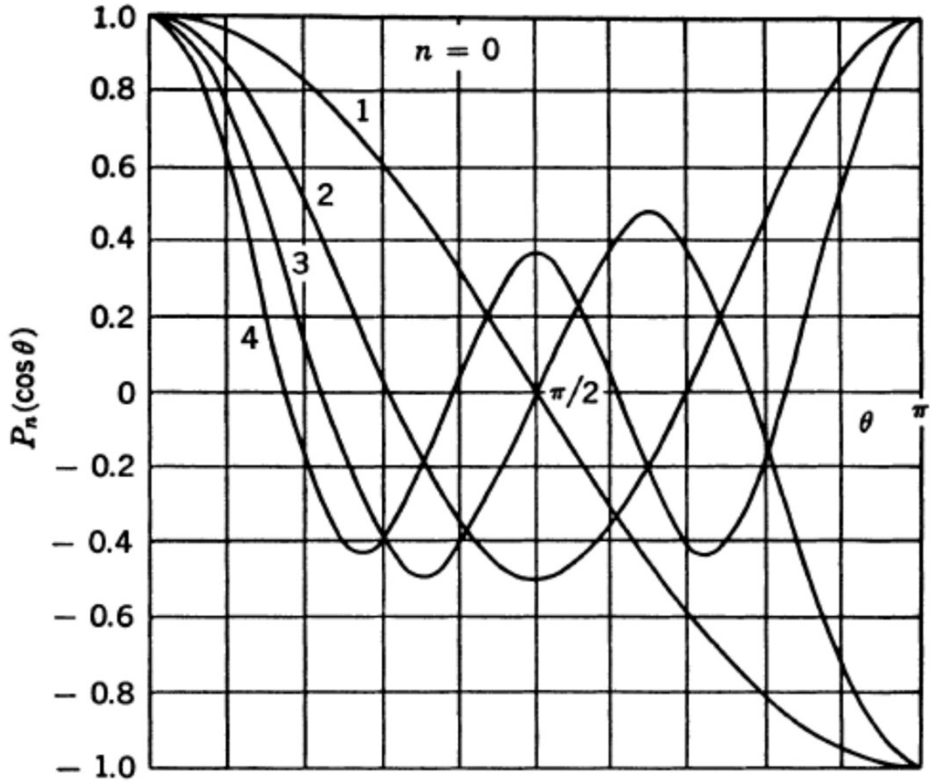
The Legendre functions with $\mu=0$ and $\nu=n$ integer are polynomials in $\cos(\theta)$ called **Legendre polyomials**: $P_n^0\left(\cos\theta\right) \doteq P_n\left(\cos\theta\right)$, solutions of

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_n}{\mathrm{d}\theta}\right)+\nu\left(\nu+1\right)P_n=0$$

$$P_0\left(\cos\theta\right)=1$$

$$P_1\left(\cos\theta\right)=\cos\theta$$

$$P_2\left(\cos\theta\right)=\frac{1}{4}\left(3\cos 2\theta+1\right)$$
$$=\frac{1}{2}\left(3\cos^2\theta-1\right)$$



θ Dependence: Legendre Functions with Integer Indices

The Legendre functions with $\mu=m>0$ and $\nu=n$ both integers are expressed in terms of the m-th derivative of the Legendre polynomial of order n as

$$P_n^m\left(u\right)=\left(-1\right)^m\left(1-u^2\right)^{m/2}\frac{d^m P_n\left(u\right)}{du^m}$$

Note that these are polynomials in u only when m is an even integer.

θ Dependence: Orthogonality and Completeness

The Legendre polyomials form a **complete and orthogonal** set in the interval from 0 to π.

As concerns the orthogonality, we have:

$$\int_0^\pi P_n\left(\cos\theta\right)P_q\left(\cos\theta\right)\mathrm{d}\cos\theta = \int_0^\pi P_n\left(\cos\theta\right)P_q\left(\cos\theta\right)\sin\theta\,\mathrm{d}\theta = \frac{2}{2n+1}\delta_{nq}$$

Completeness means that an arbitrary function $f(\theta)$ can be expanded in a series of Legendre polynomials:

$$f\left(\theta\right) = \sum_{n=0}^{+\infty} a_n P_n\left(\cos\theta\right)$$ (Fourier—Legendre series)

with

$$a_n = \frac{2n+1}{n}\int_0^\pi f\left(\theta\right)P_n\left(\cos\theta\right)\sin\theta\mathrm{d}\theta$$

The Angular ϕ Dependence

The ϕ equation is the familiar **harmonic equation**, giving rise to harmonic solutions

$$h\left(\mu\phi\right) = a_1 \cos\left(\mu\phi\right) + a_2 \sin\left(\mu\phi\right) = b_1 e^{-j\mu\phi} + b_2 e^{+j\mu\phi}$$

If a *single-valued* ψ in the range 0 to 2π on ϕ is desired, μ must be an **integer** $\mu = m$.

On the other hand, when the wave function is defined in *restricted angular ranges*, then μ may have **noninteger values** that depend on the boundary conditions of the problem.

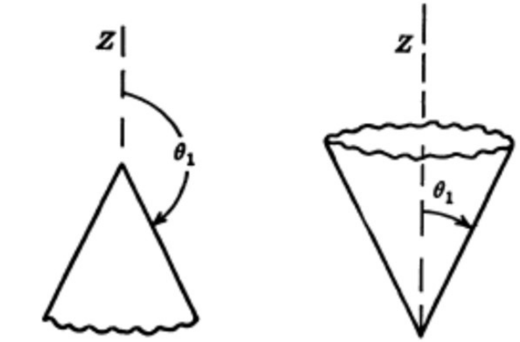
General Representation of Scalar Wavefunctions

The representation of a **general scalar wavefunction** in terms of the spherical wavefunctions derived thus far is finally

$$\boxed{\sum_{\mu,\nu} C_{\mu\nu} b_{\nu}\left(kr\right) L_{\nu}^{\mu}\left(\cos\theta\right) h\left(\mu\phi\right)}$$

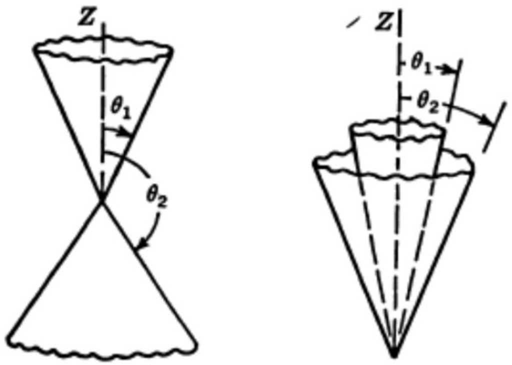
where the indices μ, ν may assume **integer** or **noninteger** values, depending on the boundary conditions of the problem at hand, and the summation can be either discrete or continuous (i.e., an integral).

Example: Radial Waveguides



$$\mu = m = 0,1,2,\ldots$$

$$L_\nu^m\left(\cos\theta\right) = P_\nu^m\left(\cos\theta\right)$$

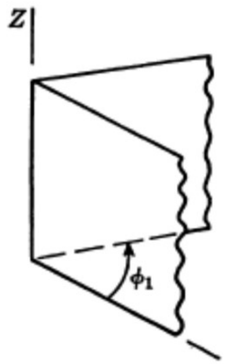


$$\mu = m = 0,1,2,\ldots$$

$$L_\nu^m\left(\cos\theta\right) = a_1 P_\nu^m\left(\cos\theta\right) + a_2 Q_\nu^m\left(\cos\theta\right)$$

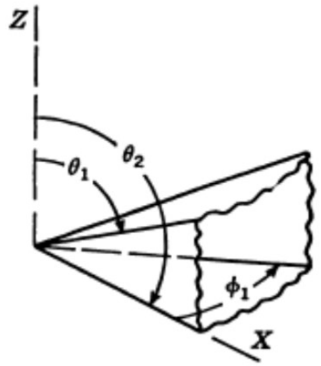
Here the noninteger indices ν are found by enforcing the boundary conditions on the PEC cones, which imply the vanishing of the function L_ν^m or its derivative, depending on the polarization of the field.

Example: Radial Waveguides



$$\nu = n = 1, 2, \ldots$$

$$L_n^{\mu}\left(\cos\theta\right) = P_n^{\mu}\left(\cos\theta\right)$$



$$L_\nu^{\mu}\left(\cos\theta\right) = a_1 P_\nu^{\mu}\left(\cos\theta\right) + a_2 Q_\nu^{\mu}\left(\cos\theta\right)$$

Similarly, the noninteger indices μ are found by enforcing the boundary conditions on the PEC half planes, which imply the vanishing of the function $h\left(\mu\phi\right)$ or its derivative, depending on the polarization of the field.

Vector Spherical Wave Functions

Construction of Potentials: TM^z/TE^z Fields

In order to represent electromagnetic (i.e., **vector**) fields in terms of scalar spherical wave functions ψ, one possibility is to let ψ be a **rectangular component** of the vector potential $\mathbf{A}$ or $\mathbf{F}$, e.g.:

$$\mathbf{A} = \mathbf{z}_0 \psi = \mathbf{r}_0 \cos\theta\psi - \boldsymbol{\theta}_0 \sin\theta\psi$$

which generates a field TM to z.

In fact, assuming $\mathbf{A}$ to be a Lorenz potential, in a source-free region it results

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = \mathbf{0}$$

and $\left(\nabla^2 \mathbf{A}\right)_z = \nabla^2 A_z$ so that $\nabla^2 A_z + k^2 A_z = 0$, i.e., $A_z = \psi$ is indeed a scalar wave function.

Construction of Potentials: TM^z/TE^z Fields

Dually, by considering an electric vector potential:

$$\mathbf{F} = \mathbf{z}_0 \psi = \mathbf{r}_0 \cos\theta\psi - \boldsymbol{\theta}_0 \sin\theta\psi$$

generates a field TE to z.

It can be shown that in a source-free region an arbitrary electromagnetic field can be constructed in this way as a superposition of its TM^z and TE^z parts.

Radially Directed Potentials

An alternative, and somewhat simpler, representation of an arbitrary electromagnetic field is also possible in spherical coordinates.

This involves using **radially directed** vector potentials:

$$\mathbf{A} = \mathbf{r}_0 A_r \qquad\qquad \mathbf{F} = \mathbf{r}_0 F_r$$

However, assuming that $\mathbf{A}$ and $\mathbf{F}$ are Lorenz potentials (and hence are solutions of the vector Helmholtz equation), the radial components A_r, F_r **do not** satisfy the scalar Helmholtz equation, i.e., they are not scalar wavefunctions, because

$$\left(\nabla^2 \mathbf{A}\right)_r \neq \nabla^2 A_r$$

A different choice of the gauge for the potentials is then necessary....

Radially Directed Potentials

Recalling the representation of the fields in terms of the mixed potentials $\mathbf{A}$ and V:

$$\mathbf{H} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -j\omega\mu\mathbf{A} - \nabla V$$

from the second Maxwell equation we have, in a source-free region:

$$\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = -j\omega\varepsilon_{\mathrm{c}}\nabla V$$

Assuming $\mathbf{A} = \mathbf{r}_0 A_r$, the θ and ϕ components of the preceding equation are

$$\frac{\partial^2 A_r}{\partial r \partial \theta} = -j\omega\varepsilon_{\mathrm{c}}\frac{\partial V}{\partial \theta} \qquad \frac{\partial^2 A_r}{\partial r \partial \phi} = -j\omega\varepsilon_{\mathrm{c}}\frac{\partial V}{\partial \phi}$$

Radially Directed Potentials

These can be satisfied by letting:

$$\frac{\partial A_r}{\partial r} = -j\omega\varepsilon_c V$$

Note that the preceding relation fixes a gauge **different** from the Lorenz gauge $\nabla \cdot \mathbf{A} = -j\omega\varepsilon_c V$, since

$$\nabla \cdot \mathbf{A} = \nabla \cdot \left(\mathbf{r}_0 A_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) \neq \frac{\partial A_r}{\partial r}$$

With this gauge, the r component of $\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = -j\omega\varepsilon_c \nabla V$ is

$$\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} + k^2 A_r = 0$$

Debye Potentials

and it is readily verified that this is equivalent to writing

$$\left(\nabla^2 + k^2\right)\frac{A_r}{r} = 0$$

Dually:

$$\left(\nabla^2 + k^2\right)\frac{F_r}{r} = 0$$

Therefore, we may let $A_r / r = \psi^a$ and $F_r / r = \psi^f$, i.e.,

$$\mathbf{A} = \mathbf{r}_0 r\psi^a = \mathbf{r}\psi^a \qquad\qquad \mathbf{F} = \mathbf{r}_0 r\psi^f = \mathbf{r}\psi^f$$

where ψ^a, ψ^f are scalar wavefunctions known as electric and magnetic **Debye potentials**, respectively.

Field Representation through Debye Potentials

In a source-free region, from Maxwell's equations we find

$$\boxed{\begin{aligned}\mathbf{E} &= -\nabla\times\left(\mathbf{r}\psi^{f}\right)+\frac{1}{j\omega\varepsilon_{\mathrm{c}}}\nabla\times\nabla\times\left(\mathbf{r}\psi^{a}\right)\\ \mathbf{H} &= \nabla\times\left(\mathbf{r}\psi^{a}\right)+\frac{1}{j\omega\mu}\nabla\times\nabla\times\left(\mathbf{r}\psi^{f}\right)\end{aligned}}$$

These relations are sufficiently general to represent an arbitrary time-harmonic electromagnetic field in a source-free region of space contained in a spherical shell $r_1 < r < r_2$.

Field Representation through Debye Potentials

Since the functions ψ^a, ψ^f are always multiplied by r, it is convenient to introduce the **alternative** spherical Bessel functions defined (originally by Schelkunoff) as

$$\hat{B}_\nu(kr) = krb_\nu(kr) = \sqrt{\frac{\pi kr}{2}} B_{\nu+1/2}(kr)$$

which satisfy the differential equation $$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{\nu(\nu+1)}{r^2}\right]\hat{B}_\nu(kr) = 0$$

so that the potentials A_r, F_r can be represented as

$$\boxed{\sum_{\mu,\nu} C_{\mu\nu}\hat{B}_\nu(kr)L_\nu^\mu(\cos\theta)h(\mu\phi)}$$

Field Representation through Debye Potentials

Explicit expressions for the field components in terms of A_r, F_r are:

$$E_r = -\frac{1}{j\omega\varepsilon_c}\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) = \frac{1}{j\omega\varepsilon_c}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r$$

$$E_\theta = \frac{-1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{1}{j\omega\varepsilon_c r}\frac{\partial^2 A_r}{\partial r\partial\theta}$$

$$E_\phi = \frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{j\omega\varepsilon_c r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi}$$

$$H_r = -\frac{1}{j\omega\mu}\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_r}{\partial\theta}\right) = \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial r^2} + k^2\right)F_r$$

$$H_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{1}{j\omega\mu r}\frac{\partial^2 F_r}{\partial r\partial\theta}$$

$$H_\phi = -\frac{1}{r}\frac{\partial A_r}{\partial\theta} + \frac{1}{j\omega\mu r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\phi}$$

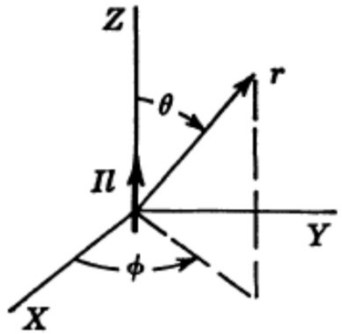
Sources of Spherical Waves

Elementary Dipole at the Origin

The sources of the lowest-order spherical waves are current elements.

For example, as is well known a *single* electric-current element (an **elementary Hertz dipole**) with current I and length l parallel to the z-axis radiates a TM^z field that can be derived from a z-directed magnetic vector potential with

$$A_z = Il\frac{e^{-jkr}}{4\pi r} = \frac{kIl}{4\pi j}h_0^{(2)}\left(kr\right)$$



$$\mathbf{H} = \nabla\times\mathbf{A} = \nabla A_z \times \mathbf{z}_0$$

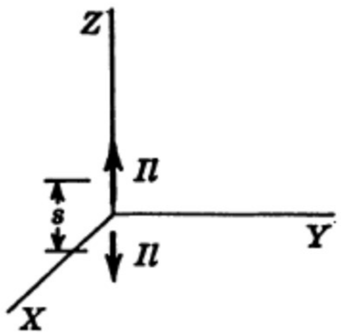
$$\mathbf{E} = -j\omega\mu\mathbf{A} + \frac{1}{j\omega\varepsilon_c}\nabla\nabla\cdot\mathbf{A} = -j\omega\mu A_z\mathbf{z}_0 + \frac{1}{j\omega\varepsilon_c}\nabla\frac{\partial A_z}{\partial z}$$

Pairs of Dipoles close to the Origin

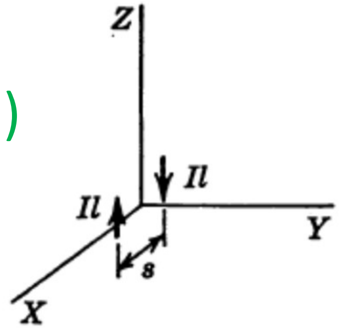
Spherical waves of order 1 are produced by *pairs* of closely spaced current elements placed in the vicinity of the origin.

E.g., for the configurations on the right:

$$A_z \xrightarrow[s\to 0]{} \frac{k^2 Ils}{4\pi j} h_1^{(2)}\left(kr\right) P_1\left(\cos\theta\right) \quad (n=1, m=0)$$



$$A_z \xrightarrow[s\to 0]{} \frac{k^2 Ils}{4\pi j} h_1^{(2)}\left(kr\right) P_1^1\left(\cos\theta\right)\cos\phi \quad (n=1, m=1)$$



Elementary Dipole at the Origin: Radial Potential

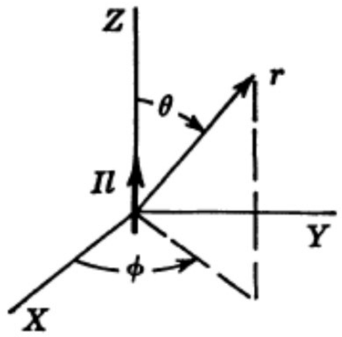
The only nonzero components of the field produced by an elementary dipole placed on and parallel to the z-axis are H_ϕ, E_r, E_θ, therefore such a field is TM^r.

As such, it can be derived from a radially directed magnetic vector potential. If the dipole is located at the origin, we need a radial wavefunction of order one:

$$A_r = \frac{jkIl}{4\pi}\hat{H}_1^{(2)}\left(kr\right)P_1\left(\cos\theta\right) = r\underbrace{\frac{jk^2Il}{4\pi}h_1^{(2)}\left(kr\right)P_1\left(\cos\theta\right)}_{=\psi^a} \qquad (n{=}1, m{=}0)$$

$$\mathbf{E} = \frac{1}{j\omega\varepsilon_c}\nabla\times\nabla\times\left(\mathbf{r}\psi^a\right)$$

$$\mathbf{H} = \nabla\times\left(\mathbf{r}\psi^a\right)$$



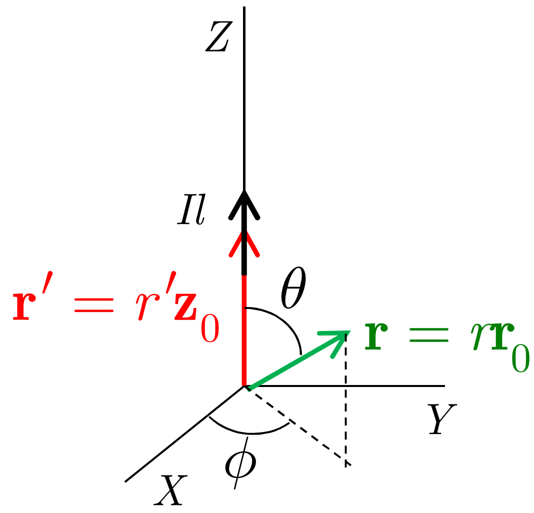
Displaced Dipole: Radial Potential

When the dipole is displaced off the origin, it is also possible to employ a radial wavefunction of *order zero*:

$$A_r = r\frac{Il}{r'}\frac{e^{-jkR}}{4\pi R} = r\underbrace{\frac{kIl}{j4\pi r'}h_0^{(2)}\left(kR\right)}_{=\psi^a} \qquad (n{=}0, m{=}0)$$

where

$$R = \left|\mathbf{r} - \mathbf{r}'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$



Exercise: show that, in the limit $r' \to 0$, this produces the correct field of a current element placed at the origin.

Some Useful Wave Transformations

Plane Waves in terms of Spherical Waves

Let us consider a scalar, harmonic traveling wave (i.e., a *scalar plane wave*)

$$e^{jkz} = e^{jkr\cos\theta}$$

This wave is finite at the origin $r=0$, regular everywhere including the z-axis ($\theta=0,\pi$), and independent of ϕ.

Therefore, it admits an expansion in spherical waves of the form

$$e^{jkr\cos\theta} = \sum_{n=0}^{+\infty} a_n j_n\left(kr\right) P_n\left(\cos\theta\right)$$

Plane Waves in terms of Spherical Waves

To evaluate the a_n, we multiply each side by $P_q(\cos\theta)\sin\theta$ and integrate from 0 to π on θ. Because of orthogonality, all terms except $q=n$ vanish and

$$\int_0^\pi e^{jkr\cos\theta} P_n\left(\cos\theta\right)\sin\theta \mathrm{d}\theta = \frac{2a_n}{2n+1} j_n\left(kr\right)$$

The n-th derivative of the left-hand side with respect to r evaluated at $r=0$ is

$$j^n k^n \int_0^\pi \cos^n \theta P_n\left(\cos\theta\right)\sin\theta \mathrm{d}\theta = \frac{j^n k^n 2^{n+1}\left(n!\right)^2}{\left(2n+1\right)!}$$

The n-th derivative of the right-hand side with respect to r evaluated at $r=0$ is

$$\frac{2^{n+1} k^n \left(n!\right)^2}{\left(2n+1\right)\left(2n+1\right)!} a_n$$

Exercise: prove these expressions

Plane Waves in terms of Spherical Waves

Equating the two preceding expressions we get

$$a_n = j^n \left(2n+1\right)$$

hence

$$\boxed{e^{jkr\cos\theta} = \sum_{n=0}^{+\infty} j^n \left(2n+1\right) j_n\left(kr\right) P_n\left(\cos\theta\right)}$$

(expansion of a plane wave in spherical waves)

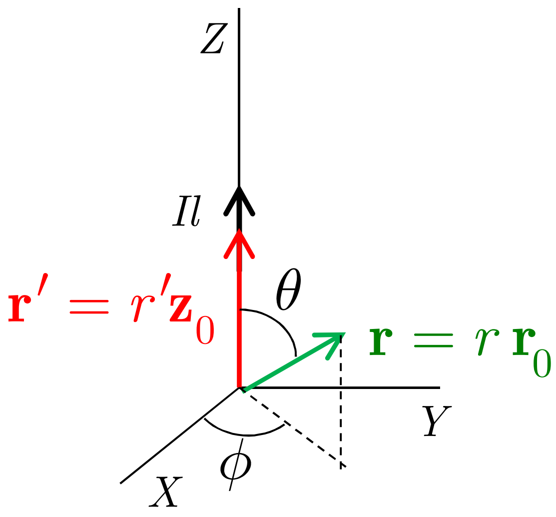
Note that we have also established the identity

$$j_n\left(kr\right) = \frac{j^{-n}}{2} \int_0^{\pi} e^{jkr\cos\theta} P_n\left(\cos\theta\right) \sin\theta \,\mathrm{d}\theta$$

Change from One Spherical Coordinate System to Another

Let us now consider a point source at $\mathbf{r}' = r'\mathbf{z}_0$, whose potentials can be expressed in terms of the wavefunction of order zero:

$$h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right) = -\frac{e^{-jk\left|\mathbf{r}-\mathbf{r}'\right|}}{jk\left|\mathbf{r}-\mathbf{r}'\right|}$$



$$A_z = \frac{kIl}{4\pi j} h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right)$$

$$A_r = r\frac{kIl}{j4\pi r'} h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right)$$

We wish to express this wavefunction in terms of spherical wave functions referred to $r=0$.

Change from One Spherical Coordinate System to Another

The wavefunction is **rotationally symmetric** w.r.t. the z-axis. Furthermore, it is **regular** at $r=0$ and outgoing in $r > r'$. Finally, it is **symmetric** in $\mathbf{r}, \mathbf{r}'$.

$$\Rightarrow \quad h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right) = \begin{cases} \sum_{n=0}^{+\infty} c_n h_n^{(2)}\left(kr'\right) j_n\left(kr\right) P_n\left(\cos\theta\right), & r < r' \\ \sum_{n=0}^{+\infty} c_n j_n\left(kr'\right) h_n^{(2)}\left(kr\right) P_n\left(\cos\theta\right), & r > r' \end{cases}$$

If we let the source recede to infinity, the field in the vicinity of the origin is a plane wave:

$$h_n^{(2)}\left(z\right) \sim \frac{j^{n+1}}{z} e^{-jz} \quad \Rightarrow \quad h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right) \xrightarrow[r' \to +\infty, \theta'=0]{} \frac{je^{-jr'}}{kr'} e^{jkr\cos\theta}$$

Change from One Spherical Coordinate System to Another

On the right-hand side we have

$$\sum_{n=0}^{+\infty} c_n h_n^{(2)}\left(kr'\right) j_n\left(kr\right) P_n\left(\cos\theta\right) \xrightarrow[\theta'=0]{r'\to+\infty} \frac{je^{-jkr'}}{kr'} \sum_{n=0}^{+\infty} c_n j^n j_n\left(kr\right) P_n\left(\cos\theta\right)$$

Equating the two sides we have

$$\frac{je^{-jr'}}{kr'} e^{jkr\cos\theta} = \frac{je^{-jkr'}}{kr'} \sum_{n=0}^{+\infty} j^n c_n j_n\left(kr\right) P_n\left(\cos\theta\right)$$

From the expansion of a plane wave in spherical waves established before:

$$e^{jkr\cos\theta} = \sum_{n=0}^{+\infty} j^n \left(2n+1\right) j_n\left(kr\right) P_n\left(\cos\theta\right)$$

Change from One Spherical Coordinate System to Another

…we find

$$c_n = 2n+1$$

hence

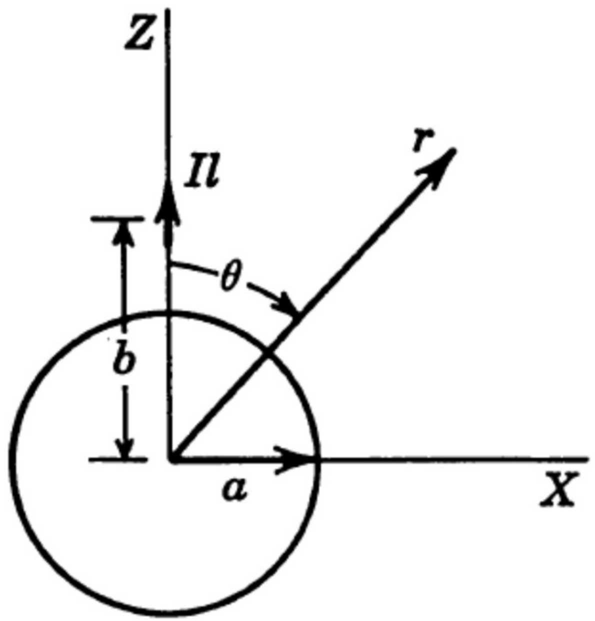
$$\boxed{h_0^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}'\right|\right) = \begin{cases} \sum_{n=0}^{+\infty}\left(2n+1\right)h_n^{(2)}\left(kr'\right)j_n\left(kr\right)P_n\left(\cos\theta\right), & r<r' \\ \sum_{n=0}^{+\infty}\left(2n+1\right)j_n\left(kr'\right)h_n^{(2)}\left(kr\right)P_n\left(\cos\theta\right), & r>r' \end{cases}}$$

(addition theorem for spherical Hankel functions)

Dipole and Conducting Sphere

The Configuration

Let us now consider a radially directed electric dipole near a conducting sphere:



By symmetry, the field has nonzero components E_r, E_θ, H_ϕ

The Incident Potential

The incident field can be represented in terms of a single radially directed (Debye) potential

$$A_r^{\mathrm{i}} = r\,\frac{Il}{b}\,\frac{e^{-jkR}}{4\pi R} = r\,\underbrace{\frac{kIl}{j4\pi b}\,h_0^{(2)}\left(kR\right)}_{=\psi^a}$$

which can be expanded in spherical wavefunctions referred to the origin:

$$A_r^{\mathrm{i}} = r\,\frac{kIl}{j4\pi b}\sum_{n=0}^{+\infty}\left(2n+1\right)h_n^{(2)}\left(kr_>\right)j_n\left(kr_<\right)P_n\left(\cos\theta\right)$$

with
$$r_> = \max\left\{r, b\right\}$$
$$r_< = \min\left\{r, b\right\}$$

The Incident Field

The incident field E_θ^{i} on the surface of the sphere $r=a$ is then

$$E_\theta^{\mathrm{i}} = \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{\partial^2 A_r^{\mathrm{i}}}{\partial r \partial\theta} = \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{\partial^2}{\partial r \partial\theta}\left[r\frac{kIl}{j4\pi b}\sum_{n=0}^{+\infty}(2n+1)h_n^{(2)}(kb)\,j_n(kr)\,P_n(\cos\theta)\right]_{r=a}\Bigg|$$

$$= \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{Il}{j4\pi b}\frac{\partial^2}{\partial r \partial\theta}\left[\sum_{n=0}^{+\infty}(2n+1)h_n^{(2)}(kb)\,\hat{J}_n(kr)\,P_n(\cos\theta)\right]_{r=a}\Bigg|$$

$$= \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{kIl}{j4\pi b}\sum_{n=0}^{+\infty}(2n+1)h_n^{(2)}(kb)\frac{\partial}{\partial(kr)}\hat{J}_n(kr)\frac{\partial}{\partial\theta}P_n(\cos\theta)\Bigg|_{r=a}$$

$$= \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{kIl}{j4\pi b}\sum_{n=1}^{+\infty}(2n+1)h_n^{(2)}(kb)\,\hat{J}_n'(ka)\,P_n^1(\cos\theta)$$

$$\hat{J}_n'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\hat{J}_n(z)$$

The term $n=0$ drops out because $P_0(\cos\theta)$ is a constant

The Scattered Potential and Field

The scattered field E_θ^{s} can be derived from a radial potential having the same form as the incident potential, but with an outgoing traveling-wave radial dependence:

$$A_r^{\mathrm{s}} = r\frac{kIl}{j4\pi b}\sum_{n=0}^{+\infty} c_n \left(2n+1\right) h_n^{(2)}\left(kb\right) h_n^{(2)}\left(kr\right) P_n\left(\cos\theta\right)$$

from which we derive the scattered field on the surface of the sphere:

$$E_\theta^{\mathrm{s}} = \frac{1}{j\omega\varepsilon_{\mathrm{c}} a}\frac{kIl}{j4\pi b}\sum_{n=1}^{+\infty} c_n \left(2n+1\right) h_n^{(2)}\left(kb\right) \hat{H}_n^{(2)\prime}\left(ka\right) P_n^1\left(\cos\theta\right)$$

Enforcing the Boundary Conditions

The coefficients c_n can be found by enforcing the boundary conditions on the surface of the sphere, assumed for simplicity to be a PEC:

$$\boxed{E_\theta = E_\theta^{\mathrm{i}} + E_\theta^{\mathrm{s}} = 0, \quad r = a,\ \theta \in \left[0, \pi\right]}$$

$$E_\theta^{\mathrm{i}} + E_\theta^{\mathrm{s}} = \frac{1}{j\omega\varepsilon_{\mathrm{c}} a} \frac{kIl}{j4\pi b} \sum_{n=1}^{+\infty} \left(2n+1\right) h_n^{(2)}\left(kb\right) \left[\hat{J}_n'\left(ka\right) + c_n \hat{H}_n^{(2)\prime}\left(ka\right)\right] P_n^1\left(\cos\theta\right) = 0$$

and from the orthogonality of the Legendre functions we deduce

$$\hat{J}_n'\left(ka\right) + c_n \hat{H}_n^{(2)\prime}\left(ka\right) = 0 \quad \Rightarrow \quad \boxed{c_n = -\frac{\hat{J}_n'\left(ka\right)}{\hat{H}_n^{(2)\prime}\left(ka\right)}}$$

The Total Potential

The total potential on the surface of the sphere is then:

$$A_r(a,\theta) = \frac{Il}{j4\pi b}\sum_{n=0}^{+\infty}(2n+1)\,h_n^{(2)}(kb)\left[\hat{J}_n(ka) - \frac{\hat{J}_n'(ka)}{\hat{H}_n^{(2)\prime}(ka)}\hat{H}_n^{(2)}(ka)\right]P_n(\cos\theta)$$

$$= \frac{\hat{J}_n(ka)\,\hat{H}_n^{(2)\prime}(ka) - \hat{J}_n'(ka)\,\hat{H}_n^{(2)}(ka)}{\hat{H}_n^{(2)\prime}(ka)} \longleftarrow \text{Wronskian}$$

$$= \frac{-j}{\hat{H}_n^{(2)\prime}(ka)}$$

$$\Rightarrow \boxed{A_r(a,\theta) = -\frac{Il}{4\pi kb^2}\sum_{n=0}^{+\infty}(2n+1)\frac{\hat{H}_n^{(2)}(kb)}{\hat{H}_n^{(2)\prime}(ka)}P_n(\cos\theta)}$$

The Total FIeld

The total field on the surface of the sphere is finally:

$$E_r(a,\theta) = -\frac{1}{j\omega\varepsilon_c}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}A_r(a,\theta)\right)$$

$$E_\theta(a,\theta) = 0$$

$$H_\phi(a,\theta) = -\frac{1}{r}\frac{\partial}{\partial\theta}A_r(a,\theta)$$

Note:

Term-by-term $\theta-$differentiation of the series that gives the total potential A_r is permissible, since the resulting series is uniformly convergent w.r.t. θ for $b>a$

The Watson Transformation; Creeping Waves

Convergence of the Partial-Wave Series

The solution obtained thus far, although **mathematically correct**, has some important **drawbacks**.

In fact, it is expressed as an infinite summation (whose addends are called partial waves) whose **convergence is very slow** if the sphere is large in terms of the free-space wavelength.

Typically, a number of terms $\boxed{N \approx ka = 2\pi \frac{a}{\lambda}}$ has to be summed.

Depending on the problem, this number can be substantially high (thousands or more).

The Need for a Different Solution

Nowadays, with the availability of digital computers, this is not a problem (but it was a century ago!).

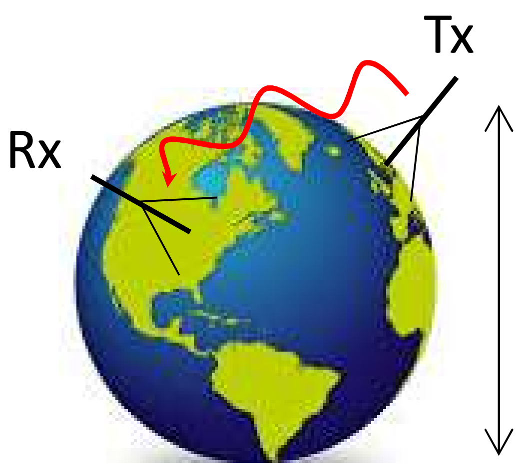
However, **no physical insight** is gained from such a form of the solution…

> *It is very nice that the computer understands the problem. But I would like to understand it, too.*

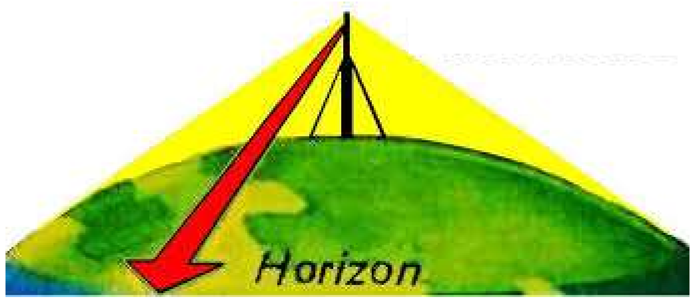
(attributed to E. Wigner)

Radio-Wave Propagation around the Earth

After the first successful experiments of transoceanic radio transmission conducted by Marconi in 1901, a number of studies appeared in the first decades of the XX century aimed at finding a satisfactory mathematical model for **radio-wave propagation beyond the horizon**.



$$2a \simeq 12700 \ \text{km}$$



Marconi 1901 experiment:

$$\lambda \simeq 350 \ \text{m} \ \left(f \simeq 850 \ \ \text{kHz}\right)$$

$$ka = \frac{2\pi}{\lambda} a \simeq 1.13 \cdot 10^5$$

The Watson Transformation

The main idea of the so-called Watson (or Sommerfeld-Watson) transformation, first proposed in 1910 by Poincaré and Nicholson, consists in **replacing the partial-wave series by an integral** and then to obtain an approximate value for the integral by means of the calculus of residues.

The Diffraction of Electric Waves by the Earth.

By G. N. WATSON, Sc.D., D.Sc., Assistant Professor of Pure Mathematics at University College, London.

(Communicated by Prof. J. W. Nicholson, F.R.S. Received May 29, 1918.)

'Roy. Soc. Proc.,' A, vol. 95, pp. 83–99 (1918).

In 1918 Watson first exposed the method in a systematic and rigorous form, without introducing unnecessary approximations.

The Key Idea

Consider the partial-wave series for the radial potential

$$A_r\left(a,\theta\right) = -\frac{Il}{4\pi k b^2}\sum_{n=0}^{+\infty}\left(2n+1\right)\frac{\hat{H}_n^{(2)}\left(kb\right)}{\hat{H}_n^{(2)\prime}\left(ka\right)}P_n\left(\cos\theta\right)$$

The **key idea** is to consider each term of the series as **the residue of a suitable function**:

$$\left(2n+1\right)\frac{\hat{H}_n^{(2)}\left(kb\right)}{\hat{H}_n^{(2)\prime}\left(ka\right)}P_n\left(\cos\theta\right) = \mathrm{Res}\left[f\left(s\right); s = n+\frac{1}{2}\right]$$

Watson used:

$$f\left(s\right) = -2\pi s\frac{P_{s-1/2}\left(-\cos\theta\right)\hat{H}_{s-1/2}^{(2)}\left(kb\right)}{\cos\left(s\pi\right)\hat{H}_{s-1/2}^{(2)\prime}\left(ka\right)}$$

(consider that $P_n\left(-\cos\theta\right) = \left(-1\right)^n P_n\left(\cos\theta\right)$ when n is an integer)

Poles of the Integrand

The poles of
$$f\left(s\right) = -2\pi s \frac{P_{s-1/2}\left(-\cos\theta\right)\hat{H}^{(2)}_{s-1/2}\left(kb\right)}{\cos\left(s\pi\right)\hat{H}^{(2)\prime}_{s-1/2}\left(ka\right)}$$

are the zeros of
$$\cos\left(s\pi\right), \quad \hat{H}^{(2)\prime}_{s-1/2}\left(ka\right)$$

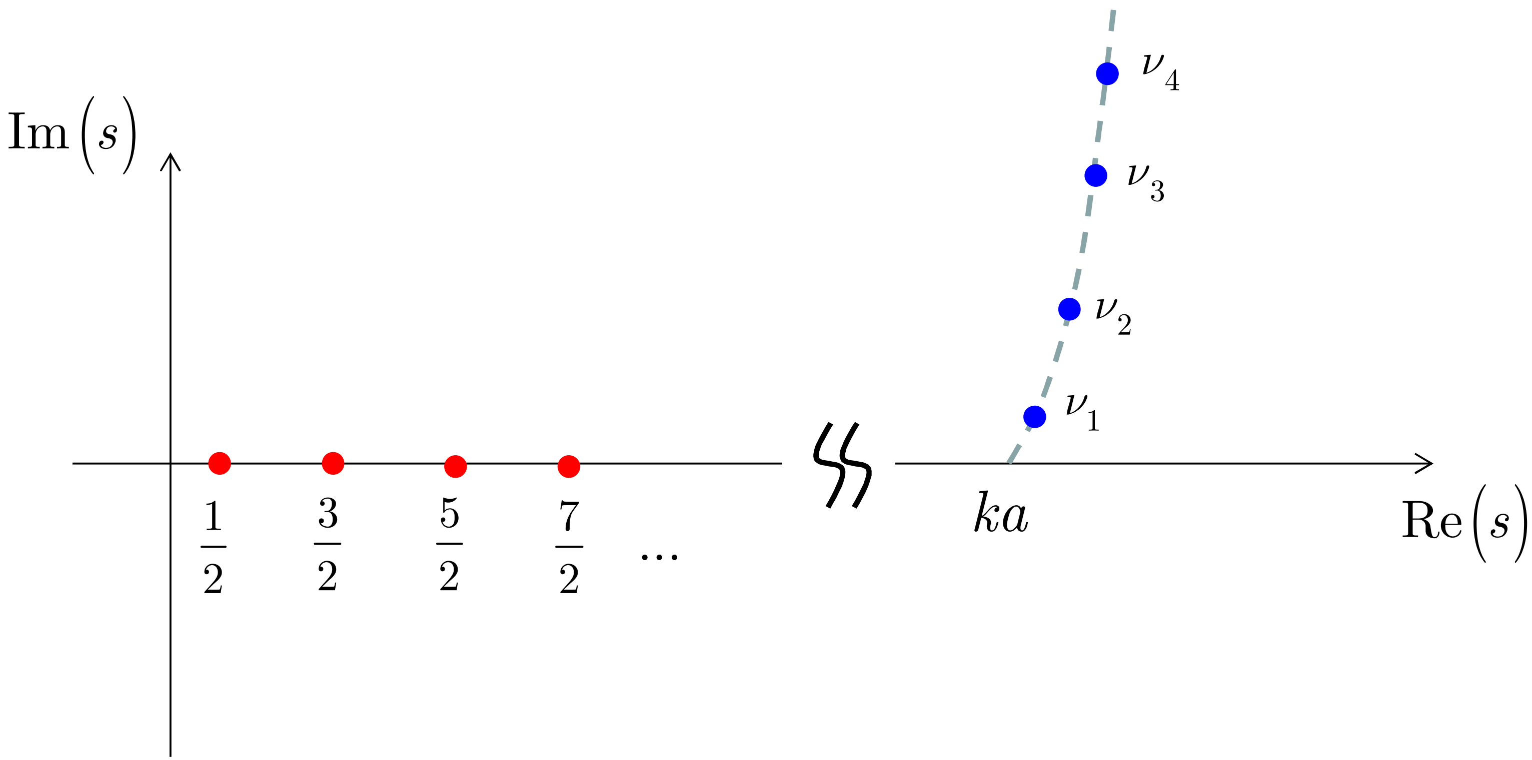
$$\left\{s = n + \frac{1}{2}\right\}_{n=0,1,2,\ldots} \qquad \left\{s = \nu_n\right\}_{n=1,2,\ldots}$$

There are no other poles, because

$$P_{s-1/2}\left(-\cos\theta\right), \quad \hat{H}^{(2)}_{s-1/2}\left(kb\right)$$

are entire functions of s.

Poles of the Integrand



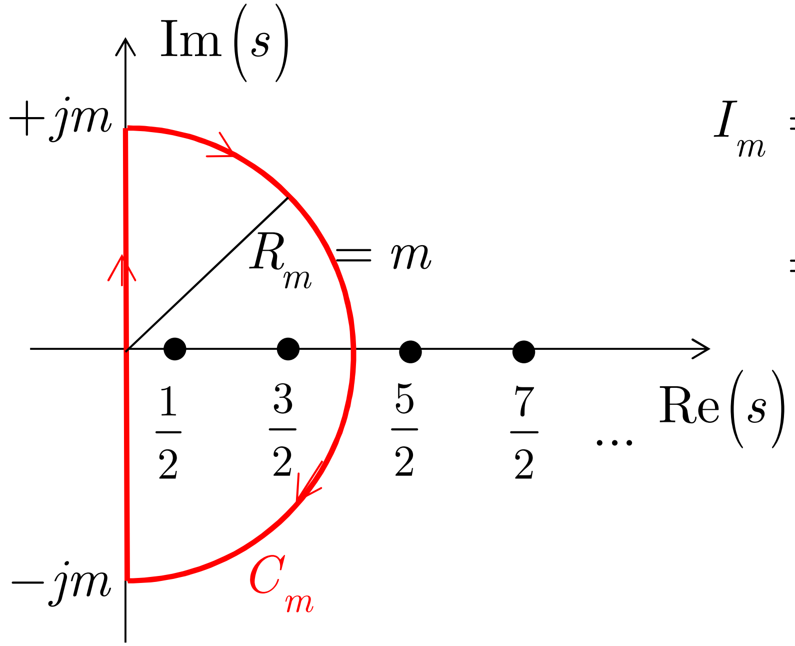
It can be shown that $\mathrm{Re}\{\nu_n\} \neq 0, \mathrm{Im}\{\nu_n\} > 0$.

Furthermore, when ka is large, the poles ν_n are close to the broken line that originates at $s = ka$ having equation

$$\mathrm{Re}\left\{\sqrt{s^2 - (ka)^2} - s\cosh^{-1}\left(\frac{s}{ka}\right)\right\} = 0$$

Contour Integrals and Residue Theorem

Consider now the sequence of integrals I_m , $m=1,2,\ldots$, along the closed contours C_m:



$$
\begin{aligned}
I_m &= \int_{C_m} f(s)\,\mathrm{d}s \\
&= -2\pi j \sum_{\substack{\text{poles} \\ \text{within } C_m}} \mathrm{Res}\left[f(s);\mathrm{pole}\right]
\end{aligned}
$$

- The integral along the imaginary axis **is zero** because the integrand is an odd function of s.
- Furthermore, it can be shown that the integral along the half circle **tends to zero** as m tends to infinity.

Letting the Radius Tend to Infinity

This means that, by letting the radius R_m tend to infinity,

$$\lim_{m\to+\infty} I_m = 0 = -2\pi j \sum_{\substack{\text{poles}\\ \text{with } \operatorname{Re}(s)>0}} \operatorname{Res}\left[f(s);\text{pole}\right]$$

hence

$$\sum_{\substack{\text{poles}\\ \text{at } s=n+1/2}} \operatorname{Res}\left[f(s);\text{pole}\right] + \sum_{\substack{\text{poles}\\ \text{at } s=\nu_n}} \operatorname{Res}\left[f(s);\text{pole}\right] = 0$$

or equivalently

$$\sum_{\substack{\text{poles}\\ \text{at } s=n+1/2}} \operatorname{Res}\left[f(s);\text{pole}\right] = -\sum_{\substack{\text{poles}\\ \text{at } s=\nu_n}} \operatorname{Res}\left[f(s);\text{pole}\right]$$

Letting the Radius Tend to Infinity

Now, on the left-hand side we have the original series

$$\sum_{\substack{\text{poles}\\ \text{at } s=n+1/2}} \mathrm{Res}\left[f\left(s\right);\mathrm{pole}\right] = \sum_{n=0}^{+\infty}\left(2n+1\right)\frac{\hat{H}_n^{(2)}\left(kb\right)}{\hat{H}_n^{(2)}\left(ka\right)}P_n\left(\cos\theta\right)$$

whereas on the right-hand side we have the **transformed series**

$$-\sum_{\substack{\text{poles}\\ \text{at } s=\nu_n}} \mathrm{Res}\left[f\left(s\right);\mathrm{pole}\right] = 2\pi\sum_{n=1}^{+\infty}\nu_n \frac{P_{\nu_n-1/2}\left(-\cos\theta\right)\hat{H}_{\nu_n-1/2}^{(2)}\left(kb\right)}{\cos\left(\nu_n\pi\right)\dfrac{\partial}{\partial s}\left[\hat{H}_{s-1/2}^{(2)\prime}\left(ka\right)\right]_{s=\nu_n}}$$

where we have used the fact that, if $\mathrm{N}(s)/\mathrm{D}(s)$ has a simple pole at s_0, then the relevant residue can be computed as $\mathrm{N}(s_0)/\mathrm{D'}(s_0)$ (exercise: prove this).

Alternative Representation for the Potential

Finally, for the radial potential we have

$$\boxed{A_r\left(a,\theta\right) = -\frac{Il}{2kb^2}\sum_{n=1}^{+\infty}\nu_n \frac{P_{\nu_n-1/2}\left(-\cos\theta\right)\hat{H}^{(2)}_{\nu_n-1/2}\left(kb\right)}{\cos\left(\nu_n\pi\right)\dfrac{\partial}{\partial s}\left[\hat{H}^{(2)\prime}_{s-1/2}\left(ka\right)\right]_{s=\nu_n}}}$$

This alternative series seems more complicated than the original one, in that one must find *numerically* the pole locations ν_n and the relevant residues.

However, for large ka this series **converges extremely fast**, so that only a few terms (possibly, only one) are sufficient to accurately represent the potential.

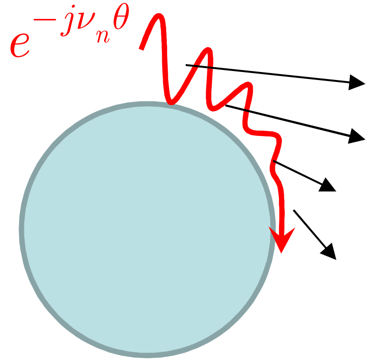
Angular Dependence: Creeping Waves

Consider the angular dependence of each term in the alternative series:

$$P_{\nu_n - 1/2}\left(-\cos\theta\right) \underset{\substack{|\nu_n| \gg 1 \\ \varepsilon \le \theta \le \pi - \varepsilon \\ |\nu_n|\varepsilon \gg 1}}{\sim} \frac{\cot\theta}{8\nu_n}\left(\frac{2}{\pi\nu_n \sin\theta}\right)^{1/2} e^{-j\left(\nu_n\theta + 3\pi/4\right)}$$

This is an **azimuthally propagating wave** known as a **creeping wave** (a terminology introduced by W. Franz).

Since $\mathrm{Im}(\nu_n) > 0$, the wave is **attenuated** (it radiates while propagating, due to the curvature of the sphere)



Final Remarks

1) In *quantum mechanics*, the integer index n in the angular terms of the partial-wave expansion

$$P_n\left(\cos\theta\right)$$

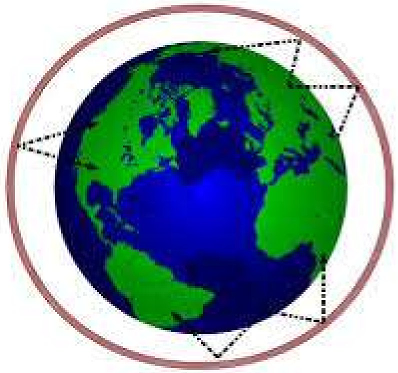
is associated with the (orbital) **angular momentum**.

Allowing n to assume complex values, as is done in the Watson's transformation, is therefore referred to as working in the **complex angular-momentum** domain.

The study of the analytic properties of quantum scattering as a function of complex angular momentum is known as **Regge theory** (started by the physicist Tullio Regge in 1959).
Therefore, the poles ν_n in the alternative representation of the potential can also be called **Regge poles**.

Final Remarks

2) As concerns the problem of radio-wave propagation beyond the horizon, it was immediately realized by Watson that the rate of azimuthal attenuation of the dominant creeping wave was **much too large** if compared with the one observed experimentally.



The attenuation is smaller because the dominant mechanism is that of ionospheric reflection, i.e., the conductive ionosphere and the conductive ground form what is known as **Earth-ionosphere waveguide**…

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G. N. Watson, "The diffraction of electromagnetic waves by the Earth," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 95, no. 666, pp. 83-99, 1918.