$$\boxed{\lim_{n} \underbrace{\left(n^2+1\right)}_{\downarrow \atop +\infty}^{\overbrace{\frac{1}{\log n}}^{\downarrow\, 0}}} = (+\infty)^0 \quad (f.i) = \lim_{n\to+\infty} e^{\frac{\log(n^2+1)}{\log n}} = e^2$$

$$\frac{\log(n^2+1)}{\log n} = \frac{\log\left(n^2\left(1+\frac{1}{n^2}\right)\right)}{\log n} = \frac{\log(n^2)+\log\left(1+\frac{1}{n^2}\right)}{\log n} =$$

$$= \frac{2\cancel{\log n}}{\cancel{\log n}} + \frac{\log\left(1+\frac{1}{n^2}\right)}{\log n} \to 2.$$

$$\frac{0}{+\infty} = 0$$

$$\boxed{\lim_{n\to+\infty} \frac{n!}{n^3+3^n} \cdot \log_2\left(\frac{n!+3^n}{n!-3^n}\right)} = (+\infty\cdot 0) = \frac{2}{\log 2}$$

$$\frac{n!}{n^3+3^n} \to +\infty \qquad \frac{n!+3^n}{n!-3^n} \to 1 \qquad \log_2\left(\frac{n!+3^n}{n!-3^n}\right) \to 0$$

$$-3^n+3^n$$

$$\log_2\left(\frac{n!+3^n}{n!-3^n}\right) = \log_2\left(\frac{(n!-3^n)+2\cdot 3^n}{n!-3^n}\right) =$$

$$= \log_2\left(1+\frac{2\cdot 3^n}{n!-3^n}\right) = \frac{\log\left(1+\frac{1}{\frac{n!-3^n}{2\cdot 3^n}}\right)}{\log 2} =$$

$$\boxed{\log(1+a_n) \sim a_n \atop a_n \to 0}$$

$$b_n \to \pm\infty \quad \log\left(1+\frac{1}{b_n}\right) \sim \frac{1}{b_n}$$

$$= \frac{1}{\log 2} \log\left(1+\frac{1}{\underbrace{\frac{n!-3^n}{2\cdot 3^n}}_{\to +\infty}}\right) \sim \frac{1}{(\log 2)} \frac{2\cdot 3^n}{n!-3^n} \sim \frac{2\cdot 3^n}{(\log 2)\, n!}$$

$$n!-3^n = n!\left(1-\underbrace{\frac{3^n}{n!}}_{\to 0}\right) = n!(1+o(1)) \sim n!$$

$$\frac{n!}{\underbrace{n^3+3^n}_{\displaystyle \parallel}} \sim \frac{n!}{3^n}$$

$$3^n\left(1+\frac{n^3}{3^n}\right)=3^n(1+o(1))\sim 3^n$$

$$\frac{n!}{n^3+3^n}\log_2\left(\text{———}\right)\sim \frac{\cancel{n!}}{\cancel{3^n}}\cdot\frac{2}{\log 2}\cdot\frac{\cancel{3^n}}{\cancel{n!}}$$

$$\lim_{n\to+\infty}\left(\frac{n-2}{n+1}\right)^{n^\alpha}$$

$n^\alpha \to \begin{cases}+\infty & \alpha>0\\ =1 & \alpha=0\\ 0 & \alpha<0\end{cases}$

$\frac{n-2}{n+1}\to 1$

$\alpha\in\mathbb{R}$.

$\boxed{\text{es. } 2.10.3 \quad d)}$

Il limite vale $1^1=1$ se $\alpha=0$

$1^0=1$ $\alpha<0$

resta il caso $\alpha>0$.

$$\left(\frac{n-2}{n+1}\right)^{n^\alpha}\to(1^{+\infty})$$

$$\parallel$$

$$\left(\frac{n+1-3}{n+1}\right)^{n^\alpha}=\left(1-\frac{3}{n+1}\right)^{n^\alpha}=\left[\underbrace{\left(1-\frac{3}{n+1}\right)^{n+1}}_{\downarrow\, e^{-3}}\right]^{\frac{n^\alpha}{n+1}}$$

$$\frac{n^\alpha}{n+1}\longrightarrow\begin{cases}+\infty & \alpha>1\\ 1 & \alpha=1\\ 0 & 0<\alpha<1\end{cases}$$

se $\alpha>1$ $\quad a_n\to(e^{-3})^{+\infty}=0$

$\alpha=1$ $\quad a_n\to(e^{-3})^1=e^{-3}$

$0<\alpha<1$ $\quad a_n\to(e^{-3})^0=1$

Risultato: il limite vale $\begin{cases} 1 & \text{se } \boxed{\alpha < 1} \\ e^{-3} & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$

$$\boxed{\lim_{n\to+\infty} \frac{7^{\sqrt{n}}}{n^{\log_3 n}}} = \left(\frac{+\infty}{+\infty}\right) = \lim_{n} \frac{e^{\sqrt{n}\log 7}}{e^{\log_3 n \log n}} =$$

$$\left[\begin{aligned} 7^{\sqrt{n}} &= e^{\sqrt{n}\log 7} \\ n^{\log_3 n} &= e^{\log_3 n \log n} \end{aligned}\right]$$

$$= \lim_{n\to+\infty} e^{\sqrt{n}\log 7 - \log_3 n \log n} = (e^{+\infty}) = +\infty$$

$$\sqrt{n}\log 7 - \log_3 n \log n = \sqrt{n}\log 7 - \frac{\log^2 n}{\log 3} =$$

$$= \sqrt{n}\Bigg(\log 7 - \frac{1}{\log 3}\underbrace{\boxed{\frac{\log^2 n}{\sqrt{n}}}}_{\substack{\downarrow \\ 0}}\Bigg) = \sqrt{n}\big(\underbrace{\log 7 + o(1)}_{\substack{\downarrow \\ 0}}\big) \to +\infty$$

$$\frac{\log^2 n}{\sqrt{n}} = \left(\frac{\log n}{\sqrt[4]{n}}\right)^2$$

$$\boxed{\lim_{n\to+\infty}\left(1+\frac{1}{n^2}-\frac{1}{n}\right)^n} = (1^{+\infty}) = \frac{1}{e}$$

$$\left(1+\frac{1}{n^2}-\frac{1}{n}\right)^n = \left(1+\frac{1-n}{n^2}\right)^n = \underbrace{\left[\left(1+\frac{1}{\underbrace{\left(\frac{n^2}{1-n}\right)}_{\substack{\downarrow \\ -\infty}}}\right)^{\frac{n^2}{1-n}}\right]}_{\substack{\downarrow \\ e}}^{\underbrace{n\frac{(1-n)}{n^2}}_{\substack{\downarrow \\ -1}}}$$

$$\frac{n^2}{1-n} = \frac{n^{\cancel{2}}}{\cancel{n}(-1+o(1))} \longrightarrow -\infty$$

In alternativa:

$$\left(1+\frac{1}{n^2}-\frac{1}{n}\right)^n = e^{\underbrace{n\log\left(1+\frac{1}{n^2}-\frac{1}{n}\right)}_{\downarrow \, -1}} \to e^{-1}$$

Studio l'esponente

$$n\log\Big(1+\underbrace{\boxed{\frac{1}{n^2}-\frac{1}{n}}}_{\downarrow\,0}\Big) \sim n\left(\frac{1}{n^2}-\frac{1}{n}\right) = n\,\frac{(1-n)}{n^2} \to -1$$

$\log(1+a_n) \sim a_n$

$$\boxed{\lim_{n\to+\infty}\Big(1+\underbrace{\log\left(1+\frac{\alpha}{n}\right)}_{\downarrow\,0}\Big)^{n^{2-\alpha}}}$$

$\alpha \in \mathbb{R}$.

$$n^{2-\alpha} \longrightarrow \begin{cases} +\infty & 2-\alpha>0 \quad \alpha<2 \\ =1 & \alpha=2 \\ 0 & \alpha>2 \end{cases}$$

Il limite vale 1 se $\alpha \geq 2$ (niente f.i.)

Da adesso $\alpha<2$.

$$\left(1+\log\left(1+\frac{\alpha}{n}\right)\right)^{n^{2-\alpha}} = 1^{(n^2)} = 1 \to 1 \text{ se } \alpha=0$$

se $0<\alpha<2$ $\quad \left(1+\log\left(1+\frac{\alpha}{n}\right)\right)^{n^{2-\alpha}} = \left(1+\frac{1}{\left(\frac{1}{\log\left(1+\frac{\alpha}{n}\right)}\right)}\right)^{n^{2-\alpha}}$

$$\downarrow \; +\infty$$

se $\alpha<0$ $\quad \left(1+\log\left(1+\frac{\alpha}{n}\right)\right)^{n^{2-\alpha}} = \left(1+\frac{1}{\left(1/\log\left(1+\frac{\alpha}{n}\right)\right)}\right) \to -\infty$

In ogni caso $\boxed{\alpha<2 \quad \alpha\neq 0}$

$$\left(1+\log\left(1+\frac{\alpha}{n}\right)\right)^{n^{2-\alpha}} = \left[\underbrace{\left(1+\frac{1}{\left(\frac{1}{\log\left(1+\frac{\alpha}{n}\right)}\right)}\right)^{\frac{1}{\log\left(1+\frac{\alpha}{n}\right)}}}_{\to e}\right]^{n^{2-\alpha}\log\left(1+\frac{\alpha}{n}\right)}$$

$$n^{2-\alpha}\underbrace{\log\left(1+\frac{\alpha}{n}\right)}_{\sim\frac{\alpha}{n}} \sim \alpha\frac{n^{2-\alpha}}{n} = \alpha n^{1-\alpha}$$

se $1<\alpha<2 \Rightarrow \alpha n^{1-\alpha} \to 0 \Rightarrow$ Il limite vale $e^0=1$

$\alpha=1 \Rightarrow$ l'esponente tende a 1 $\Rightarrow$ il limite vale $e^1=e$

$0<\alpha<1 \Rightarrow \alpha n^{1-\alpha} \to +\infty \Rightarrow$ il limite vale $e^{+\infty}=+\infty$

$\alpha<0 \Rightarrow \alpha n^{1-\alpha} \to -\infty \Rightarrow$ il limite vale $e^{-\infty}=0$

$$\left(1+\log\left(1+\frac{\alpha}{n}\right)\right)^{n^{2-\alpha}} = e^{n^{2-\alpha}\log\left(1+\log\left(1+\frac{\alpha}{n}\right)\right)}$$

$$n^{2-\alpha}\underbrace{\log\left(1+\overbrace{\log\left(1+\frac{\alpha}{n}\right)}^{\to 0}\right)}_{\sim \log\left(1+\frac{\alpha}{n}\right)\sim\frac{\alpha}{n}} \sim \alpha n^{1-\alpha}$$

$$\boxed{\lim_{n\to+\infty}\left(n^{\sqrt{n}}-\left(\sqrt{n}\right)^n\right)} = \lim_{n\to+\infty}\left(e^{\sqrt{n}\log n} - e^{\overbrace{n\log\sqrt{n}}^{=\frac{1}{2}n\log n}}\right) = -\infty$$

$$\boxed{\begin{aligned} n^{\sqrt{n}} &= e^{\sqrt{n}\log n}\\ (\sqrt{n})^n &= e^{n\log\sqrt{n}}\end{aligned}} \qquad = \lim_{n\to+\infty} e^{\frac{1}{2}n\log n}\left(\underbrace{e^{\sqrt{n}\log n-\frac{1}{2}n\log n}}_{\to 0}-1\right)$$

$$\sqrt{n}\log n-\frac{1}{2}n\log n = \log n\left(\sqrt{n}-\frac{n}{2}\right) = n\log n\underbrace{\left(\frac{1}{\sqrt{n}}-\frac{1}{2}\right)}_{\to -1/2}$$

$$\lim_{n\to+\infty}\left(n^{\sqrt{n}}-(\sqrt{n})^{n}\right)=\lim_{n\to+\infty}\left[n^{\sqrt{n}}-n^{\frac{n}{2}}\right]=$$

$$=\lim_{n\to+\infty} n^{n/2}\left(\underbrace{n^{\overbrace{\sqrt{n}-\frac{n}{2}}^{\to-\infty}}}_{\to 0}-1\right)=-\infty$$

$n \to +\infty$

Verifiche di limite

$$\lim_{x\to 4} x^2=16 \qquad x_0=4\ ,\ \ell=16$$

significa:

$\forall \varepsilon>0 \ \exists \delta>0$ t.c. $|x^2-16|<\varepsilon$ $\forall x$ verificante $0<|x-4|<\delta$

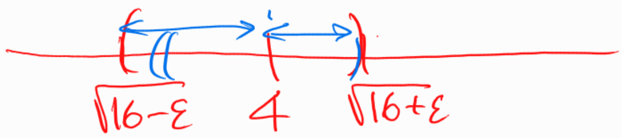
$$|x^2-16|<\varepsilon$$
$$\Updownarrow$$

1°) $16-\varepsilon < x^2 < 16+\varepsilon$

$\boxed{\text{Posso imporre } \varepsilon<16}$

$$\Uparrow$$

$$\cancel{\left(-\sqrt{16+\varepsilon}<x<-\sqrt{16-\varepsilon}\right)}\ \cancel{\vee}\left(\sqrt{16-\varepsilon}<x<\sqrt{16+\varepsilon}\right)$$



Basta prendere $\delta=\min\left\{\sqrt{16+\varepsilon}-4,\ 4-\sqrt{16-\varepsilon}\right\}=$

$$=\sqrt{16+\varepsilon}-4$$

$$|x^2-16|<\varepsilon$$

$$|x^2-16|=|(x-4)(x+4)|=|x-4|\,\overbrace{|x+4|}^{9 \text{ se } |x-4|<1}\underset{\boxed{|x-4|<1}}{\leq} 9\,|x-4|<\varepsilon$$

$$\boxed{|x-4|<\frac{\varepsilon}{9}}$$

[OSS se $|x-4|<1 \Rightarrow$ $3<x<5 \Rightarrow 7<x+4<9$
$\Rightarrow$ $|x+4|<9$]

Basta prendere $\delta = \min\left\{1, \frac{\varepsilon}{9}\right\}$

se $|x-4| < \delta \Rightarrow \begin{cases} |x-4| < 1 \\ |x-4| < \frac{\varepsilon}{9} \end{cases}$