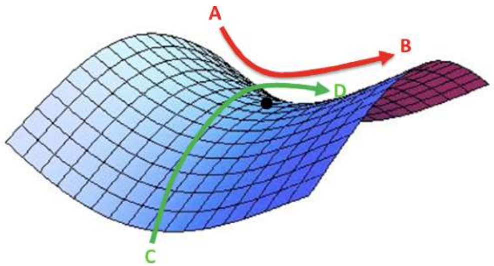
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 10

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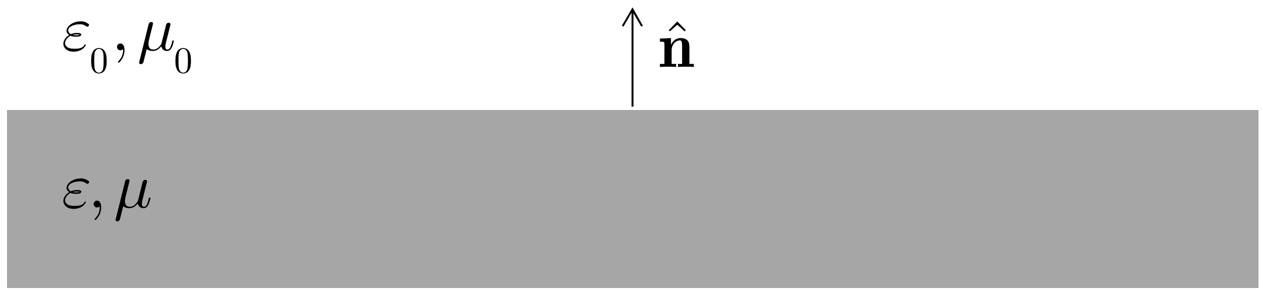
First-Order Impedance/Transition Conditions

First-Order Impedance Conditions

As is well known, the interface between air and a highly conducting medium can be modeled by introducing the *Leontovich boundary condition*, also known as **Standard Impedance Boundary Condition** (SIBC):

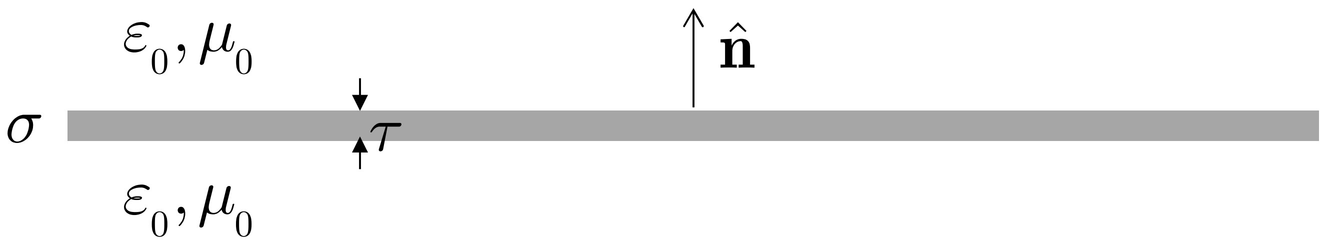
$$\boxed{\hat{\mathbf{n}} \times \mathbf{E} = \eta\, \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H} \right)} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$



The condition can be derived under the assumption $\left| N \right| = \left| \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} \right| \gg 1$

First-Order Transition Conditions: Resistive Sheet

Consider now a thin layer of a highly conducting, nonmagnetic dielectric:



The bulk current density $\mathbf{J}$ can be replaced by an **equivalent surface current** $\mathbf{J}_\mathrm{s}$:

$$\left.\begin{array}{c} \mathbf{J} = \sigma\,\mathbf{E}'_\mathrm{tan} \\ \tau \ll \lambda_0 \end{array}\right\} \rightarrow \mathbf{J}_\mathrm{s} = \tau\,\mathbf{J}$$

$$\Rightarrow \qquad \mathbf{E}'_\mathrm{tan} = \frac{1}{\sigma\tau}\mathbf{J}_\mathrm{s}$$

First-Order Transition Conditions: Resistive Sheet

and since the tangential electric field is continuous $\mathbf{E}_{\tan} = \dfrac{1}{\sigma\tau}\mathbf{J}_{\mathrm{s}}$, i.e.,

$$\boxed{\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}\right) = -R_{\mathrm{e}}\,\mathbf{J}_{\mathrm{s}}}$$

where

$$R_{\mathrm{e}} = \frac{1}{\sigma\tau} \quad \left(\mathrm{ohm}\right)$$

is the (electrical) **resistivity** of the sheet.

First-Order Transition Conditions: Resistive Sheet

Consider now a thin layer of lossy, nonmagnetic material with complex permittivity $\varepsilon_c = \varepsilon - j\sigma/\omega$ immersed in free space. The *volume equivalence principle* allows us to replace the layer by the **equivalent polarisation current**

$$\mathbf{J}_e = j\frac{k_0}{\eta_0}\left(\frac{\varepsilon_c}{\varepsilon_0} - 1\right)\mathbf{E}'$$

Assuming $k_0\tau \ll 1$, the component of $\mathbf{J}_e$ normal to the layer can be neglected and the tangential component replaced by the surface current $\mathbf{J}_s = \tau\mathbf{J}_e$, hence

$$\boxed{\mathbf{E}_{tan} = R_e\mathbf{J}_s} \qquad R_e = -\frac{j\eta_0}{k_0\tau\left(\frac{\varepsilon}{\varepsilon_0} - 1\right)} \quad \left(\xrightarrow[\sigma \gg \omega\varepsilon]{} \frac{1}{\sigma\tau}\right)$$

First-Order Transition Conditions: Conductive Sheet

The **dual** of an (electrically) resistive sheet is a (magnetically) **conductive** one simulating a lossy material with $\varepsilon_c = \varepsilon_0$. The corresponding conditions are

$$\boxed{\hat{\mathbf{n}} \times \mathbf{H} = R_m \hat{\mathbf{n}} \times \mathbf{J}_{ms}}$$

with

$$\mathbf{J}_{ms} = -\left[\hat{\mathbf{n}} \times \mathbf{E}\right]_-^+ \qquad \left[\hat{\mathbf{n}} \times \mathbf{H}\right]_-^+ = 0$$

and

$$R_m = -\frac{j}{\eta_0 k_0 \tau \left(\dfrac{\mu}{\mu_0} - 1\right)} \qquad (\text{siemens})$$

Combination Sheets

The conditions for resistive and conductive sheets can be written as:

$$\hat{\mathbf{n}} \times \left(\mathbf{E}^{+} + \mathbf{E}^{-}\right) = 2R_{\mathrm{e}}\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \mathbf{H}\right]_{-}^{+}$$

$$\left[\hat{\mathbf{n}} \times \mathbf{E}\right]_{-}^{+} = 0$$

$$\hat{\mathbf{n}} \times \left(\mathbf{H}^{+} + \mathbf{H}^{-}\right) = -2R_{\mathrm{m}}\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \mathbf{E}\right]_{-}^{+}$$

$$\left[\hat{\mathbf{n}} \times \mathbf{H}\right]_{-}^{+} = 0$$

By addition and subtraction we get the transition condition for a **combination sheet**:

$$\hat{\mathbf{n}} \times \mathbf{E}^{\pm} = \left(R_{\mathrm{e}} \pm \frac{1}{4R_{\mathrm{m}}}\right)\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^{+}\right) - \left(R_{\mathrm{e}} \mp \frac{1}{4R_{\mathrm{m}}}\right)\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^{-}\right)$$

Impedance Condition via Transition Conditions

A combination sheet is generally **partially transparent**. However, if

$$R_{\mathrm{e}} = \frac{1}{4R_{\mathrm{m}}}$$

then the combination sheet becomes **opaque** and **its transition conditions reduce to the SIBC on the two sides of the sheet**:

$$\hat{\mathbf{n}} \times \mathbf{E}^{\pm} = \pm\eta\, \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^{\pm}\right)$$

where

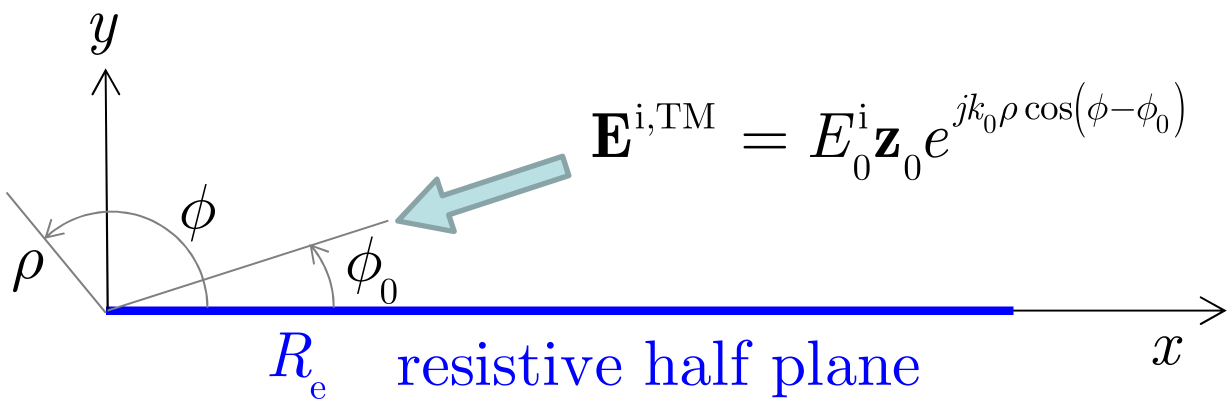
$$\eta = 2R_{\mathrm{e}} = \frac{1}{2R_{\mathrm{m}}}$$

is the surface impedance on both sides of the sheet.

The Sommerfeld Half-Plane Problem: Resistive Sheet

Resistive Half Plane

Let us then consider a TM-polarized plane wave impinging on a resistive half plane:



The boundary conditions satisfied by the half plane are

$$
\begin{aligned}
E_z &= -R_{\mathrm{e}} \left[H_x \right]_-^+ \\
\left[E_z \right]_-^+ &= 0
\end{aligned}
\qquad y = 0, x > 0
$$

Dual Integral Equations

As in the PEC case, we introduce the **angular-spectrum representation** of the fields:

$$E_z^{\mathrm{s}}\left(\rho,\phi\right) = E_0^{\mathrm{i}} \int_C P_{\mathrm{e}}\left(\cos\alpha\right) e^{-jk_0\rho\cos\left(\phi\mp\alpha\right)} \mathrm{d}\alpha, \quad y \gtrless 0$$

$$H_x^{\mathrm{s}}\left(\rho,\phi\right) = \pm\frac{E_0^{\mathrm{i}}}{\eta_0} \int_C \sin\alpha P_{\mathrm{e}}\left(\cos\alpha\right) e^{-jk_0\rho\cos\left(\phi\mp\alpha\right)} \mathrm{d}\alpha, \quad y \gtrless 0$$

or, equivalently,

$$E_z^{\mathrm{s}}\left(x,y\right) = E_0^{\mathrm{i}} \int_{-\infty}^{+\infty} \frac{P_{\mathrm{e}}\left(\lambda\right)}{\sqrt{1-\lambda^2}} e^{-jk_0x\lambda} e^{-jk_0\left|y\right|\sqrt{1-\lambda^2}} \mathrm{d}\lambda$$

$$H_x^{\mathrm{s}}\left(x,y\right) = \pm\frac{E_0^{\mathrm{i}}}{\eta_0} \int_{-\infty}^{+\infty} P_{\mathrm{e}}\left(\lambda\right) e^{-jk_0x\lambda} e^{-jk_0\left|y\right|\sqrt{1-\lambda^2}} \mathrm{d}\lambda$$

Dual Integral Equations

By enforcing the **boundary condition** on the resistive sheet we find:

$$\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{1-\lambda^2}}+\frac{2R_{\mathrm{e}}}{\eta_0}\right)P_{\mathrm{e}}(\lambda)e^{-jk_0x\lambda}\mathrm{d}\lambda=-e^{jk_0x\lambda_0},\qquad x>0$$

where $\lambda_0=\cos\phi_0$.

On the other hand by enforcing that the scattered magnetic field is zero (or, equivalently, that the current density is zero) on $y=0,\ x<0$:

$$\int_{-\infty}^{+\infty}P_{\mathrm{e}}(\lambda)e^{-jk_0x\lambda}\mathrm{d}\lambda=0,\qquad x<0$$

This is the sought pair of **dual integral equations**

Upper and Lower Functions

As in the PEC case, the second integral equation is satisfied by letting the unknown angular spectrum be an "upper function":

$$P_{\mathrm{e}}(\lambda) = U(\lambda)$$

whereas the first equation is satisfied provided that:

$$\left(\frac{1}{\sqrt{1-\lambda^2}} + \frac{2R_{\mathrm{e}}}{\eta_0}\right) U(\lambda) = \frac{1}{2\pi j} \frac{L_1(\lambda)}{L_1(-\lambda_0)} \frac{1}{\lambda + \lambda_0} + L_2(\lambda)$$

(functional equation)

where $L_{1,2}(\lambda)$ are unknown "lower" functions.

Splitting Procedure

According to the Wiener-Hopf procedure, it is now necessary to factorise

$$\left(\frac{1}{\sqrt{1-\lambda^2}}+\frac{2R_{\mathrm{e}}}{\eta_0}\right)^{-1}=K_+\left(\overline{\eta},\lambda\right)K_-\left(\overline{\eta},\lambda\right)$$

where $K_\pm\left(\overline{\eta},\lambda\right)$ are **upper/lower split functions** with $K_+\left(\overline{\eta},-\lambda\right)=K_-\left(\overline{\eta},\lambda\right)$ and

$$\overline{\eta}=\frac{2R_{\mathrm{e}}}{\eta_0}$$

Generally, the factorization of a function into a product (or sum) of upper (+) and lower (-) split functions is a **difficult task** if analytical results are desired; however, direct integral expressions can be employed that can be evaluated numerically (this is referred to as a *numerical splitting*).

Splitting for the Resistive Half Plane

For the present case of a resistive half plane, **explicit expressions** for the split functions $K_{\pm}(\lambda)$ where first obtained by Senior in 1952, who later rewrote them in terms of the more convenient **Maljuzhinets function**:

$$K_{+}\left(\bar{\eta}, \cos\alpha\right) = \frac{4}{\sqrt{\bar{\eta}}} \sin\frac{\alpha}{2} \left\{ \frac{\psi_{\pi}\left(\frac{3\pi}{2} - \alpha - \theta\right) \psi_{\pi}\left(\frac{\pi}{2} - \alpha + \theta\right)}{\left(\psi_{\pi}\left(\frac{\pi}{2}\right)\right)^{2}} \right\}^{2}$$

$$\cdot \left\{ \left[1 + \sqrt{2} \cos\left(\frac{\frac{\pi}{2} - \alpha + \theta}{2} \right) \right] \left[1 + \sqrt{2} \cos\left(\frac{\frac{3\pi}{2} - \alpha - \theta}{2} \right) \right] \right\}^{-1}$$

The Maljuzhinets Function

In the previous expression $\sin\theta = \dfrac{1}{\overline{\eta}}$ and $\psi_{\pi}(\alpha)$ is the **Maljuzhinets half-plane function**, given by

$$\psi_{\pi}\left(\alpha\right) = \exp\left\{-\frac{1}{8\pi}\int_{0}^{\alpha}\left(\frac{\pi\sin u - 2\sqrt{2}\pi\sin\dfrac{u}{2} + 2u}{\cos u}\right)\mathrm{d}u\right\}$$

in which α may be complex. Note that

$$K_{+}\left(\overline{\eta},\lambda\right) \underset{\overline{\eta}\to\infty}{\sim} \frac{1}{\sqrt{\overline{\eta}}}$$ (sheet absent)

$$K_{+}\left(\overline{\eta},\lambda\right) \xrightarrow[\overline{\eta}\to 0]{} \sqrt{1-\lambda} = \sqrt{2}\sin\frac{\alpha}{2}$$ (PEC sheet)

Solution for the Resistive Half Plane

Having achieved the factorization, the solution for the spectrum is obtained by proceeding as in the PEC case: by inserting the factorized form into the functional equation we have

$$\frac{U(\lambda)}{K_+(\bar{\eta},\lambda)} = \frac{1}{2\pi j}\frac{L_1(\lambda)}{L_1(-\lambda_0)}\frac{K_-(\bar{\eta},\lambda)}{\lambda+\lambda_0} + K_-(\bar{\eta},\lambda)L_2(\lambda)$$

which can be written alternatively as

$$\frac{U(\lambda)}{K_+(\bar{\eta},\lambda)} - \frac{1}{2\pi j}\frac{K_+(\bar{\eta},\lambda_0)}{\lambda+\lambda_0} = \frac{1}{2\pi j}\left[\frac{L_1(\lambda)}{L_1(-\lambda_0)}K_-(\bar{\eta},\lambda) - K_-(\bar{\eta},-\lambda_0)\right] \cdot \frac{1}{\lambda+\lambda_0} + K_-(\bar{\eta},\lambda)L_2(\lambda)$$

Solution for the Resistive Half Plane

Since the left hand side is an upper function and the right hand side is a lower function, both must be **entire functions**.

The **asymptotic behavior** of such a function at infinity can be deduced from the property

$$\lim_{|\mathrm{Im}\,\alpha \to \infty|} \psi_\pi(\alpha) = O\left(\exp\left\{ \frac{|\mathrm{Im}\,\alpha|}{8\pi} \right\} \right)$$

which implies $K_+(\overline{\eta}, \lambda \to \infty) = O(1)$.

On the other hand, from the edge condition we find $U(\lambda \to \infty) = O(\lambda^{-1})$

As in the PEC case, we conclude that the left hand side of the modified functional equation is **infinitesimal at infinity**. The **Liouville Theorem** can now be invoked to conclude that such a function is **identically zero**.

Solution for the Resistive Half Plane

Therefore

$$\frac{U(\lambda)}{K_+(\overline{\eta},\lambda)} - \frac{1}{2\pi j}\frac{K_+(\overline{\eta},\lambda_0)}{\lambda+\lambda_0} = 0$$

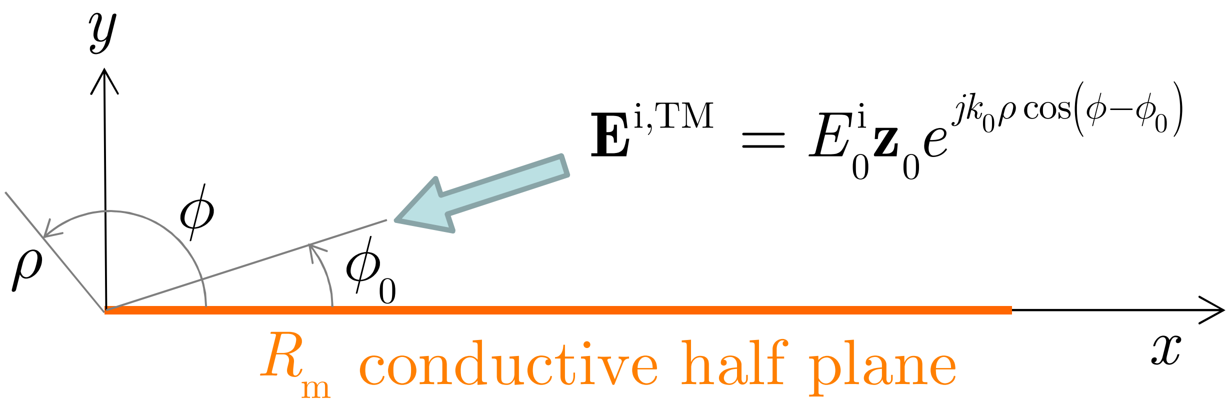
or

$$P_e(\lambda) = U(\lambda) = \frac{1}{2\pi j}\frac{K_+(\overline{\eta},\lambda)K_+(\overline{\eta},\lambda_0)}{\lambda+\lambda_0}$$

i.e.,

$$P_e(\cos\alpha) = \frac{1}{2\pi j}\frac{K_+(\overline{\eta},\cos\alpha)K_+(\overline{\eta},\cos\phi_0)}{\cos\alpha+\cos\phi_0}$$

Conductive Half Plane



In this case we let

$$E_z^{\mathrm{s}}(\rho,\phi) = \pm E_0^{\mathrm{i}} \int_C P_{\mathrm{m}}(\cos\alpha) e^{-jk_0\rho\cos(\phi\mp\alpha)} \mathrm{d}\alpha, \quad y \gtrless 0$$

$$H_x^{\mathrm{s}}(\rho,\phi) = \frac{E_0^{\mathrm{i}}}{\eta_0} \int_C \sin\alpha P_{\mathrm{m}}(\cos\alpha) e^{-jk_0\rho\cos(\phi\mp\alpha)} \mathrm{d}\alpha, \quad y \gtrless 0$$

where now the spectral function $P_{\mathrm{m}}(\cos\alpha)$ is proportional to the spectrum of the **magnetic current density** supported by the conductive half plane.

Dual Integral Equations

By applying the boundary conditions on the conductive half plane

$$H_x = -R_\mathrm{m}\left[E_z\right]_-^+ \qquad y = 0, x > 0$$
$$\left[H_x\right]_-^+ = 0$$

and the condition that the equivalent surface magnetic current density is zero on $y = 0, x < 0$ we get a **pair of dual integral equations**:

$$\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{1-\lambda^2}} + \frac{1}{2\eta_0 R_\mathrm{m}}\right)P_\mathrm{m}\left(\lambda\right)e^{-jk_0x\lambda}\mathrm{d}\lambda = \frac{1}{2\eta_0 R_\mathrm{m}}\sqrt{1-\lambda_0^2}e^{jk_0x\lambda_0}, \qquad x > 0$$

$$\int_{-\infty}^{+\infty}\frac{P_\mathrm{e}\left(\lambda\right)}{\sqrt{1-\lambda^2}}e^{-jk_0x\lambda}\mathrm{d}\lambda = 0, \qquad x < 0$$

Wiener-Hopf Solution

By proceeding as in the previous cases we let

$$\frac{P_{\mathrm{e}}(\lambda)}{\sqrt{1-\lambda^2}} = U(\lambda) \qquad \text{("upper function")}$$

and

$$\left\{K_+(\overline{\eta}_{\mathrm{m}},\lambda)K_-(\overline{\eta}_{\mathrm{m}},\lambda)\right\}^{-1}U(\lambda) = -\frac{1}{2\pi j}\overline{\eta}_{\mathrm{m}}\frac{L_1(\lambda)}{L_1(-\lambda_0)}\frac{\sqrt{1-\lambda_0^2}}{\lambda+\lambda_0} + L_2(\lambda)$$

(functional equation)

where $\overline{\eta}_{\mathrm{m}} = \frac{1}{2R_{\mathrm{m}}\eta_0}$ and $K_\pm(\overline{\eta}_{\mathrm{m}},\lambda)$ are the same upper/lower split functions as in the resistive case.

Wiener-Hopf Solution

The Wiener-Hopf procedure then gives

$$U(\lambda) = -\frac{\overline{\eta}_{\mathrm{m}}}{2\pi j}\frac{\sqrt{1+\lambda_0}}{\sqrt{1-\lambda}}\frac{K_+(\overline{\eta}_{\mathrm{m}},\lambda_0)K_-(\overline{\eta}_{\mathrm{m}},\lambda)}{\lambda+\lambda_0}$$

hence

$$P_{\mathrm{m}}(\cos\alpha) = -\frac{\overline{\eta}_{\mathrm{m}}}{2\pi j}\sqrt{1+\cos\phi_0}\sqrt{1+\cos\alpha}\,\frac{K_+(\overline{\eta}_{\mathrm{m}},\cos\phi_0)K_-(\overline{\eta}_{\mathrm{m}},\cos\alpha)}{\cos\alpha+\cos\phi_0}$$

The Sommerfeld Half-Plane Problem: Diffracted Field

Scattered Field Representation: PEC Case

Let us consider a resistive half plane and in particular a **PEC half plane** ($R_e=0$).
The angular spectrum representation of the scattered electric field is

$$E_z^s(\rho,\phi) = E_0^i \int_C P_e(\cos\alpha)\, e^{-jk_0\rho\cos(\phi\pm\alpha)}\mathrm{d}\alpha, \quad y \gtrless 0$$

where, as we found in the previous lesson,

$$P_e(\cos\alpha) = \frac{1}{2\pi j}\frac{\sqrt{1-\cos\alpha}\sqrt{1-\cos\phi_0}}{\cos\alpha+\cos\phi_0} = \frac{1}{\pi j}\frac{\sin\frac{\alpha}{2}\sin\frac{\phi_0}{2}}{\cos\alpha+\cos\phi_0}$$

$$= \frac{1}{4\pi j}\left[\sec\frac{\alpha+\phi_0}{2} - \sec\frac{\alpha-\phi_0}{2}\right]$$

Scattered Field Representation: PEC Case

We thus have

$$E_z^{\mathrm{s}}\left(\rho,\phi\right)=\frac{E_0^{\mathrm{i}}}{4\pi j}\int_C\left[\sec\frac{\alpha+\phi_0}{2}-\sec\frac{\alpha-\phi_0}{2}\right]e^{-jk_0\rho\cos\left(\phi\mp\alpha\right)}\mathrm{d}\alpha,\quad y\gtrless 0$$

Here the singularities of the integrand are seen to be **a pair of simple poles**:

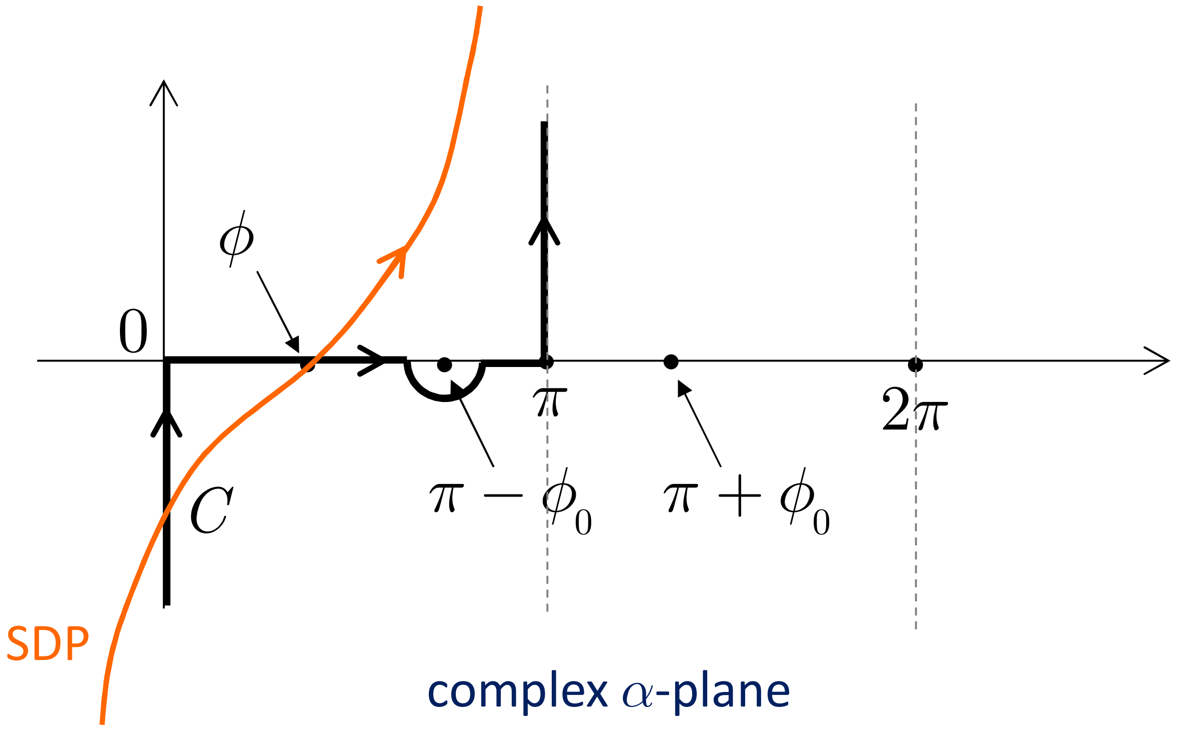
$$\alpha_{\mathrm{p1}}=\pi-\phi_0$$

$$\alpha_{\mathrm{p2}}=\pi+\phi_0$$

As we will see shortly, their residue contribution allows for *recovering the reflected wave* (α_{p1}) and *offsetting the incident wave* (α_{p2}) in the geometrical shadow region.

Asyptotic Evaluation of the Scattered FIeld

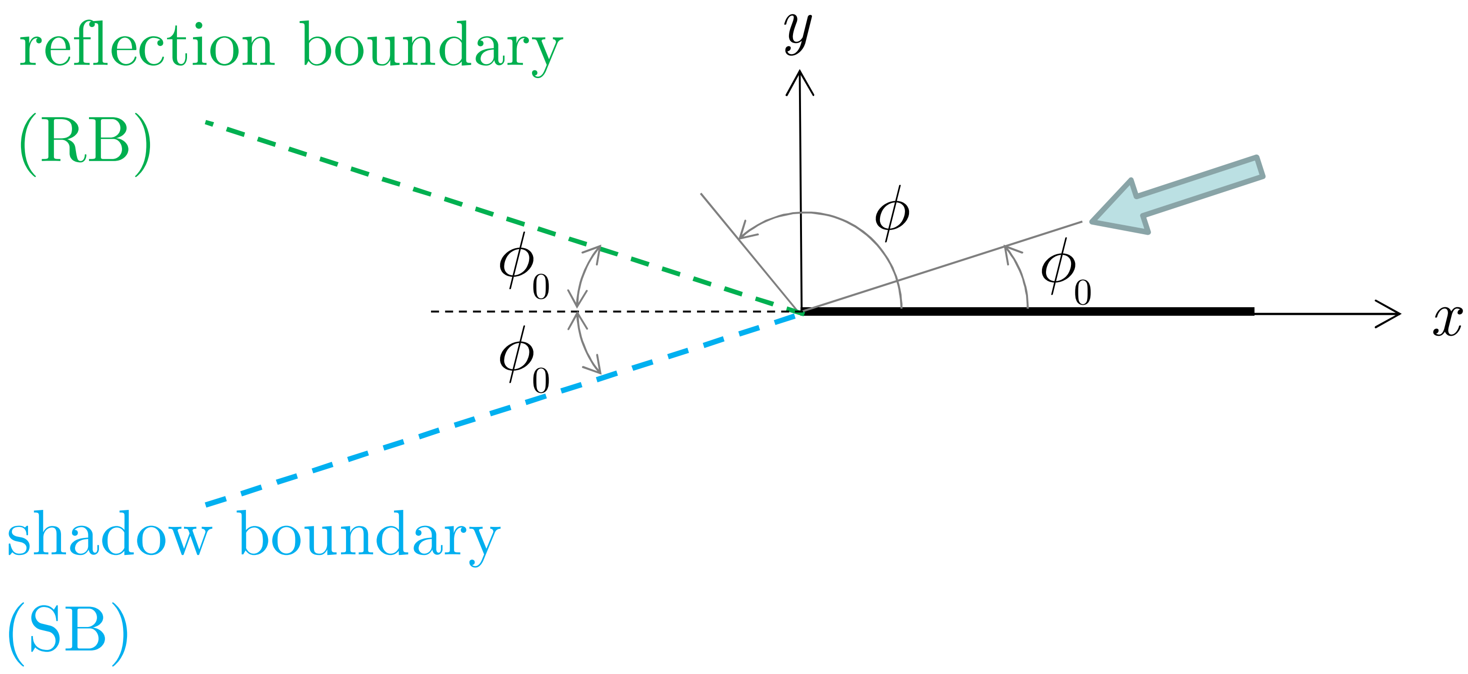
The scattered field can be asymptotically evaluated with the **steepest descent method** by deforming the integration path C to the Steepest Descent Path (SDP):



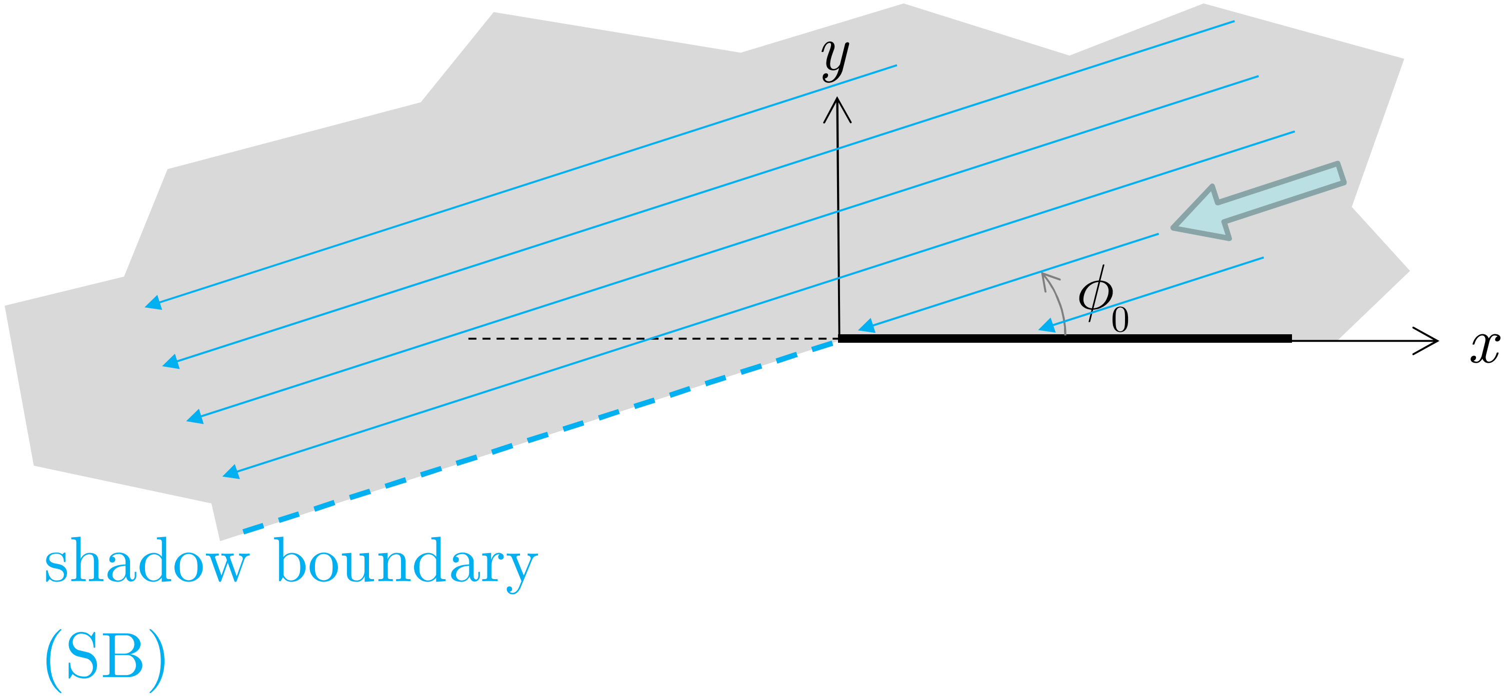
(in the PEC case it turns out that the SDP contribution can be evaluated *exactly*)

GO Boundaries

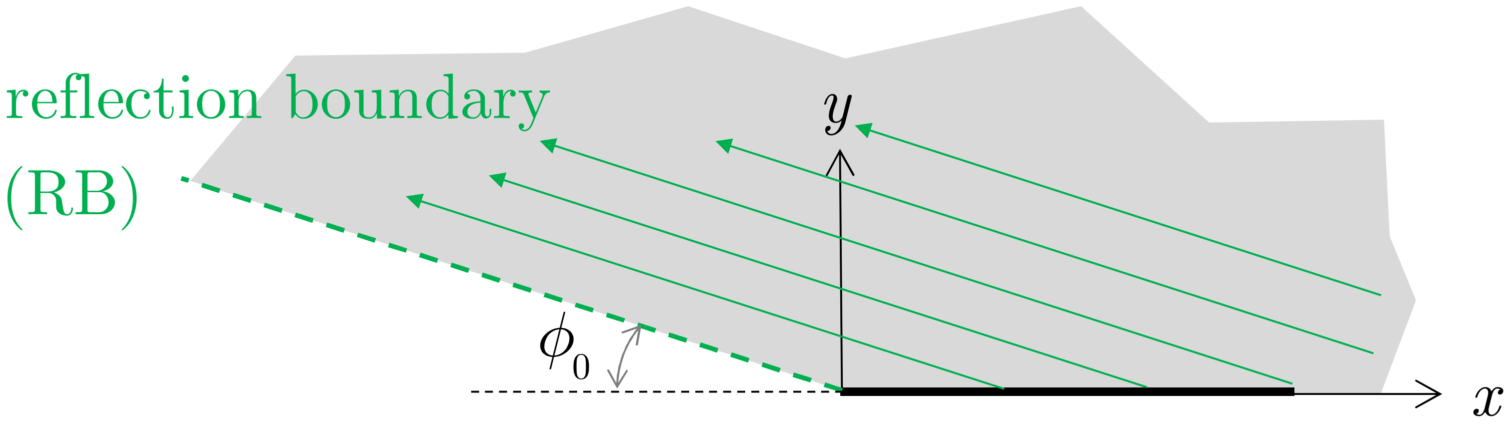
The angles corresponding to the poles at $\pi \pm \phi_0$ define the boundaries of **three distinct angular regions**, each characterized, in ray optical terms, by the presence of different **GO ray congruences**:



GO: Incident Rays

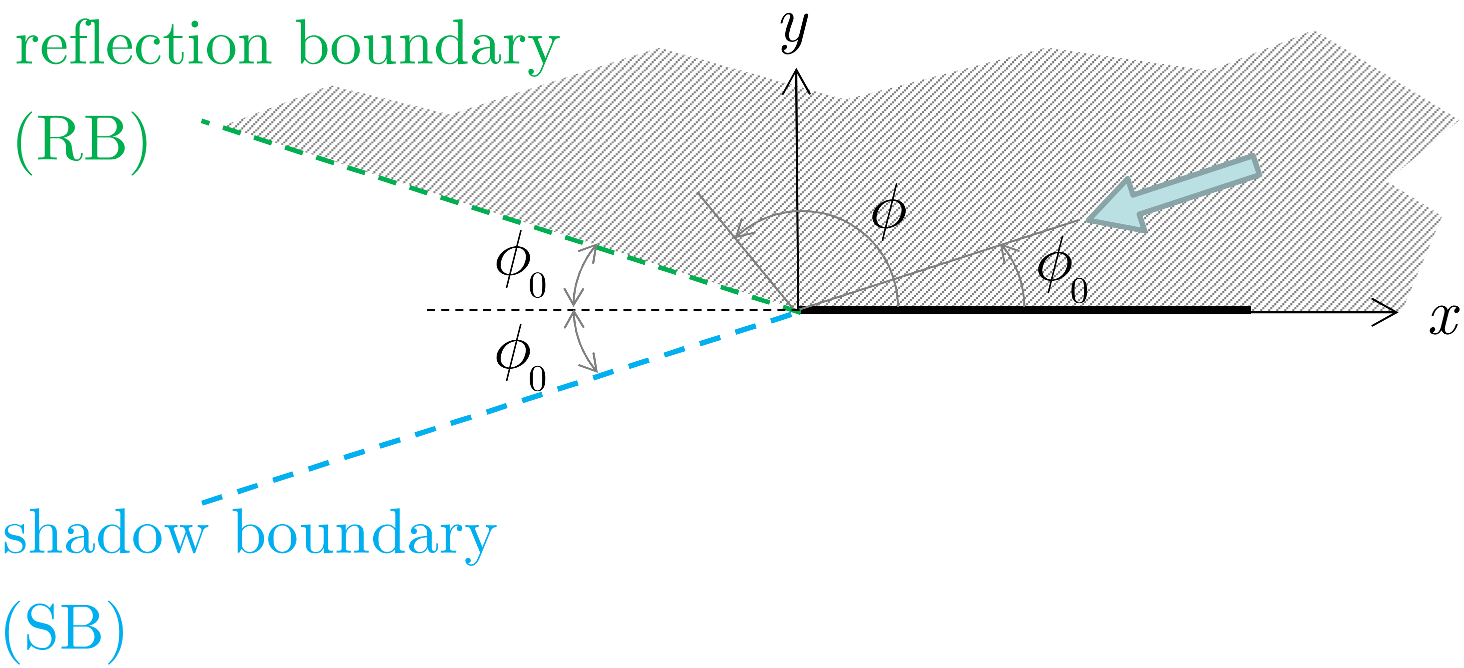


GO: Reflected Rays



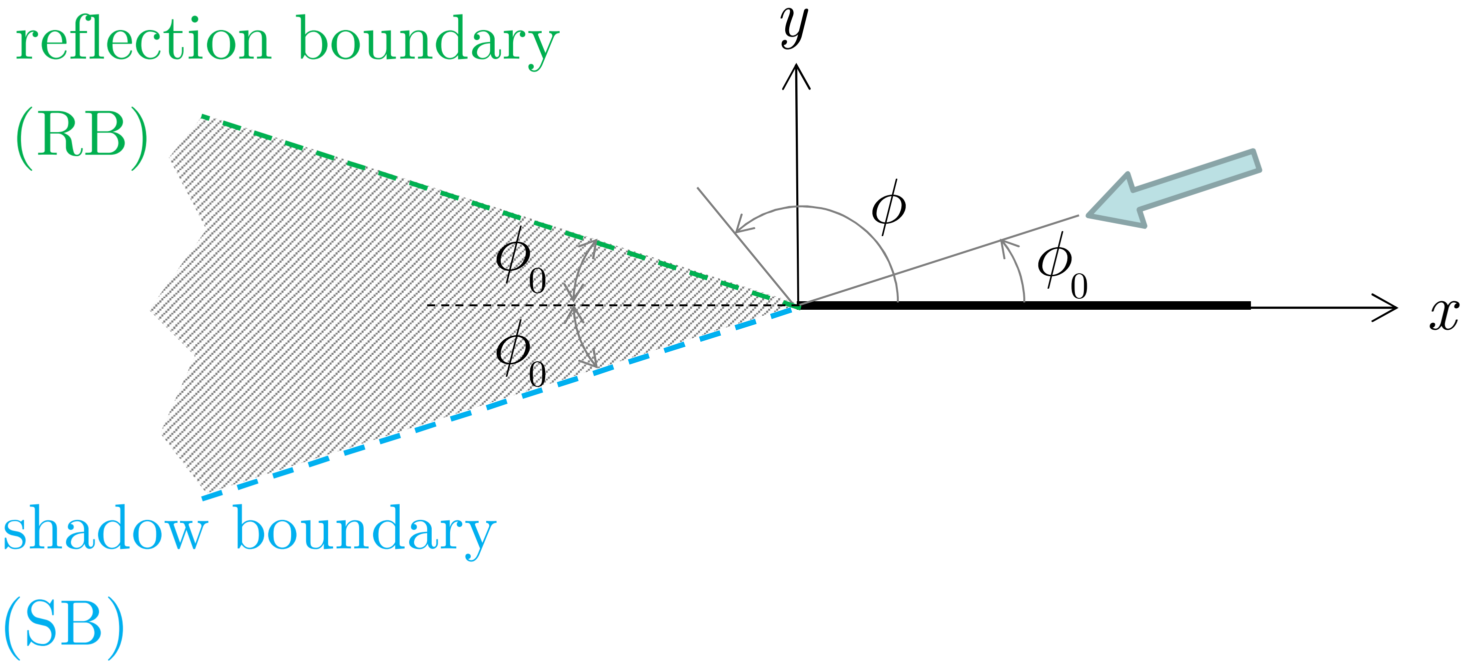
GO: Region 1 (Incident + Reflected Rays)

$$0 \le \phi < \pi - \phi_0$$



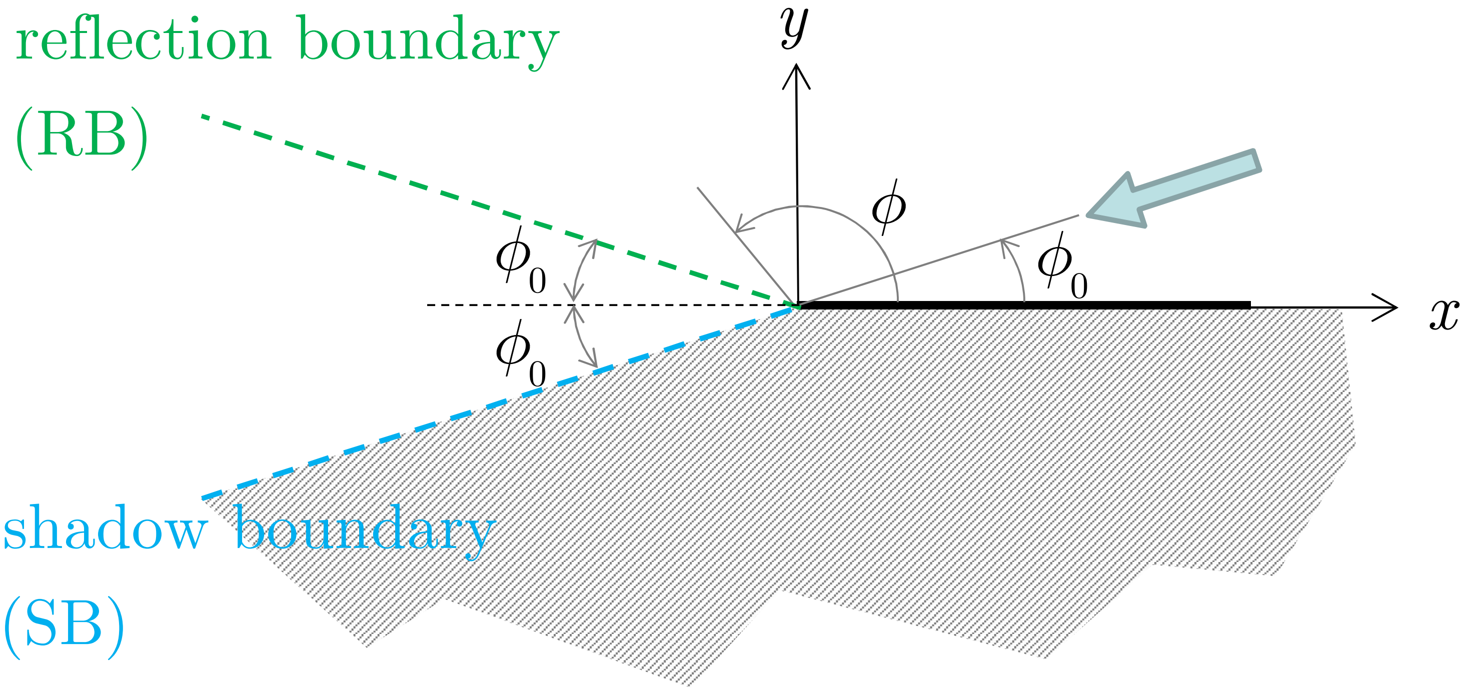
GO: Region 2 (Incident Rays)

$$\pi - \phi_0 < \phi < \pi + \phi_0$$

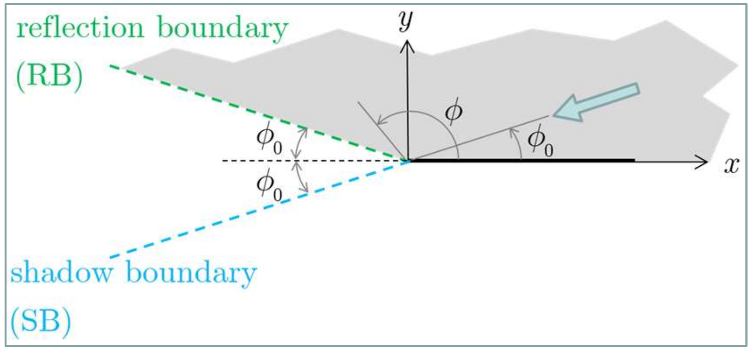


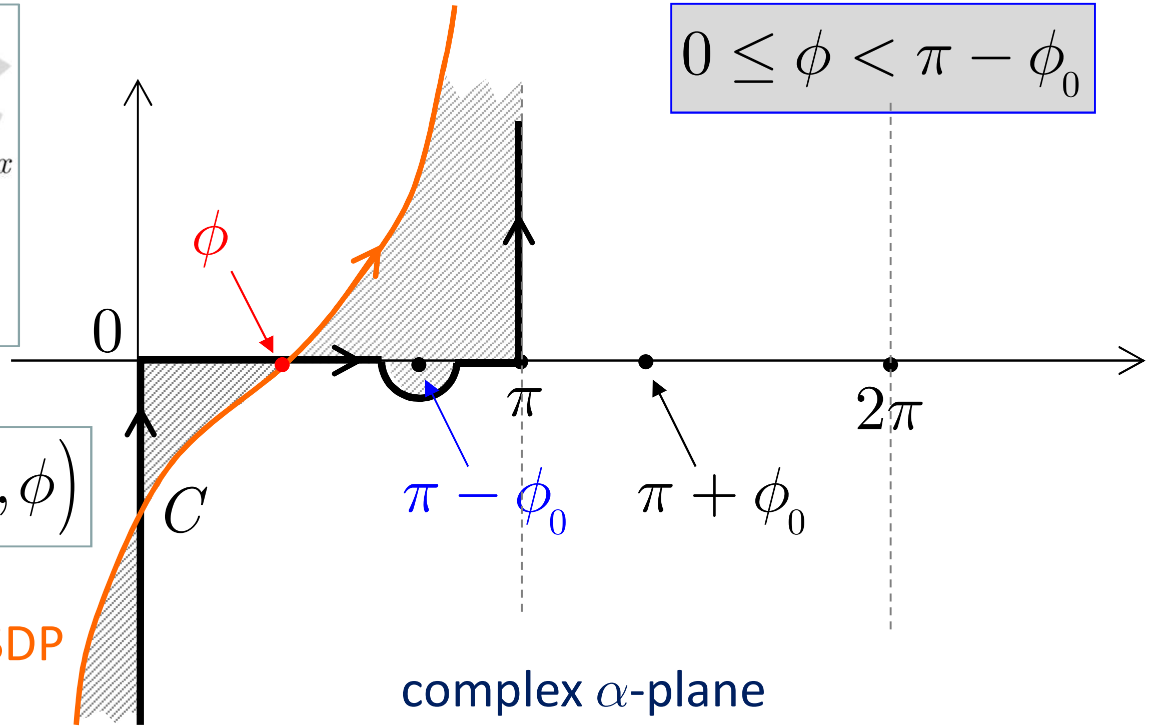
GO: Region 3 (Shadow)

$$\pi + \phi_0 < \phi$$



Region 1: Incident + Reflected + Diffracted Wave





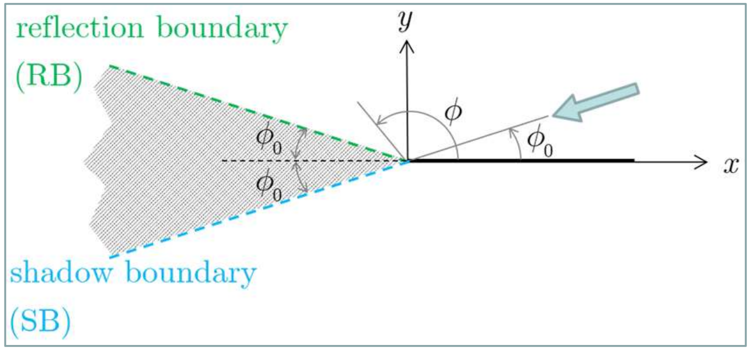
$$E_z(\rho,\phi) = E_z^{\mathrm{i}}(\rho,\phi) + E_z^{\mathrm{s}}(\rho,\phi)$$

$$E_z^{\mathrm{s}}(\rho,\phi) = \frac{E_0^{\mathrm{i}}}{4\pi j}\int_{\mathrm{SDP}} \ldots \mathrm{d}\alpha - E_0^{\mathrm{i}} e^{jk_0\rho\cos(\phi+\phi_0)} u(\pi - \phi_0 - \phi)$$

SP contribution
(**diffracted wave**)

residue contribution
of the pole at $\pi - \phi_0$ (**reflected wave**)

Region 2: Incident + Diffracted Wave

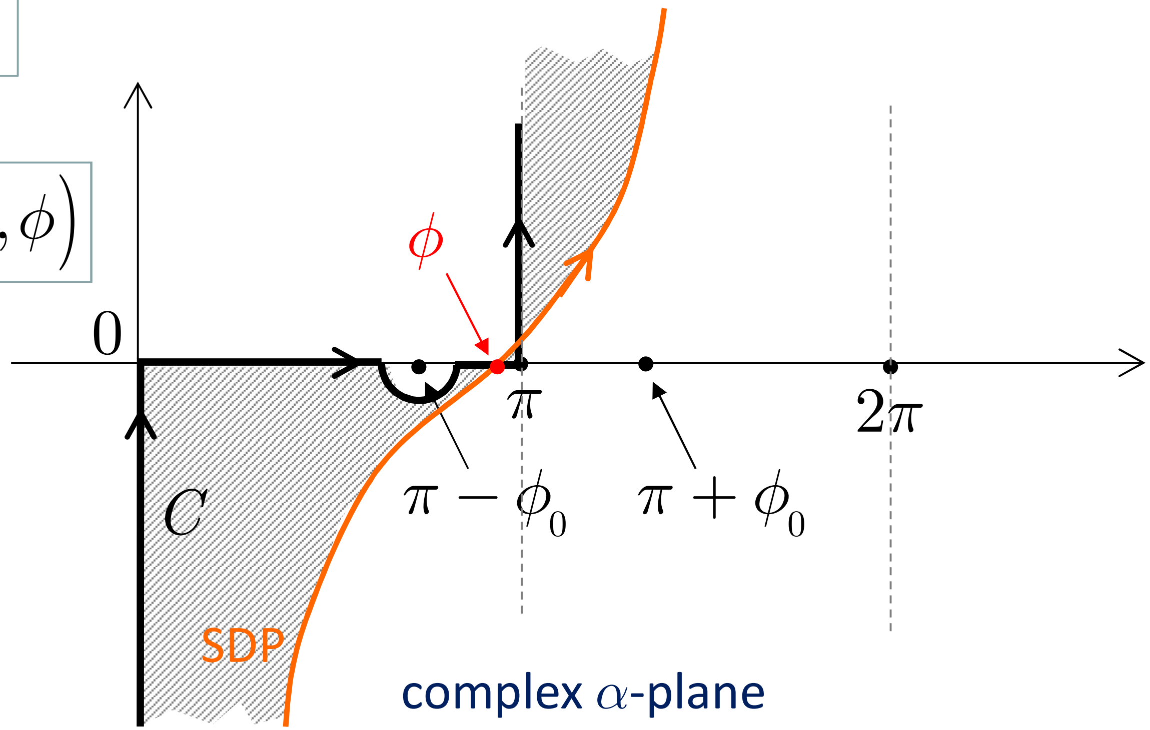


$$\pi - \phi_0 < \phi < \pi + \phi_0$$

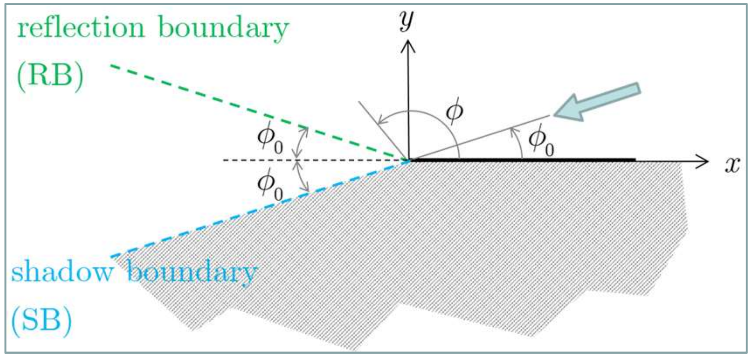
$$E_z(\rho,\phi) = E_z^{\mathrm{i}}(\rho,\phi) + E_z^{\mathrm{s}}(\rho,\phi)$$

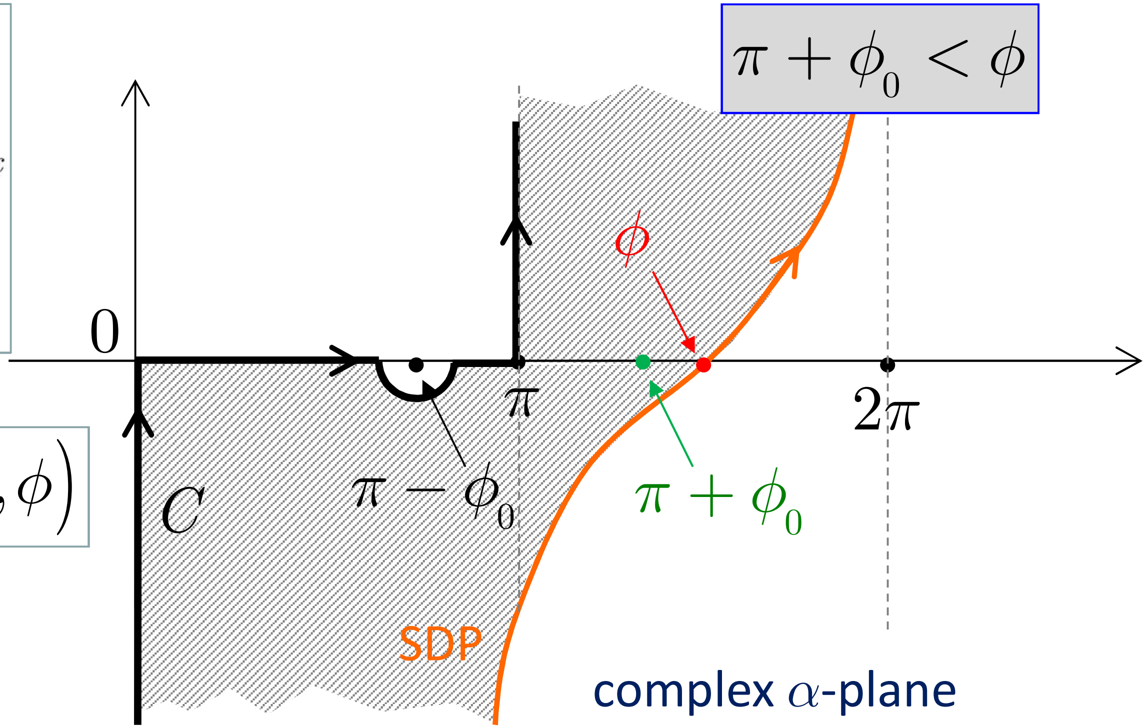
$$E_z^{\mathrm{s}}(\rho,\phi) = \frac{E_0^{\mathrm{i}}}{4\pi j}\int_{\mathrm{SDP}} \ldots \mathrm{d}\alpha$$

SP contribution
(**diffracted wave**)



Region 3: Diffracted Wave





$$E_z\left(\rho,\phi\right) = E_z^{\mathrm{i}}\left(\rho,\phi\right) + E_z^{\mathrm{s}}\left(\rho,\phi\right)$$

$$E_z^{\mathrm{s}}\left(\rho,\phi\right) = \frac{E_0^{\mathrm{i}}}{4\pi j}\int_{\mathrm{SDP}} \ldots \mathrm{d}\alpha - E_0^{\mathrm{i}} e^{+jk_0\rho\cos\left(\phi-\phi_0\right)} u\left(\pi+\phi_0-\phi\right)$$

SP contribution
(**diffracted wave**)

residue contribution
of the pole at $\pi+\phi_0$ (=—**incident wave**)

Non-Uniform Diffracted Field

An application of the steepest descent method in its basic form gives:

$$E_z^{\mathrm{s}}\left(\rho,\phi\right) \sim E_z^{\mathrm{d,nu}}\left(\rho,\phi\right) = E_0^{\mathrm{i}}\sqrt{\frac{2\pi}{k_0}}e^{j\pi/4}\,\frac{e^{-jk_0\rho}}{\sqrt{\rho}}\,P_{\mathrm{e}}\left(\cos\phi\right)$$

$$= E_0^{\mathrm{i}}\,\frac{e^{-jk_0\rho}}{\sqrt{\rho}}\,D_{E\mathrm{e}}^{\mathrm{nu}}\left(\phi,\phi_0\right)$$

cylindrical wave

non-uniform diffraction coefficient

where

$$D_{E\mathrm{e}}^{\mathrm{nu}}\left(\phi,\phi_0\right) = \frac{e^{-j\pi/4}}{\sqrt{2\pi k_0}}\,\frac{1}{2}\left[\sec\frac{\phi+\phi_0}{2} - \sec\frac{\phi-\phi_0}{2}\right]$$

Non-Uniform Diffracted Field

However, there are two issues:

1) the representation of the total field is **discontinuous** across the RB and SB.

In fact, the residue contribution of the optical poles has a step discontinuity at $\pi \pm \phi_0$.

2) For any ρ, the accuracy of the asymptotic expansion **decreases in the vicinity of the RB and SB**, where the non-uniform diffraction coefficient tends to infinity

In fact, when ϕ is close to $\pi \pm \phi_0$ there is **a pole close to the SP**, hence the radius of convergence of the Taylor series used to derive the asymptotic expansion of the integral tends to zero

Uniform Diffracted Field

Both problems can be solved by performing a **uniform asymptotic evaluation** of the SDP integral, which explicitly takes into account the presence of the optical poles.

The essential ingredient is the formula:

$$\int_{\mathrm{SDP}(\phi)} \sec\frac{\alpha-\alpha_0}{2}\, e^{-jk_0\rho\cos(\phi-\alpha)}\mathrm{d}\alpha = \mp 4\sqrt{\pi}e^{-j\pi/4}e^{-jk_0\rho} \cdot F_{\mathrm{C}}\left[\pm\sqrt{2k_0\rho}\cos\left(\frac{\phi-\alpha_0}{2}\right)\right], \qquad \phi-\alpha_0 \lessgtr \pi$$

where $F_{\mathrm{C}}(.)$ is a modified Fresnel integral: $F_{\mathrm{C}}(z) = e^{jz^2}\int_z^{\infty} e^{-j\tau^2}\mathrm{d}\tau$

(*Clemmow transition function*)

Uniform Evaluation of the Field

By using this formula we find

$$E_z(\rho,\phi) = E_z^{\mathrm{i}}(\rho,\phi) + E_z^{\mathrm{s}}(\rho,\phi) = -E_0^{\mathrm{i}} \frac{e^{j\pi/4}}{\sqrt{\pi}} e^{-jk_0\rho}$$

$$\cdot\left[\mp F_{\mathrm{C}}\left(\pm\sqrt{2k_0\rho}\cos\left(\frac{\phi+\phi_0}{2}\right)\right) \pm F_{\mathrm{C}}\left(\pm\sqrt{2k_0\rho}\cos\left(\frac{\phi-\phi_0}{2}\right)\right)\right]$$

$$+ e^{jk_0\rho\cos(\phi+\phi_0)} u(\pi-\phi-\phi_0) + e^{jk_0\rho\cos(\phi-\phi_0)} u(\pi-\phi+\phi_0)$$

In spite of the presence of the unit-step functions, this expression is now **continuous everywhere**…

Uniform Evaluation of the Field

This is a consequence of the crucial property of the Clemmow transition function

$$F_{\mathrm{C}}\left(-z\right) + F_{\mathrm{C}}\left(z\right) = \sqrt{\pi} e^{-j\pi/4} e^{jz^2}$$

By using this property, the total field may be cast in the form

$$E_z\left(\rho,\phi\right) = E_z^{\mathrm{i}}\left(\rho,\phi\right) + E_z^{\mathrm{s}}\left(\rho,\phi\right) = E_0^{\mathrm{i}} \frac{e^{j\pi/4}}{\sqrt{\pi}} e^{-jk_0\rho} \cdot \left[F_{\mathrm{C}}\left(-\sqrt{2k_0\rho}\cos\left(\frac{\phi-\phi_0}{2}\right)\right) - F_{\mathrm{C}}\left(-\sqrt{2k_0\rho}\cos\left(\frac{\phi+\phi_0}{2}\right)\right)\right]$$

which is an **exact expression** for the total field in the presence of a PEC half plane and is manifestly continuous everywhere.

Total Field: UTD Transition Function

The **total field** is often written also in terms of the transition function $F_{\mathrm{KP}}(.)$ adopted in 1974 by R. G. Koyoumjian and P. H. Pathak in their **Uniform Teory of Diffraction** (UTD), as

$$E_z\left(\rho,\phi\right) = E_0^{\mathrm{i}} \frac{e^{-jk_0\rho}}{\sqrt{\rho}} D_{E\mathrm{tot}}^{\mathrm{u}}\left(\phi,\phi_0\right)$$

$$D_{E\mathrm{tot}}^{\mathrm{u}}\left(\phi,\phi_0\right) = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi k_0}}\left[\sec\left(\frac{\phi-\phi_0}{2}\right)F_{\mathrm{KP}}\left(2k_0\rho\cos^2\left(\frac{\phi-\phi_0}{2}\right)\right)\right.$$
$$\left. -\sec\left(\frac{\phi+\phi_0}{2}\right)F_{\mathrm{KP}}\left(2k_0\rho\cos^2\left(\frac{\phi+\phi_0}{2}\right)\right)\right]$$

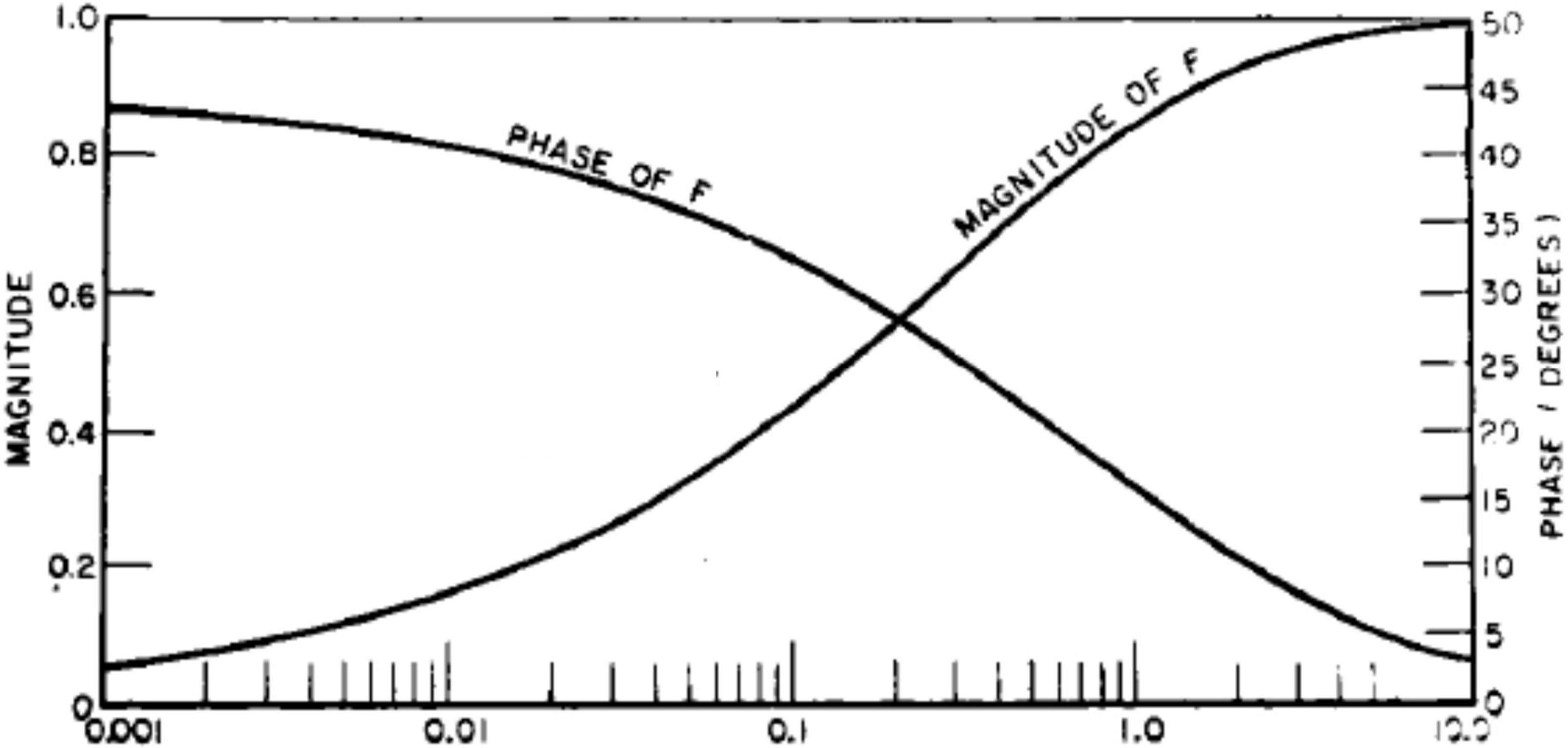
where

$$F_{\mathrm{KP}}\left(z^2\right) = \pm 2jzF_{\mathrm{C}}\left(\pm z\right)$$

(the minus sign is chosen for $\pi/4 < \arg z < 5\pi/4$)

UTD Transition Function

$$F_{\mathrm{KP}}\left(X\right) = 2j\sqrt{X}e^{jX}\int_{\sqrt{X}}^{\infty} e^{-j\tau^2}\mathrm{d}\tau$$



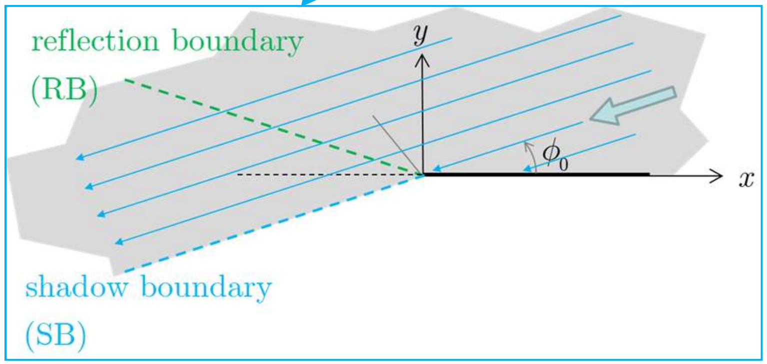
X

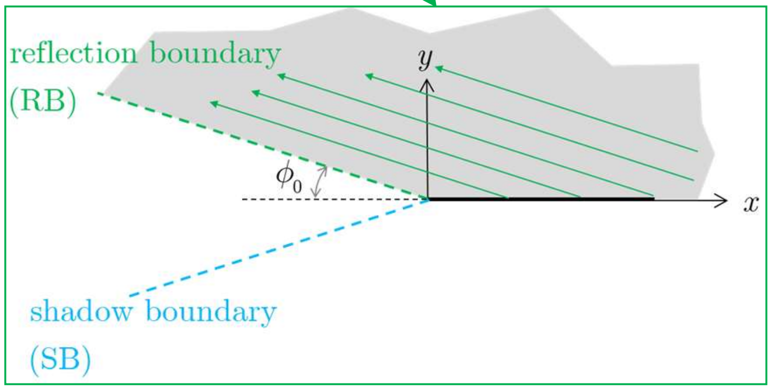
Total Field = GO Field + Diffracted Field

The total field can be expressed in **several different ways**. The one most commonly used is

$$E_z(\rho,\phi) = \underbrace{E_z^{\mathrm{i}'}(\rho,\phi) + E_z^{\mathrm{r}}(\rho,\phi)}_{=E_z^{\mathrm{GO}}(\rho,\phi)} + E_z^{\mathrm{d}}(\rho,\phi)$$

GO field





Diffracted Field

The **diffracted field** can also be written in terms of the transition function $F_{\mathrm{KP}}(.)$:

$$E_z^{\mathrm{d}}\left(\rho,\phi\right) = E_0^{\mathrm{i}}\,\frac{e^{-jk_0\rho}}{\sqrt{\rho}}\,D_E^{\mathrm{u}}\left(\phi,\phi_0\right)$$

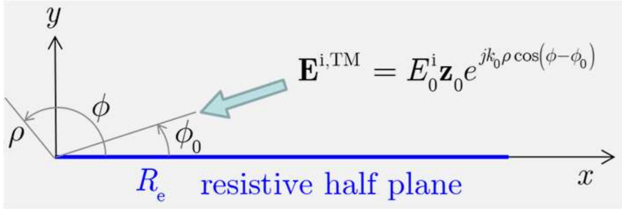
$$D_E^{\mathrm{u}}\left(\phi,\phi_0\right) = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi k_0}}\left[\sec\left(\frac{\phi-\phi_0}{2}\right)F_{\mathrm{KP}}\left(\left|\sqrt{2k_0\rho}\cos\left(\frac{\phi-\phi_0}{2}\right)\right|^2\right) - \sec\left(\frac{\phi+\phi_0}{2}\right)F_{\mathrm{KP}}\left(\left|\sqrt{2k_0\rho}\cos\left(\frac{\phi+\phi_0}{2}\right)\right|^2\right)\right]$$

where now the upper sign is used in the definition of $F_{\mathrm{KP}}(.)$:

$$F_{\mathrm{KP}}\left(z^2\right) = +2jzF_{\mathrm{C}}\left(+z\right)$$

Resistive Half Plane: Scattered Field

For a **resistive half plane** the scattered field can be written as



$$E_z^{\mathrm{s}}(\rho,\phi) = E_0^{\mathrm{i}} \int_C \left[\sec\frac{\alpha+\phi_0}{2} - \sec\frac{\alpha-\phi_0}{2} \right] \cdot \underbrace{\frac{K_+(\bar{\eta},\cos\alpha)\,K_+(\bar{\eta},\cos\phi_0)}{4\sin\frac{\alpha}{2}\sin\frac{\phi_0}{2}}}_{=1/2 \text{ in the PEC case}} e^{-jk_0\rho\cos(\phi\mp\alpha)}\,\mathrm{d}\alpha, \quad y \gtrless 0$$

The uniform evaluation of the solution for the PEC half plane relied on an identity that cannot be used in the case of a resistive half plane.

A different (*asymptotic, not exact*) approach will thus be followed for the management of the pole singularities, known as the **additive approach**, first proposed by van der Waerden in 1951 (an alternative *multiplicative approach* also exists, proposed by Pauli and Clemmow).

Resistive Half Plane: Surface-Wave Pole

We recall:

$$K_+\left(\overline{\eta},\cos\alpha\right)=\frac{4}{\sqrt{\overline{\eta}}}\sin\frac{\alpha}{2}\left\{\frac{\psi_\pi\left(\frac{3\pi}{2}-\alpha-\theta\right)\psi_\pi\left(\frac{\pi}{2}-\alpha+\theta\right)}{\left(\psi_\pi\left(\frac{\pi}{2}\right)\right)^2}\right\}^2\frac{1+\sqrt{2}\cos\left(\frac{\frac{\pi}{2}-\alpha+\theta}{2}\right)}{1+\sqrt{2}\cos\left(\frac{\frac{3\pi}{2}-\alpha-\theta}{2}\right)}$$

In addition to the optical poles at $\pi \pm \phi_0$ there is a **third pole singularity** arising from the split function $K_+\left(\overline{\eta},\cos\alpha\right)$:

$$\alpha_{\mathrm{p3}}=-\theta=-\arcsin\frac{1}{\overline{\eta}}$$

associated with a **surface wave** supported by the resistive sheet.

Resistive Half Plane: Additive Approach

Considering for instance the half space $y > 0$, we write the scattered field as

$$E_z^{\mathrm{s}}\left(\rho,\phi\right) = \frac{E_0^{\mathrm{i}}}{2\pi j}\int_C \left[Q\left(\alpha\right) - \sum_{i=1}^{3}\tilde{Q}\left(\alpha_{\mathrm{p}i}\right)\sec\frac{\alpha-\alpha_{\mathrm{p}i}\pm\pi}{2}\right]e^{-jk_0\rho\cos\left(\phi-\alpha\right)}\mathrm{d}\alpha$$

$$+\sum_{i=1}^{3}\tilde{Q}\left(\alpha_{\mathrm{p}i}\right)\int_C \sec\frac{\alpha-\alpha_{\mathrm{p}i}\pm\pi}{2}e^{-jk_0\rho\cos\left(\phi-\alpha\right)}\mathrm{d}\alpha$$

where

$$\tilde{Q}\left(\alpha_{\mathrm{p}i}\right) = \left.\frac{Q\left(\alpha\right)}{\sec\dfrac{\alpha-\alpha_{\mathrm{p}i}\pm\pi}{2}}\right|_{\alpha\to\alpha_{\mathrm{p}i}}$$

Resistive Half Plane: Additive Approach

and

$$Q(\alpha) = \frac{1}{2\pi j}\left[\sec\frac{\alpha+\phi_0}{2} - \sec\frac{\alpha-\phi_0}{2}\right]$$

$$\cdot\left[1+\sqrt{2}\cos\left(\frac{\frac{3\pi}{2}-\alpha-\theta}{2}\right)\right]^{-1}\frac{K_{u+}(\overline{\eta},\cos\alpha)K_{+}(\overline{\eta},\cos\phi_0)}{4\sin\frac{\alpha}{2}\sin\frac{\phi_0}{2}}$$

with

$$K_{u+}(\overline{\eta},\cos\alpha) = K_{+}(\overline{\eta},\cos\alpha)\left[1+\sqrt{2}\cos\left(\frac{\frac{3\pi}{2}-\alpha-\theta}{2}\right)\right]$$

Resistive Half Plane: Additive Approach

Since the terms added and subtracted contain the same pole singularities as the original integrand, the new ('regularized') integrand is **free of pole singularities** and thus can be asymptotically evaluated via the **usual steepest-descent method**.

The relevant contribution to scattered field is

$$E_z^{\mathrm{d,nu}}\left(\rho,\phi\right) = E_0^{\mathrm{i}}\sqrt{\frac{2\pi}{k_0}}e^{j\pi/4}\,\frac{e^{-jk_0\rho}}{\sqrt{\rho}}P_{\mathrm{e}}\left(\cos\phi\right)$$

$$= E_0^{\mathrm{i}}\,\frac{e^{-jk_0\rho}}{\sqrt{\rho}}D_{E\mathrm{e}}^{\mathrm{nu}}\left(\phi,\phi_0\right)$$

where

$$D_E^{\mathrm{nu}}\left(\phi,\phi_0,\overline{\eta}\right) = \underbrace{-\frac{e^{-j\pi/4}}{2\sqrt{2\pi_0}}\left[\sec\frac{\phi+\phi_0}{2}-\sec\frac{\phi-\phi_0}{2}\right]}_{=D_E^{\mathrm{nu}}\left(\phi,\phi_0,\overline{\eta}=0\right)\ (\text{PEC case})}\frac{K_{u+}\left(\overline{\eta},\cos\phi\right)K_{+}\left(\overline{\eta},\cos\phi_0\right)}{2\sin\frac{\phi}{2}\sin\frac{\phi_0}{2}}$$

Resistive Half Plane: Additive Approach

On the other hand, the **additional integrals** can be evaluated using the **identity** used in the case of a PEC half plane.

The result for the **uniform diffraction coefficient** is:

$$D_E^{\mathrm{u}}\left(\phi,\phi_0,\overline{\eta}\right) = D_E^{\mathrm{nu}}\left(\phi,\phi_0,\overline{\eta}\right) - \sqrt{\frac{2\pi}{k_0}} e^{j\pi/4}$$

$$\cdot \sum_{i=1}^{3} \tilde{Q}\left(\alpha_{\mathrm{p}i}\right) \sec\frac{\alpha - \alpha_{\mathrm{p}i} + \pi}{2}\left[1 - F_{\mathrm{KP}}\left(2k_0\rho\cos^2\left(\frac{\phi - \alpha_{\mathrm{p}i} + \pi}{2}\right)\right)\right]$$

References

T. B. A. Senior and J. L. Volakis, *Approximate boundary conditions in electromagnetics*. London, UK: The IEE, 1995, ch. 3.