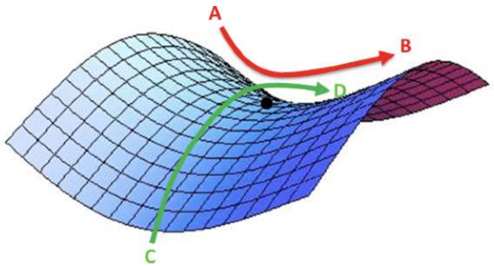
Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 8

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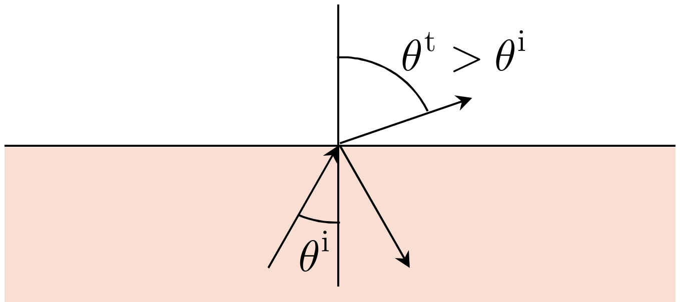
Lateral Waves

Introduction: Plane-Wave Reflection and Refraction

When a plane wave strikes a plane interface between two different homogeneous, lossless, isotropic dielectrics, there arise reflected and refracted plane waves in addition to the incident wave.

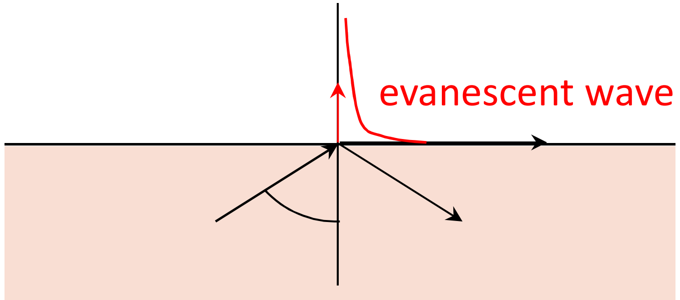
If the field impinges from the denser medium (having the lower wave speed), the refracted wave emerges at a **steeper** angle w.r.t. the normal to the interface:

$\theta^{\mathrm{i}} < \theta_{\mathrm{c}}$



For still steeper directions of incidence, **total reflection** obtains and no propagating field is transmitted:

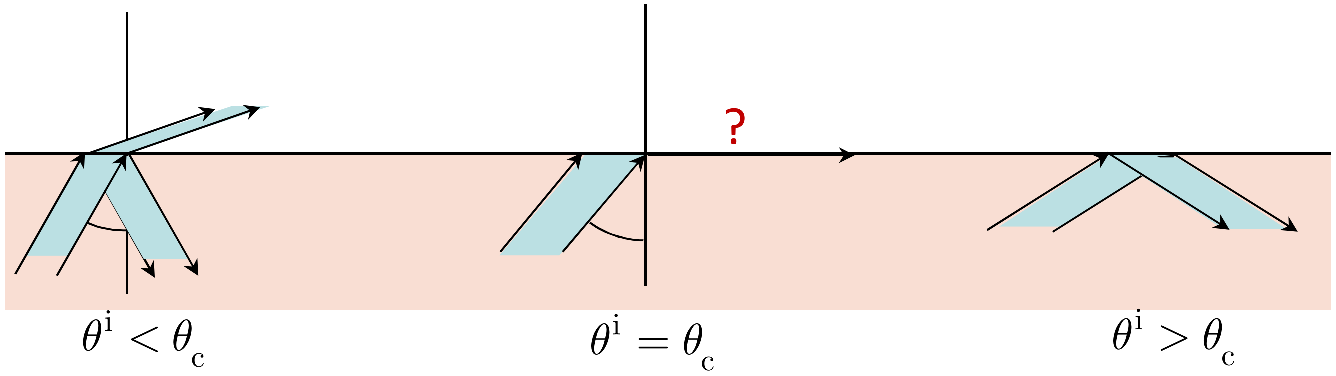
$\theta^{\mathrm{i}} > \theta_{\mathrm{c}}$



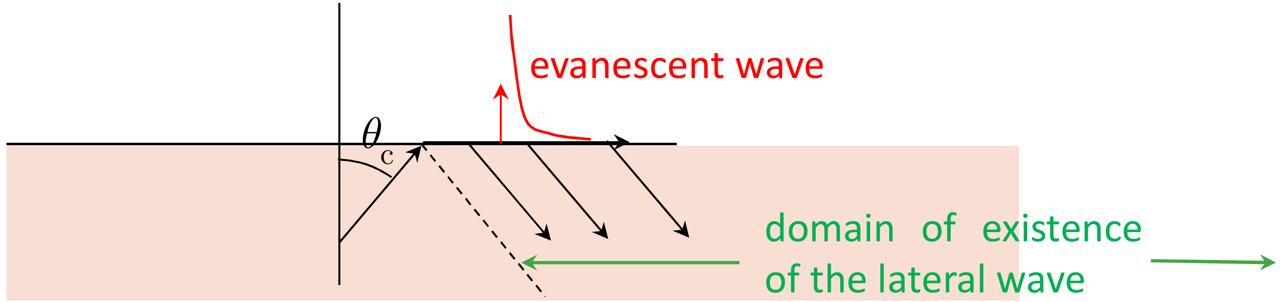
$$\theta^{\mathrm{t}} = \frac{\pi}{2} + j\theta^{\mathrm{t}}_{\mathrm{j}}$$

Introduction: The Ray Tube Dilemma

While the processes of reflection and refraction may generally be interpreted in simple ray-optical terms involving the concepts of *wavefronts*, *rays*, and *ray tubes*, this mechanism **fails** when the incidence is at the critical angle:

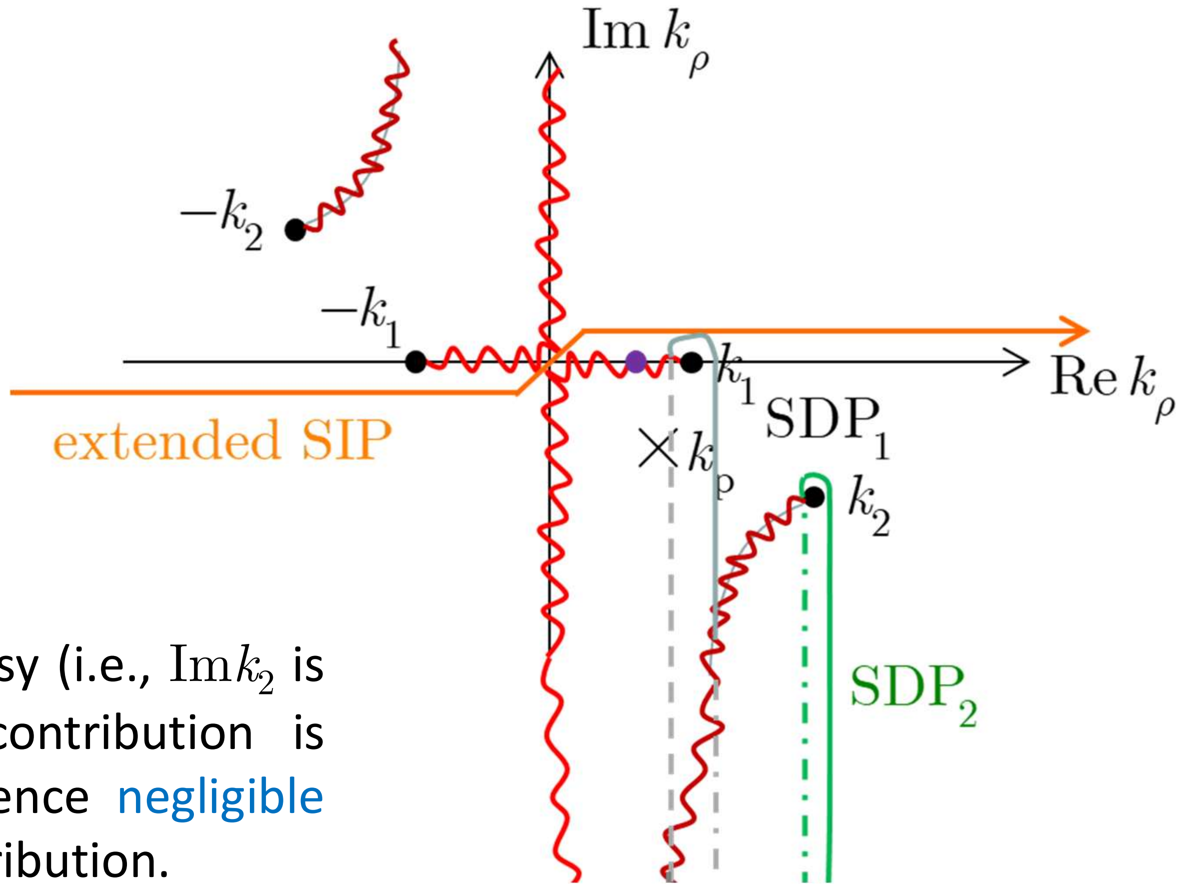


It appears plausible that the critically refracted ray may react back in the denser medium…



The Lateral Wave in the Sommerfeld Problem

In the Sommerfeld problem (source in air above a *lossy* dielectric half space), we defined the **lateral wave** as the SDP integral around the complex wavenumber k_2 in the dielectric:

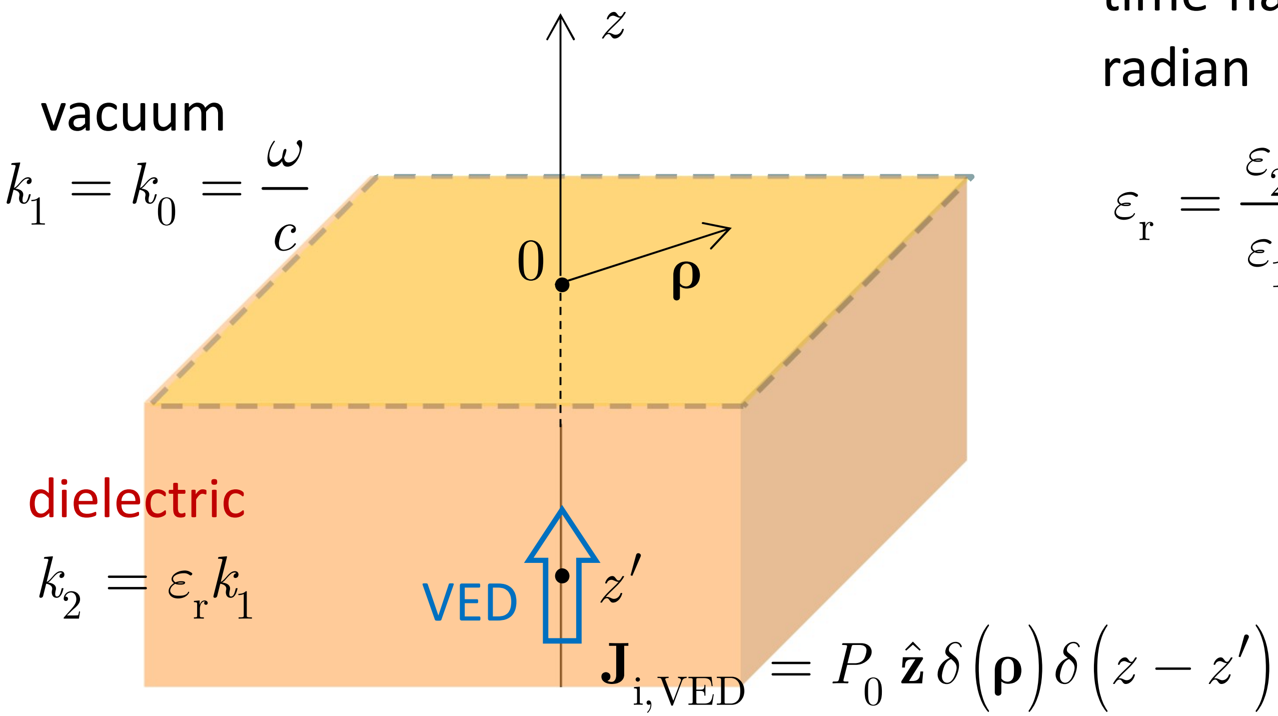


If the dielectric is enough lossy (i.e., $\mathrm{Im} k_2$ is not negligible), the SDP$_2$ contribution is exponentially smaller and hence negligible with respect to the SDP$_1$ contribution.

But if the dielectric is low-loss or lossless and the source is inside it the lateral wave may not be negligible…

Reference Configuration

Let us consider again a plane interface between two half spaces, now comprising a **lossless** dielectric ($\varepsilon_{\mathrm{r}} \in \mathbb{R},\ \varepsilon_{\mathrm{r}} > 1$) and a source **inside** it:



time-harmonic regime: $e^{j\omega t}$

radian frequency: ω

$$\varepsilon_{\mathrm{r}} = \frac{\varepsilon_2}{\varepsilon_1}$$

VED amplitude: $P_0\ [\mathrm{A} \cdot \mathrm{m}]$

Total Potential (Axial Transmission Representation)

The *Axial Transmission Representation* of the total potential is:

$$A_z\left(\rho,z\right)=P_0\begin{cases}\dfrac{e^{-jk_2R}}{4\pi R}+\dfrac{1}{2\pi}\displaystyle\int_0^{+\infty}\left(-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z2}\left(z+z'\right)}}{2jk_{z2}}J_0\left(k_\rho\rho\right)k_\rho\mathrm{d}k_\rho,\ z\le 0\\[2ex]\dfrac{1}{2\pi}\displaystyle\int_0^{+\infty}\left(1-\Gamma^{\mathrm{TM}}\right)\frac{e^{jk_{z2}z'}e^{-jk_{z1}z}}{2jk_{z2}}J_0\left(k_\rho\rho\right)k_\rho\mathrm{d}k_\rho,\ z\ge 0\end{cases}$$

where now Γ^{TM} is the reflection coefficient for incidence from the dielectric:

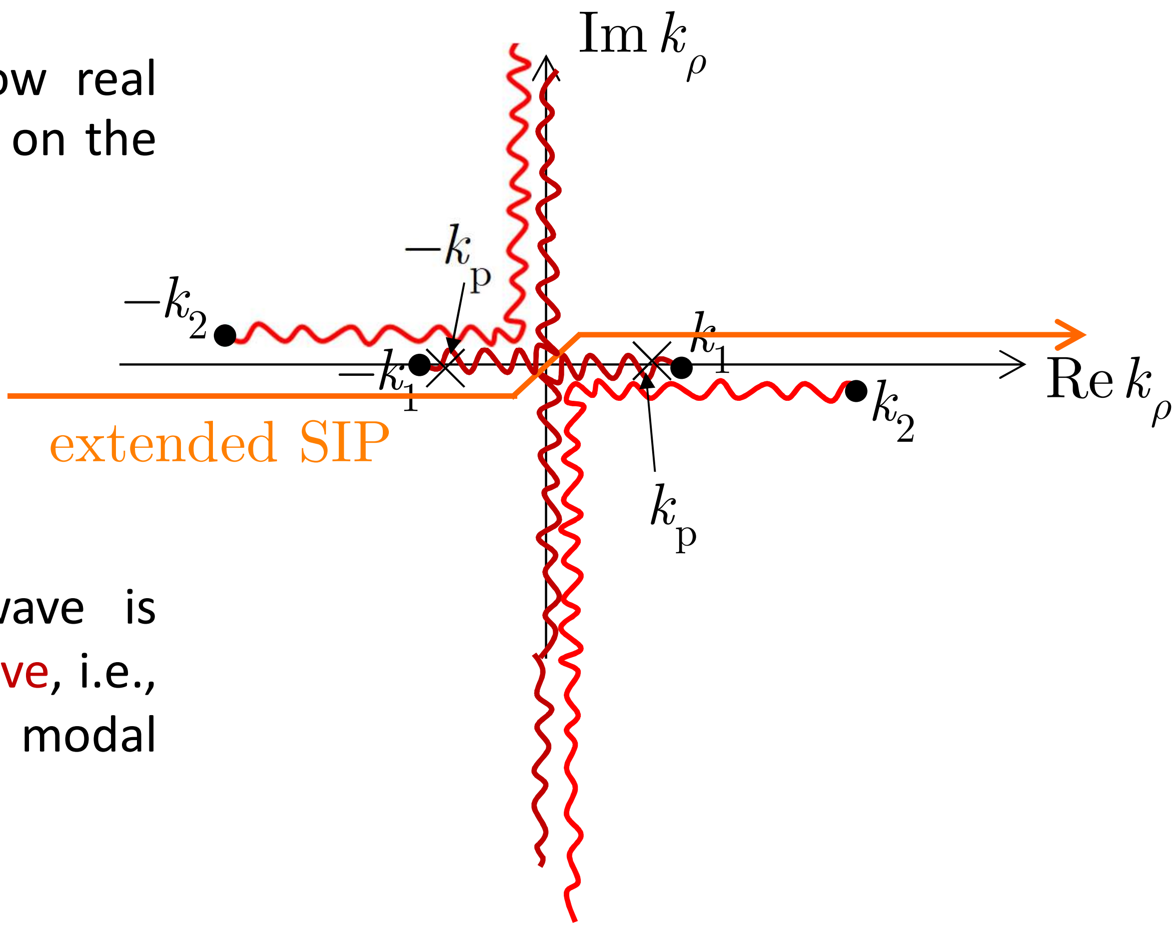
$$\Gamma^{\mathrm{TM}}=\frac{Z_{01}^{\mathrm{TM}}-Z_{02}^{\mathrm{TM}}}{Z_{01}^{\mathrm{TM}}+Z_{02}^{\mathrm{TM}}}=\frac{k_{z1}/\left(\omega\varepsilon_1\right)-k_{z2}/\left(\omega\varepsilon_2\right)}{k_{z1}/\left(\omega\varepsilon_1\right)+k_{z2}/\left(\omega\varepsilon_2\right)}=\frac{\varepsilon_{\mathrm{r}}k_{z1}-k_{z2}}{\varepsilon_{\mathrm{r}}k_{z1}+k_{z2}}$$

Singularities in the k_ρ-Plane

Since the branch points k_1 and k_2 are real, the relevant Sommerfeld branch cuts **both** run along the real and imaginary axes:

The Sommerfeld pole is now real and located to the left of k_1 on the lower rim of the k_1-BC:

$$k_p = k_1 \sqrt{\frac{\varepsilon_r}{\varepsilon_r + 1}} < k_1$$

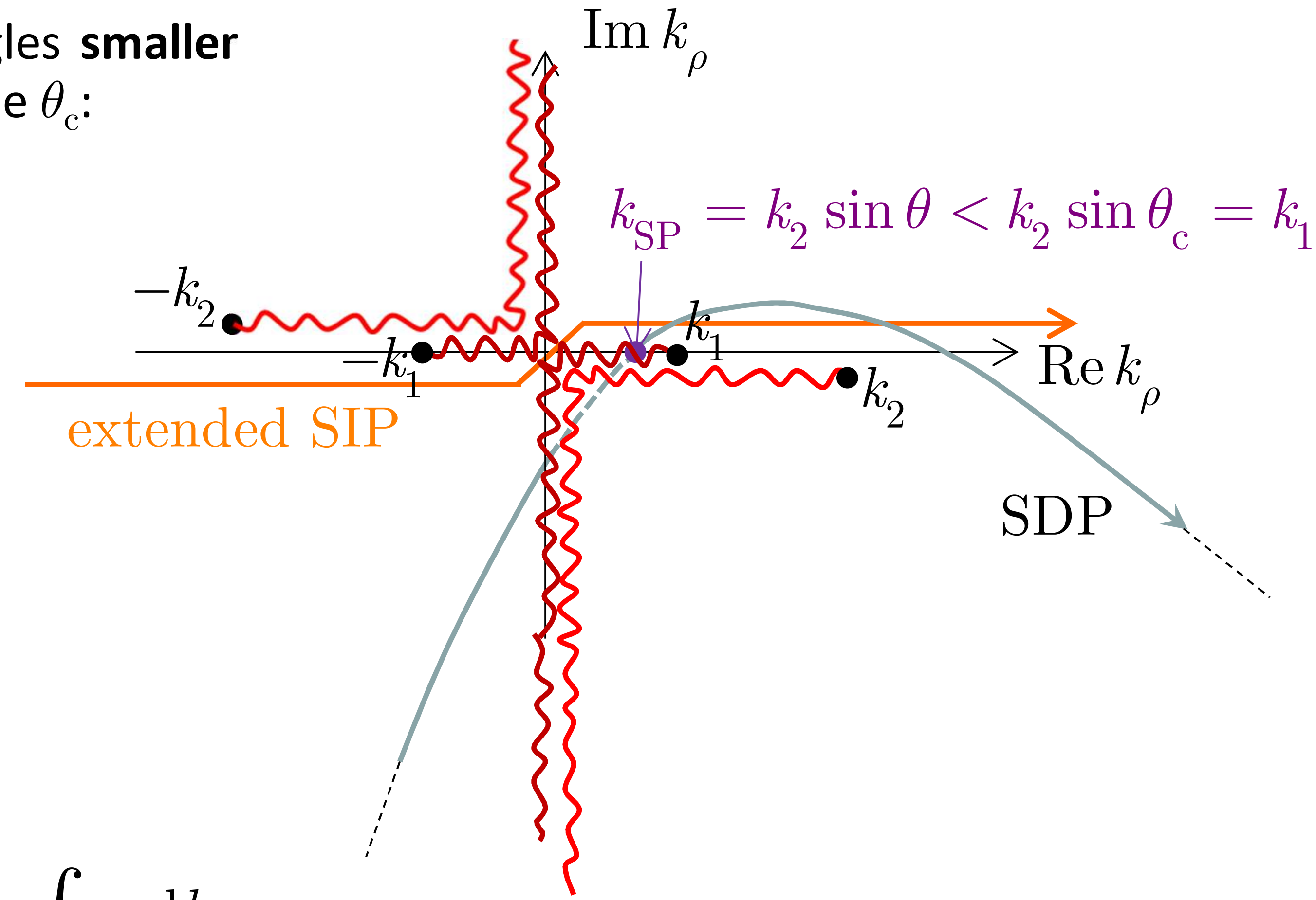


The associated Zenneck wave is now an improper surface wave, i.e., it does not belong to the modal spectrum of the structure.

Nonspectral Representation, $\theta < \theta_c$

For observation angles **smaller** than the critical angle θ_c:

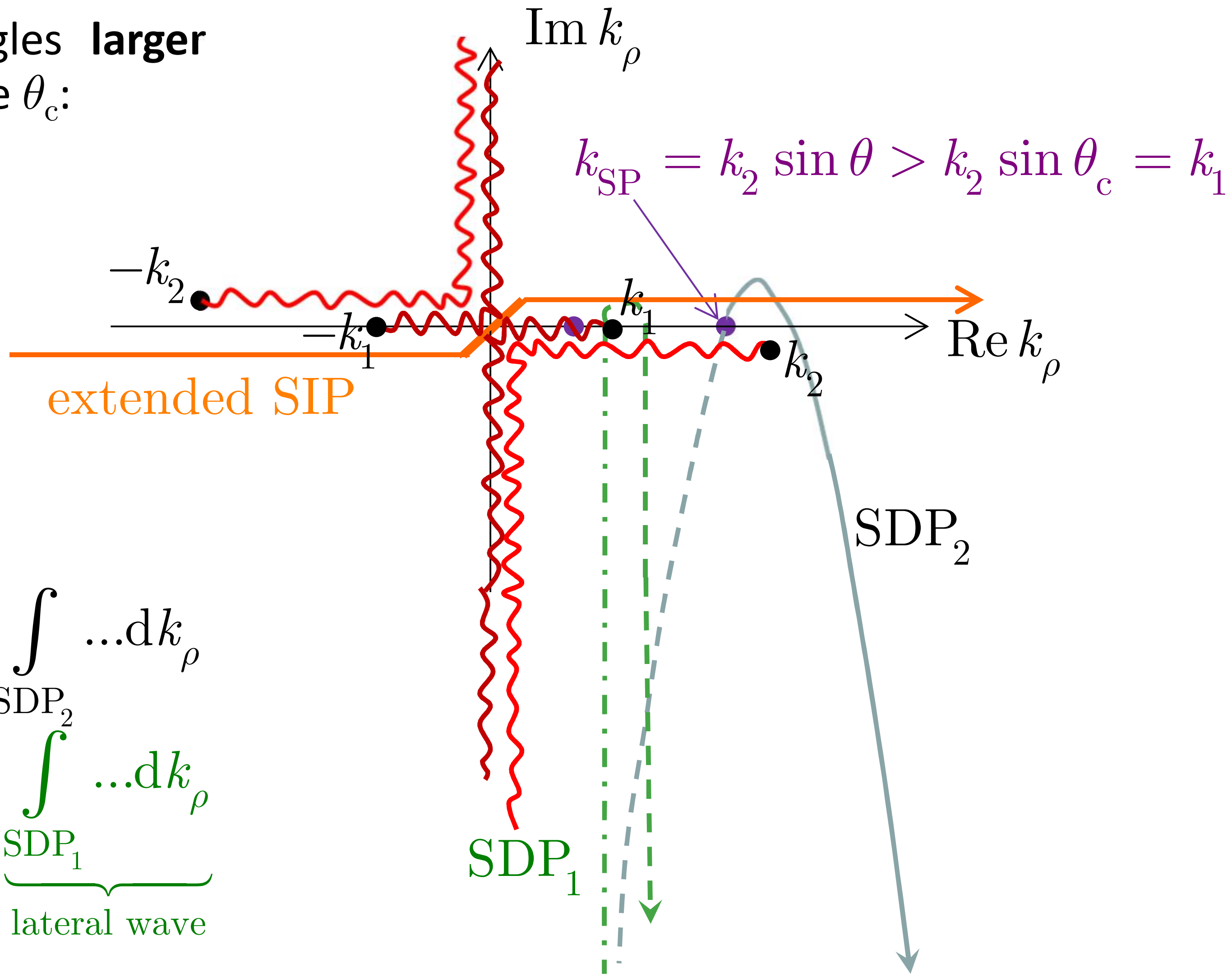
$$\theta_c = \arcsin\left(\frac{k_1}{k_2}\right) = \arcsin\frac{1}{\varepsilon_r}$$



Here $\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}} \ldots \mathrm{d}k_\rho$

Nonspectral Representation, $\theta > \theta_c$

For observation angles **larger** than the critical angle θ_c:



Here $\int_{\infty e^{-j\pi}}^{\infty} \ldots \mathrm{d}k_\rho = \int_{\mathrm{SDP}_2} \ldots \mathrm{d}k_\rho + \underbrace{\int_{\mathrm{SDP}_1} \ldots \mathrm{d}k_\rho}_{\text{lateral wave}}$

Asymptotic Evaluation of the Lateral Wave

The analytical expression of the lateral wave is:

$$A_z^{\mathrm{LatW}} = \frac{1}{4\pi} \int_{\mathrm{SDP}_1} \left(-\Gamma^{\mathrm{TM}}\right) \frac{e^{jk_{z2}\left(z+z'\right)}}{2jk_{z2}} H_0^{(2)}\left(k_\rho \rho\right) k_\rho \mathrm{d}k_\rho$$

By letting $k_\rho = k_1 - js$ this integral along the folded SDP_1 path becomes

$$A_z^{\mathrm{LatW}} = \frac{1}{4\pi} \int_0^{+\infty} \left[\!\left[-\Gamma^{\mathrm{TM}}\right]\!\right] \left[\frac{e^{jk_{z2}\left(z+z'\right)}}{2jk_{z2}} H_0^{(2)}\left(k_\rho \rho\right) k_\rho \right]_{k_\rho = k_1 - js} \left(-j\right) \mathrm{d}s$$

where

$$\left[\!\left[-\Gamma^{\mathrm{TM}}\right]\!\right] = \left(-\Gamma^{\mathrm{TM}}\right)^{+} - \left(-\Gamma^{\mathrm{TM}}\right)^{-}$$

downward leg of the SDP_1: k_{z1} proper, k_{z2} improper

upward leg of the SDP_1: k_{z1} improper, k_{z2} improper

Asymptotic Evaluation of the Lateral Wave

Now,

$$-\Gamma^{\mathrm{TM}} = -\frac{\varepsilon_{\mathrm{r}} k_{z1} - k_{z2}}{\varepsilon_{\mathrm{r}} k_{z1} + k_{z2}} = -\frac{\left(\varepsilon_{\mathrm{r}} k_{z1} - k_{z2}\right)^2}{\left(\varepsilon_{\mathrm{r}} k_{z1}\right)^2 - \left(k_{z2}\right)^2} = -\frac{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 + k_{z2}^2 - 2\varepsilon_{\mathrm{r}} k_{z1} k_{z2}}{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 - k_{z2}^2}$$

$$= \underbrace{-\frac{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 + k_{z2}^2}{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 - k_{z2}^2}}_{\text{without BP at } k_{z1}} + \frac{2\varepsilon_{\mathrm{r}} k_{z1} k_{z2}}{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 - k_{z2}^2}$$

hence

$$\left[\!\left[-\Gamma^{\mathrm{TM}} \right]\!\right] = \frac{2\varepsilon_{\mathrm{r}} k_{z2}}{\varepsilon_{\mathrm{r}}^2 k_{z1}^2 - k_{z2}^2} \left[\!\left[k_{z1} \right]\!\right]$$

where

$$k_{z1} = \sqrt{k_1^2 - k_\rho^2} = \sqrt{k_1^2 - \left(k_1 - js\right)^2} = \sqrt{2js + s^2}$$

$$k_{z2} = \sqrt{k_2^2 - k_\rho^2} = \sqrt{k_2^2 - \left(k_1 - js\right)^2} = \sqrt{k_2^2 - k_1^2 + 2js + s^2}$$

Asymptotic Evaluation of the Lateral Wave

We will use Watson's lemma to evaluate the SDP$_1$ integral, so we need to determine the local behavior of the integrand in the neighborhood of the BP $k_\rho = k_1$, i.e., of $s = 0$.

We thus approximate

$$k_{z1} = \sqrt{2jk_1 s + s^2} \sim \sqrt{2jk_1 s} = k_1^{1/2} s^{1/2} \begin{cases} -1-j & \text{proper} \\ 1+j & \text{improper} \end{cases}$$

and

$$k_{z2} = \sqrt{k_2^2 - k_1^2 + 2js + s^2} \sim \sqrt{k_2^2 - k_1^2}$$

so that

$$\left[\!\left[-\Gamma^{\mathrm{TM}} \right]\!\right] \sim \frac{2\varepsilon_{\mathrm{r}} \sqrt{k_2^2 - k_1^2}}{-\left(k_2^2 - k_1^2\right)} k_1^{1/2} s^{1/2} \left[\left(-1-j\right) - \left(1+j\right) \right]$$

Asymptotic Evaluation of the Lateral Wave

i.e., $\left[\!\left[-\Gamma^{\mathrm{TM}} \right]\!\right] \sim \Gamma_0 s^{1/2}$ with $\Gamma_0 = \dfrac{4\sqrt{2} e^{j\pi/2} \varepsilon_{\mathrm{r}} k_1^{1/2}}{\left(\varepsilon_{\mathrm{r}} - 1\right)^{1/2}}$

Coming back to the integral, in the asymptotic limit $\rho \to +\infty$ we have

$$A_z^{\mathrm{LatW}} \sim \frac{1}{4\pi} \sqrt{\frac{2j}{\pi\rho}} \Gamma_0 \frac{e^{-j\sqrt{k_2^2 - k_1^2}\left|z+z'\right|}}{2j\sqrt{k_2^2 - k_1^2}} e^{-jk_1\rho} k_1^{1/2} \underbrace{\int_0^{+\infty} s^{1/2} e^{-s\rho} \mathrm{d}s}_{=\frac{\Gamma(3/2)}{\rho^{3/2}}}$$

$$= -\frac{j}{\pi} \frac{\varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}} - 1} e^{-j\left[k_1\rho + \sqrt{k_2^2 - k_1^2}\left|z+z'\right|\right]} \frac{1}{\rho^2}$$

Analysis of the Phase Term

The **phase** in the exponential term can be rewritten as

$$k_1\rho + \sqrt{k_2^2 - k_1^2}\,|z+z'| = k_1\rho + \frac{k_2^2 - k_1^2}{\sqrt{k_2^2 - k_1^2}}|z+z'|$$

$$= k_1\left[\rho - \frac{k_1}{\sqrt{k_2^2 - k_1^2}}|z+z'|\right] + k_2\frac{k_2}{\sqrt{k_2^2 - k_1^2}}|z+z'|$$

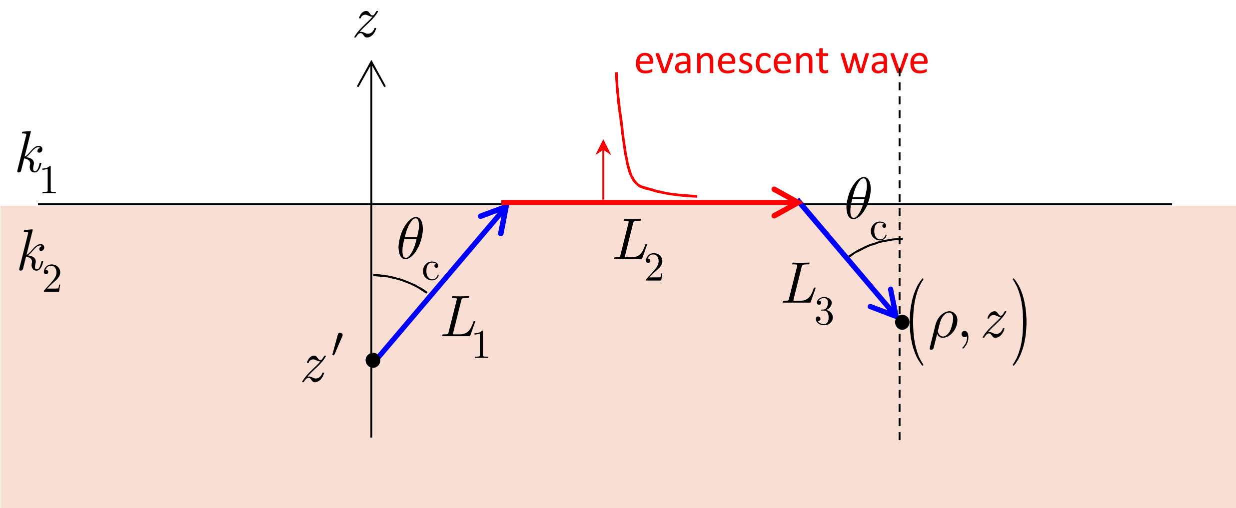
Now, from $k_1 = k_2 \sin\theta_c$ we have

$$\frac{k_1}{\sqrt{k_2^2 - k_1^2}} = \tan\theta_c, \qquad \frac{k_2}{\sqrt{k_2^2 - k_1^2}} = \frac{1}{\cos\theta_c}$$

Physical Interpretation

Therefore, the **phase** of the lateral wave can be cast in the form

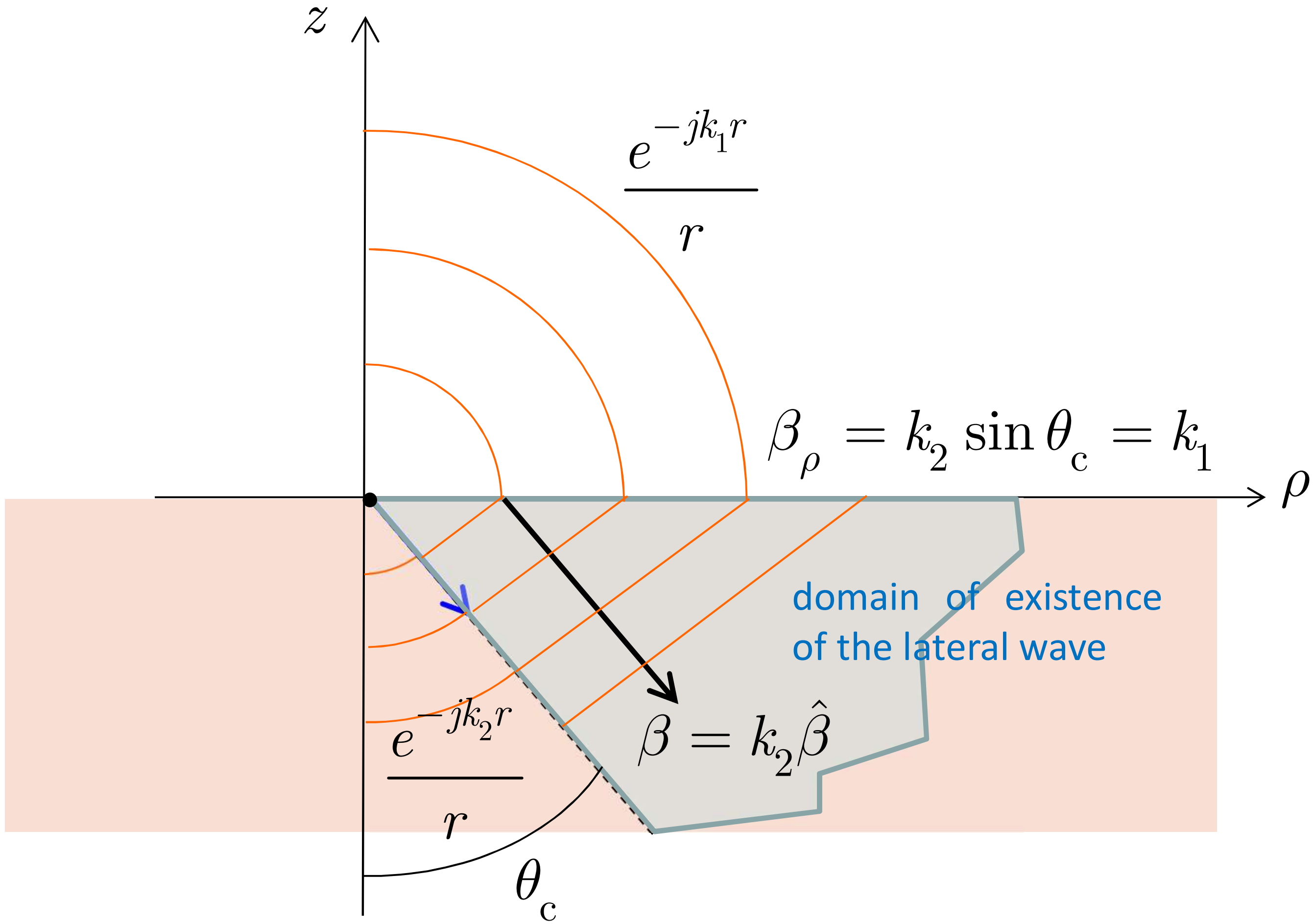
$$k_1 \underbrace{\left[\rho - \tan\theta_c \left|z + z'\right|\right]}_{L_2} + k_2 \underbrace{\frac{\left|z + z'\right|}{\cos\theta_c}}_{L_1 + L_3} = k_1 L_2 + k_2 \left(L_1 + L_3\right)$$



This substantiates the physical mechanism underlying the lateral wave alluded to in the introductory slides.

Phase Matching at the Shadow Boundaries

Note that on the boundaries of the region of existence of the lateral wave (i.e., its *shadow boundaries*, in ray-optical terms) there is phase matching between the lateral wave and the (spherical) GO waves propagating in both media:

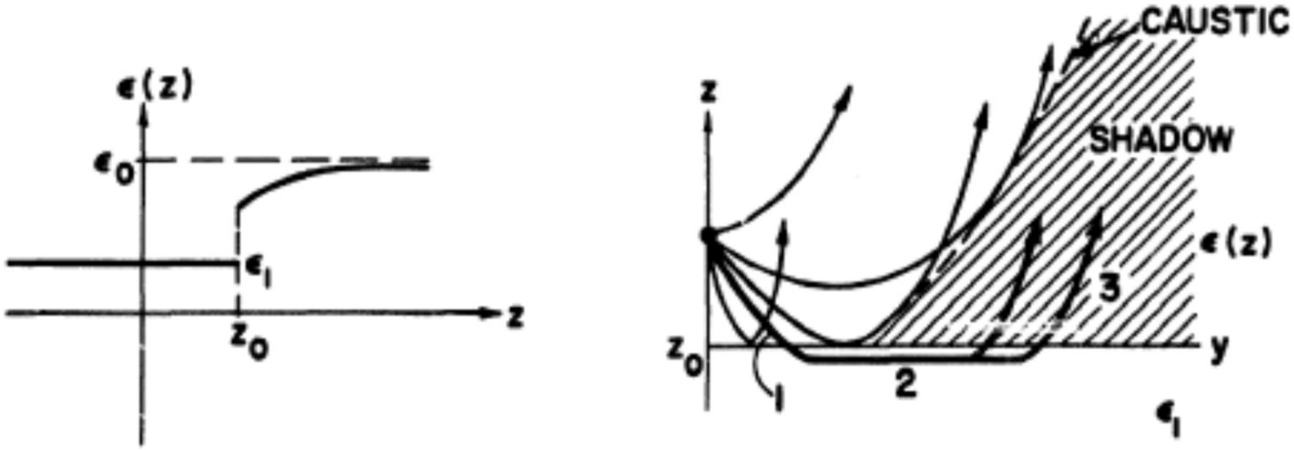


Amplitude Term: the Asymptotic Order

Note that the order of decay of the lateral wave ($1/\rho^2$) is **larger** than that of a spherical wave ($1/\rho$), hence the lateral wave is a diffraction effect generally *weaker* than the direct and reflected GO fields.

For observation points at the interface, the direct and reflected GO terms tend to cancel for large radial distances. In this case the SDP_2 integral gives rise to a wave decaying as $1/\rho^{3/2}$, still *dominant* over the lateral wave.

An exception occurs when the direct and reflected GO fields are excluded from certain domains which are nevertheless accessible to the lateral wave. This situation may arise if the denser medium is inhomogeneous:

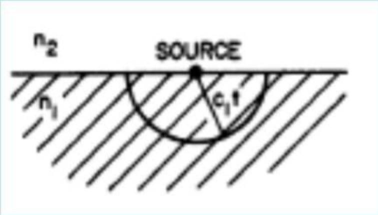


Lateral Waves in the Time Domain

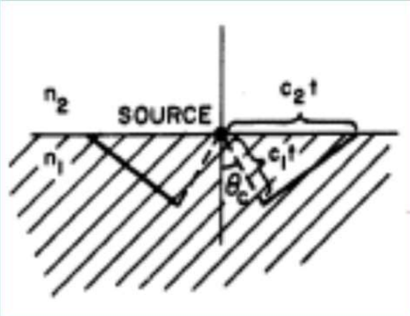
Whereas the lateral wave may be difficult to detect in the time-harmonic regime, the situation is completely different in the transient regime.

In fact, in transient conditions the different wave constituents at an observation point may be distinguished by their different arrival times.

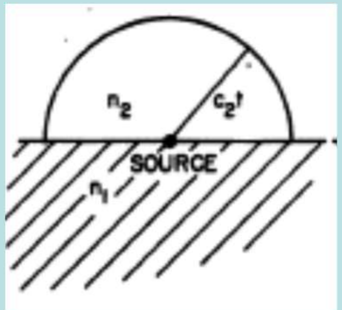
The lateral wave may furnish the **first response** in certain regions of the medium containing the source (and for this reason it is also known as the **head wave**):



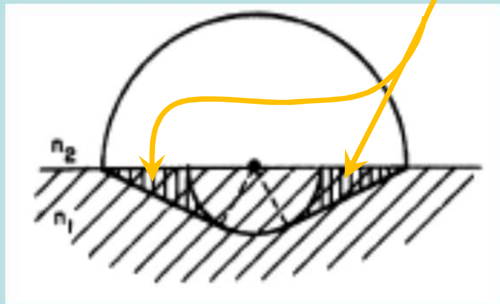
Direct-Reflected



Lateral



Refracted



Composite

Wavefronts

Fermat Principle: Feynman Revisited

In a famous passage of the *Feynman Lectures on Physics*, Snell's law for light refraction is derived from Fermat's Principle of minimum time in a colorful way:

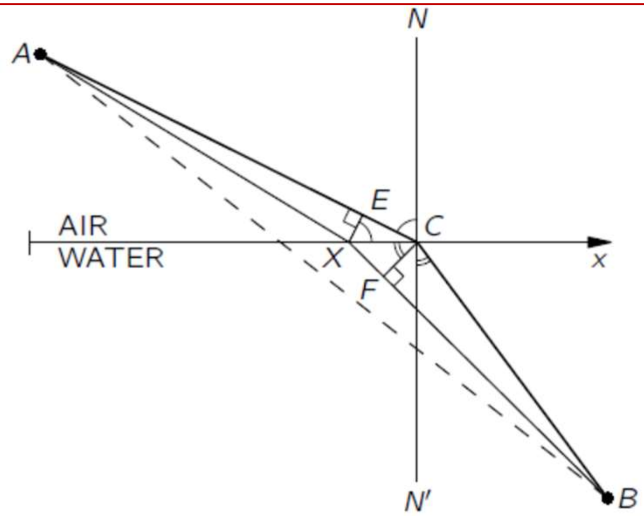


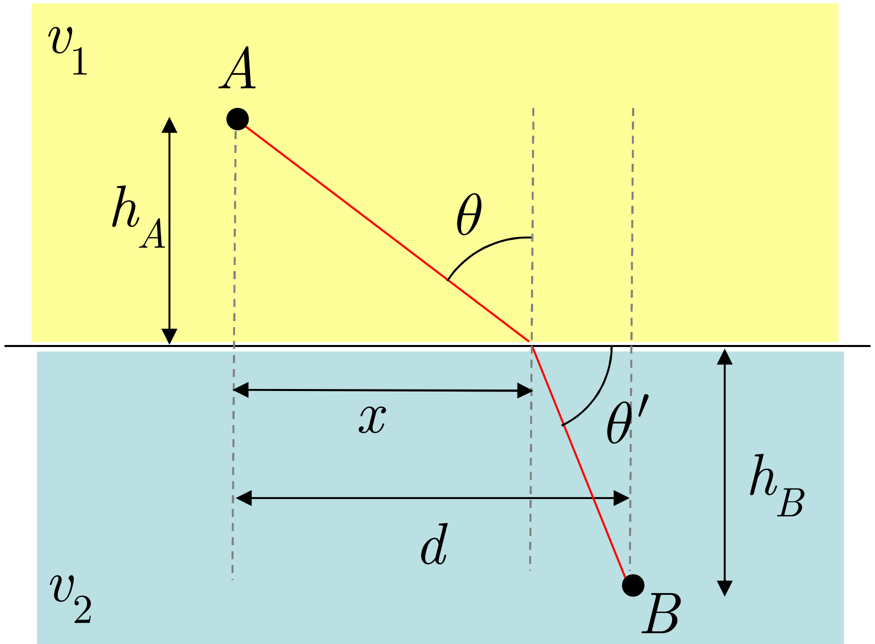
Fig. 26-4. Illustration of Fermat's principle for refraction.

In Fig. 26-4, our problem is again to go from A to B in *the shortest time*. To illustrate that the best thing to do is not just to go in a straight line, let us imagine that a beautiful girl has fallen out of a boat, and she is screaming for help in the water at point B. The line marked x is the shoreline. We are at point A on land, and we see the accident, and we can run and can also swim.

R. P. Feynman, *The Feynman Lectures on Physics*. New York, NY: Basic Books, the New Millennial Edition, 2011, vol. I, sec. 26-3.

Fermat Principle: Feynman Revisited

Let us find the value of x for which the travel time from A to B is minimum:



$$t(x) = \frac{\sqrt{h_A^2 + x^2}}{v_1} + \frac{\sqrt{h_B^2 + (d-x)^2}}{v_2}$$

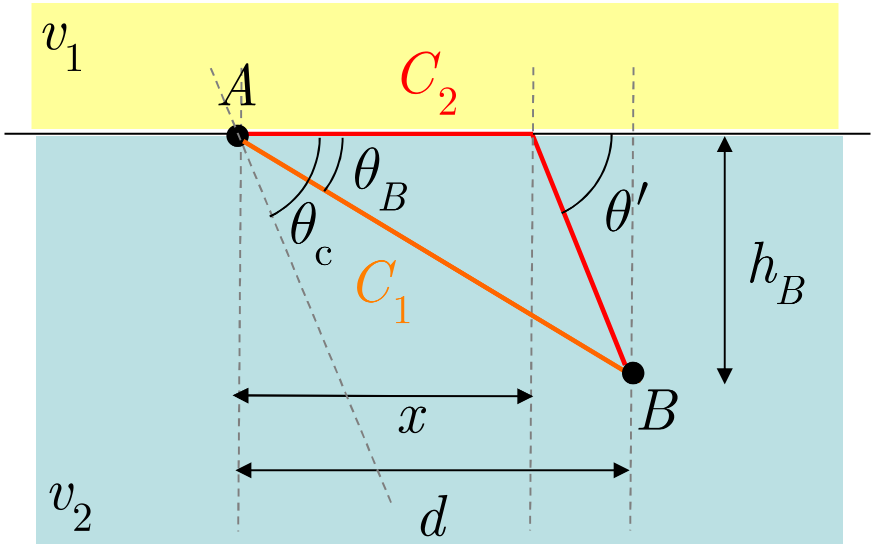
$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{v_1} \underbrace{\frac{x}{\sqrt{h_A^2 + x^2}}}_{\sin\theta} - \frac{1}{v_2} \underbrace{\frac{(d-x)}{\sqrt{h_B^2 + (d-x)^2}}}_{\sin\theta'}$$

$$\frac{1}{v_1}\sin\theta - \frac{1}{v_2}\sin\theta' = 0 \quad \Rightarrow \quad \boxed{\sin\theta = \frac{v_1}{v_2}\sin\theta' = n\sin\theta'} \qquad n = \frac{v_1}{v_2}$$

Snell's law

Fermat Principle: Feynman Revisited

But what if A is *exactly on the shoreline*? Now we have two competing paths:



$$t_1 = \frac{\sqrt{h_B^2 + d^2}}{v_2}$$

$$t_2(x) = \frac{x}{v_1} + \frac{\sqrt{h_B^2 + (d-x)^2}}{v_2}$$

$$\frac{\mathrm{d}t_2}{\mathrm{d}x} = \frac{1}{v_1} - \frac{1}{v_2} \underbrace{\frac{(d-x)}{\sqrt{h_B^2 + (d-x)^2}}}_{\sin\theta'}$$

$$\frac{1}{v_1} - \frac{1}{v_2}\sin\theta' = 0 \Rightarrow \sin\theta' = \frac{v_2}{v_1} = \frac{1}{n} = \sin\theta_c \Rightarrow \boxed{\theta' = \theta_c}$$

and it is simple to show that, if $\theta_B < \theta_c$ (as in the above figure), then the shortest time from A to B is achieved choosing path C_2 (i.e., the *lateral wave*).

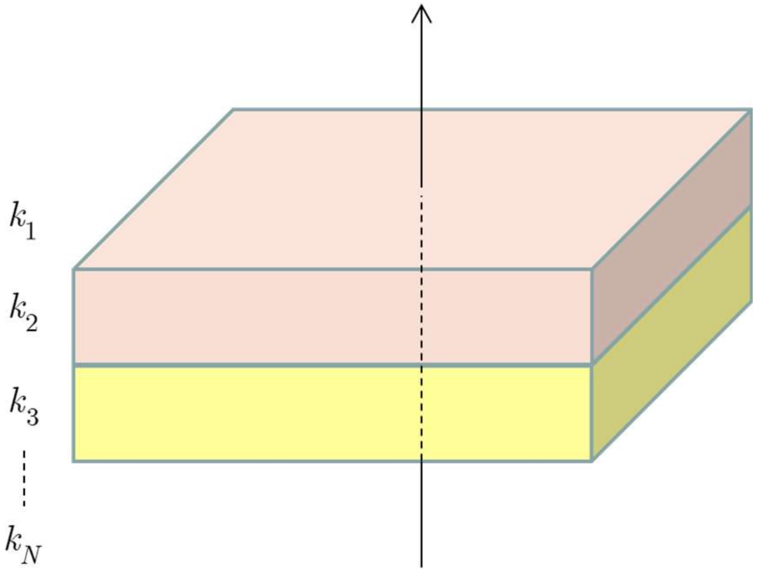
Surface and Leaky Waves

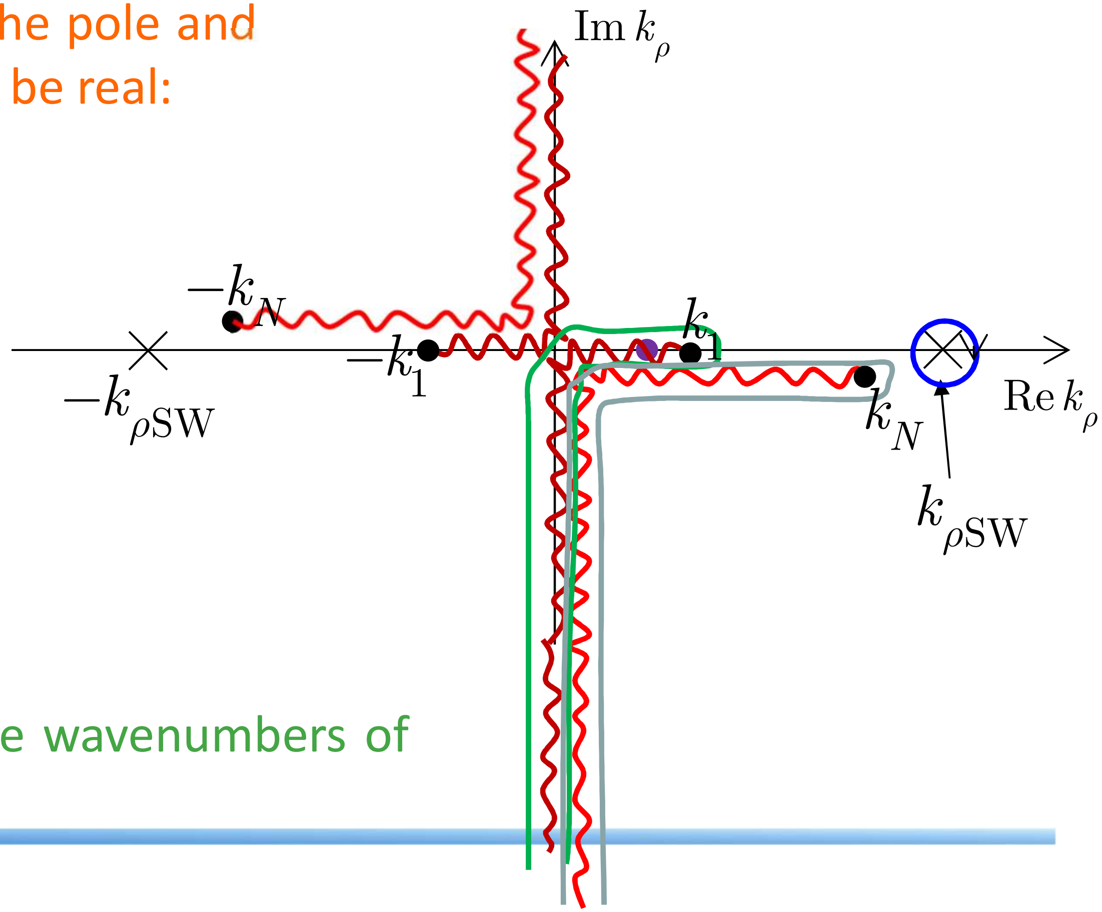
Surface Waves: Definition

Surface waves (SWs) are wave constituents that arise in a **spectral** representation of the field as residue contributions of **poles located on the proper sheet**.

The Zenneck wave is an example of surface wave; in that case, due to the losses in the dielectric medium, the wavenumber is complex.

In a lossless multilayer structure the pole and hence the SW wavenumber would be real:

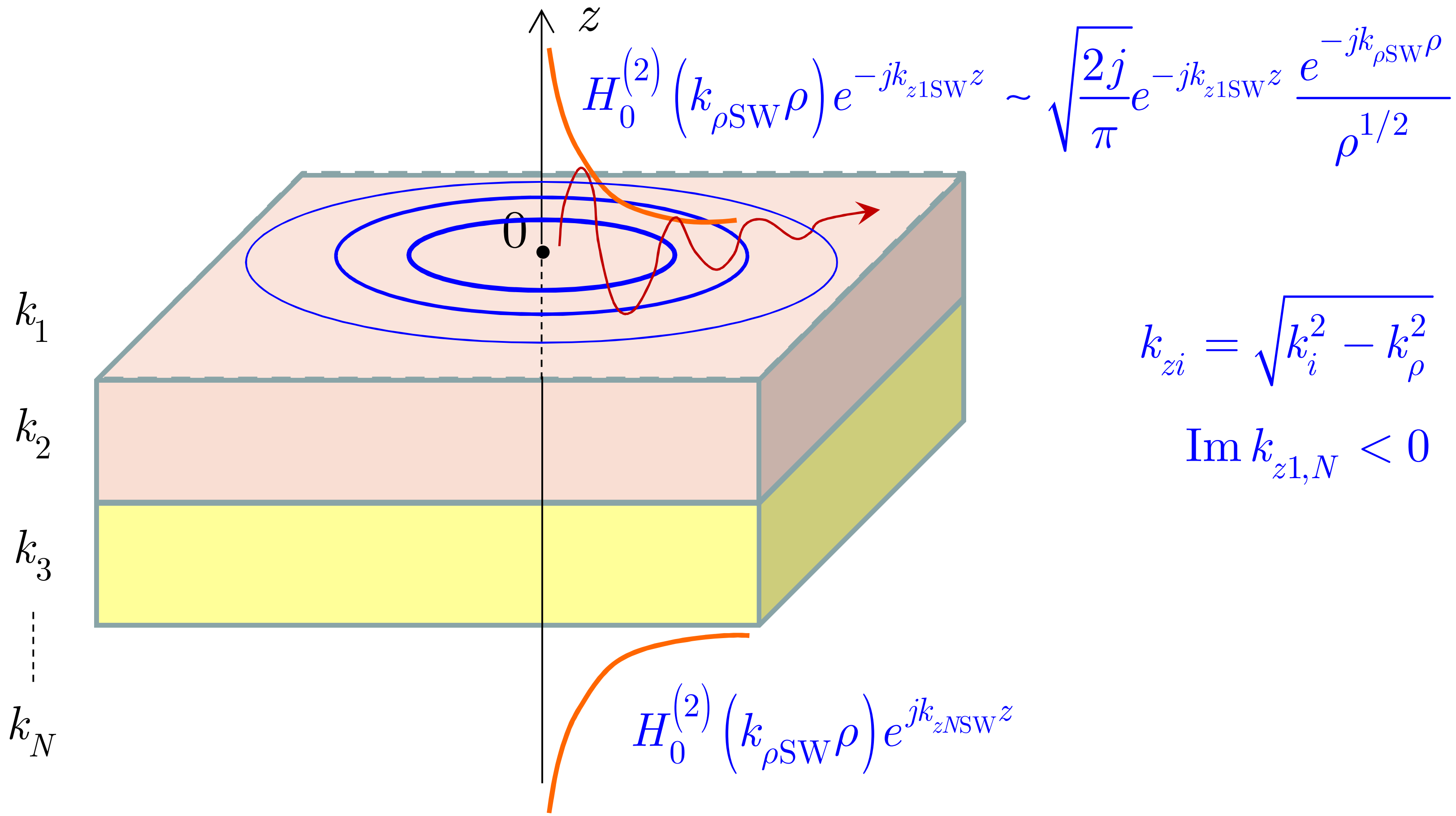




Remark: the BPs are located at the wavenumbers of the outer media k_1 and k_N.

Surface Waves: Behavior at Infinity

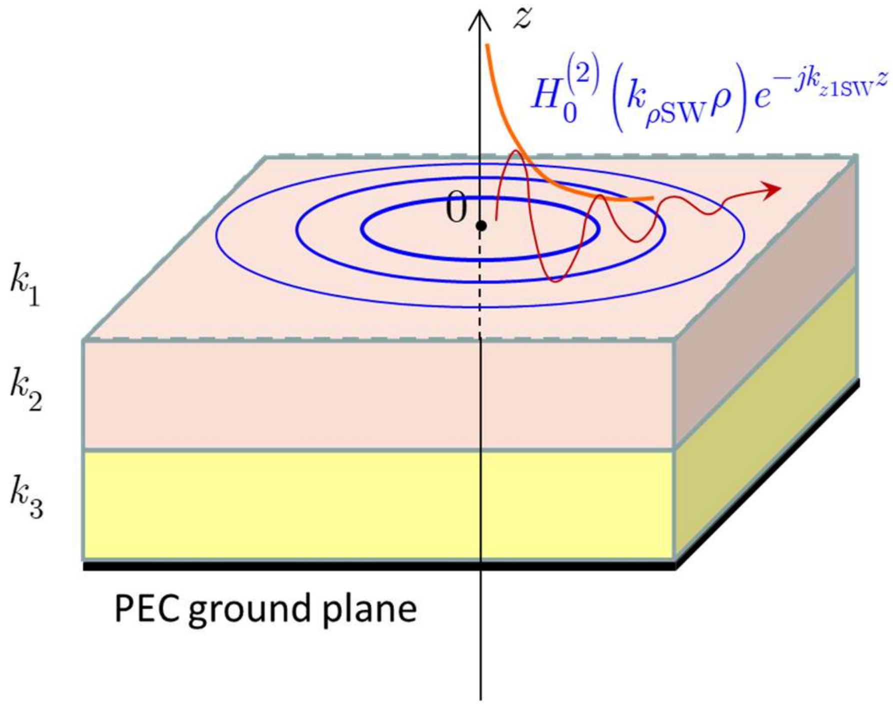
For a real SW pole it results $k_{\rho SW} > k_{1,N}$, hence the field attenuates exponentially at infinity in the outer media:

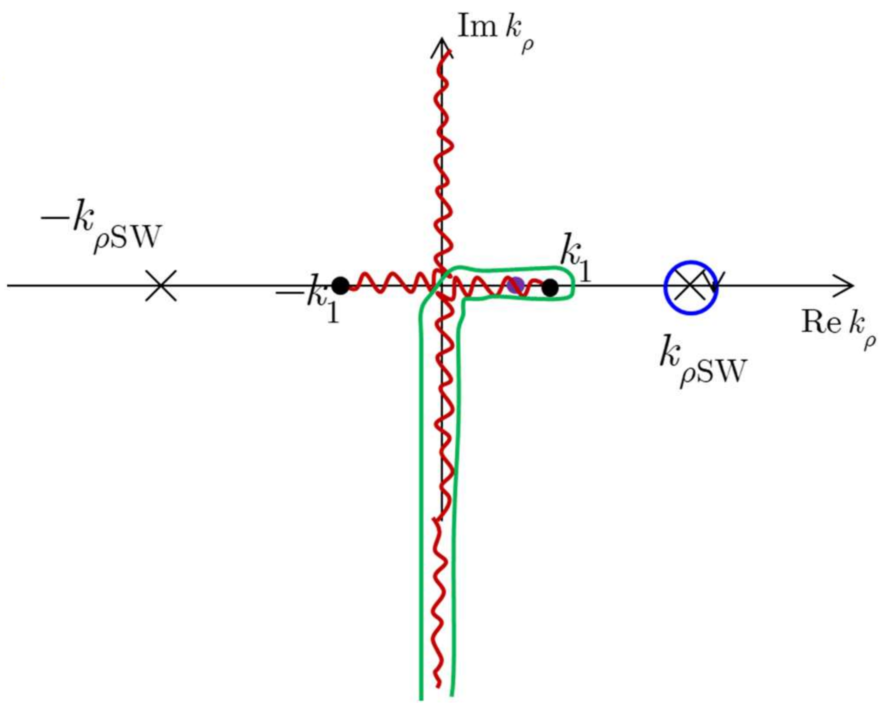


As we know, this is in fact the behavior of the Zenneck or SPP waves.

Surface Waves on Grounded Multilayers

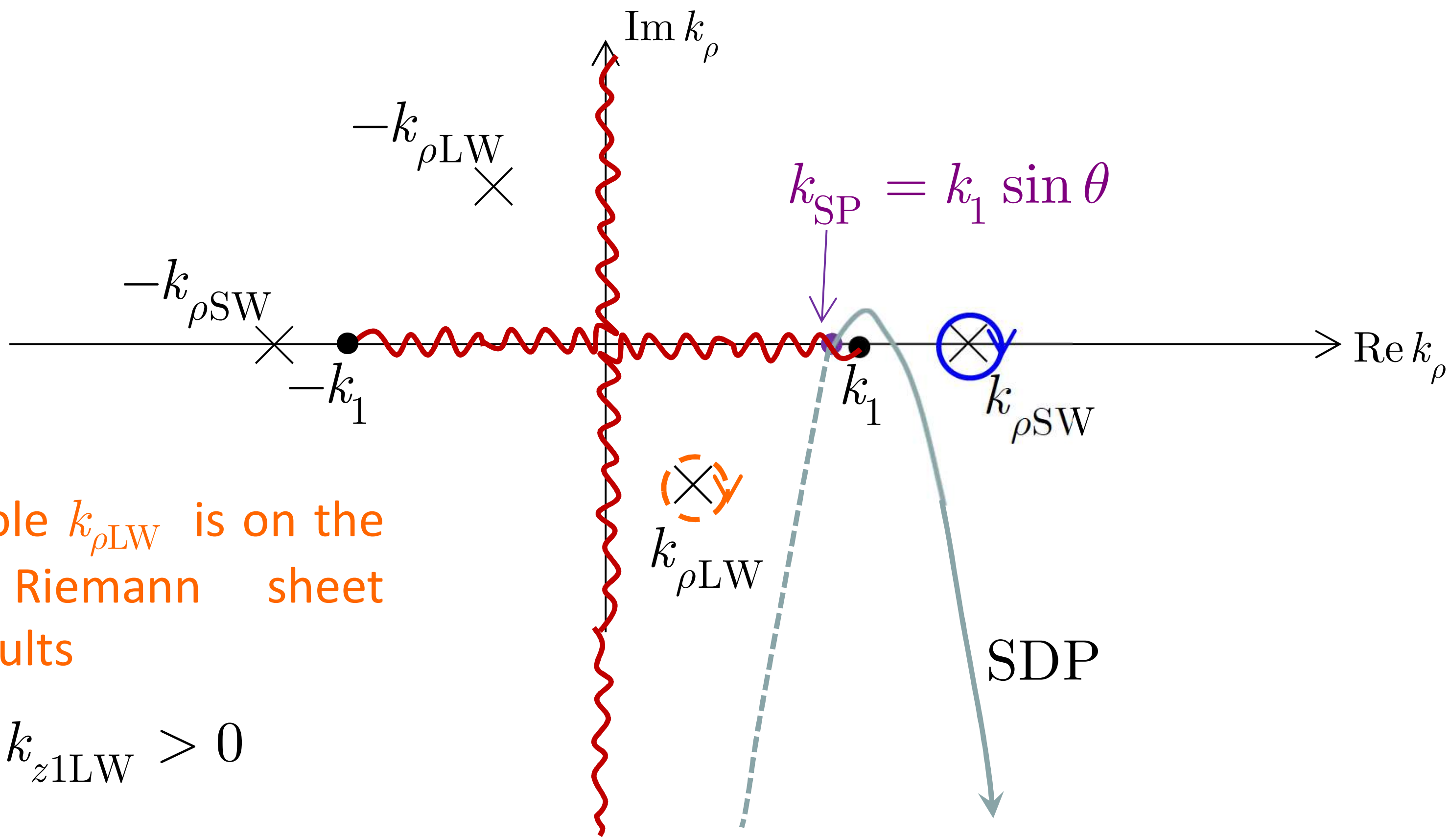
If a PEC ground plane is present at the bottom of the multilayer, a single pair of BPs at $\pm k_1$ is present, and the SWs attenuate exponentially only for $z \to +\infty$.





Leaky Waves: Definition

When the integration path is deformed to the SDP in the nonspectral representation, further **complex poles** (usually) located on **improper** sheet(s) may give their residue contribution; the relevant wave constituent is called a **leaky wave (LW)**.



Here the pole $k_{\rho\mathrm{LW}}$ is on the improper Riemann sheet hence it results

$$\mathrm{Im}\, k_{z1\mathrm{LW}} > 0$$

Leaky Waves: Phase and Attenuation Constants

The wavenumber of a leaky wave is **always complex**:

$$k_{\rho \mathrm{LW}} = \beta - j\alpha$$

phase constant (β)

attenuation (or leakage) constant (α)

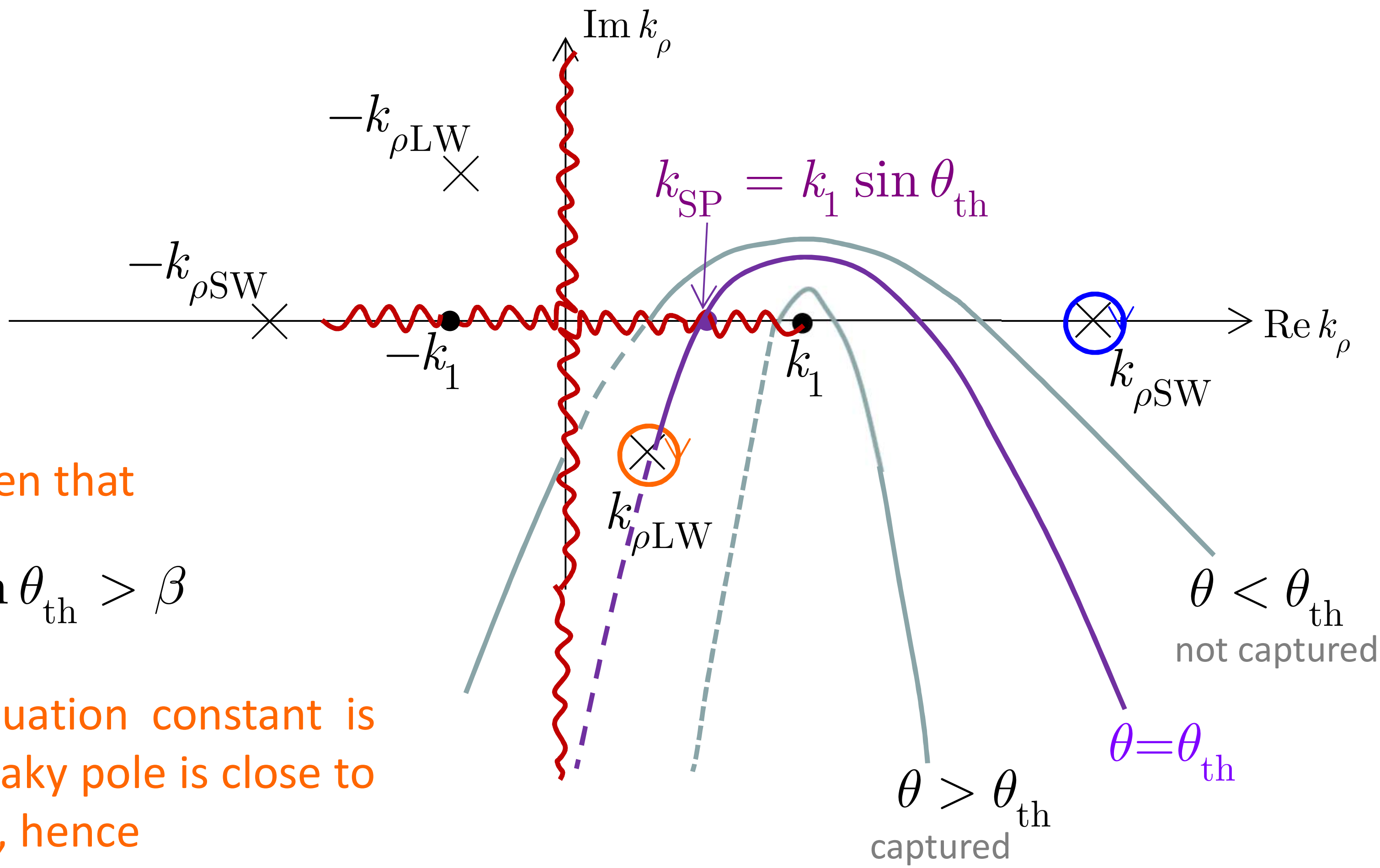
The imaginary part is present also if all the media are lossless.

If it is not too large it accounts for **power leakage via radiation** (hence the name *leaky or radiation modes*) concomitant to the propagation of the wave (in addition to power dissipation in lossy media, if present).

If it is large, it mostly represents **reactive effects** (much as in metal waveguide modes being below cutoff).

Leaky Waves: Angular Region of Definition

A leaky pole is captured only if the observation angle is larger than a threshold value θ_{th}:



It can be seen that

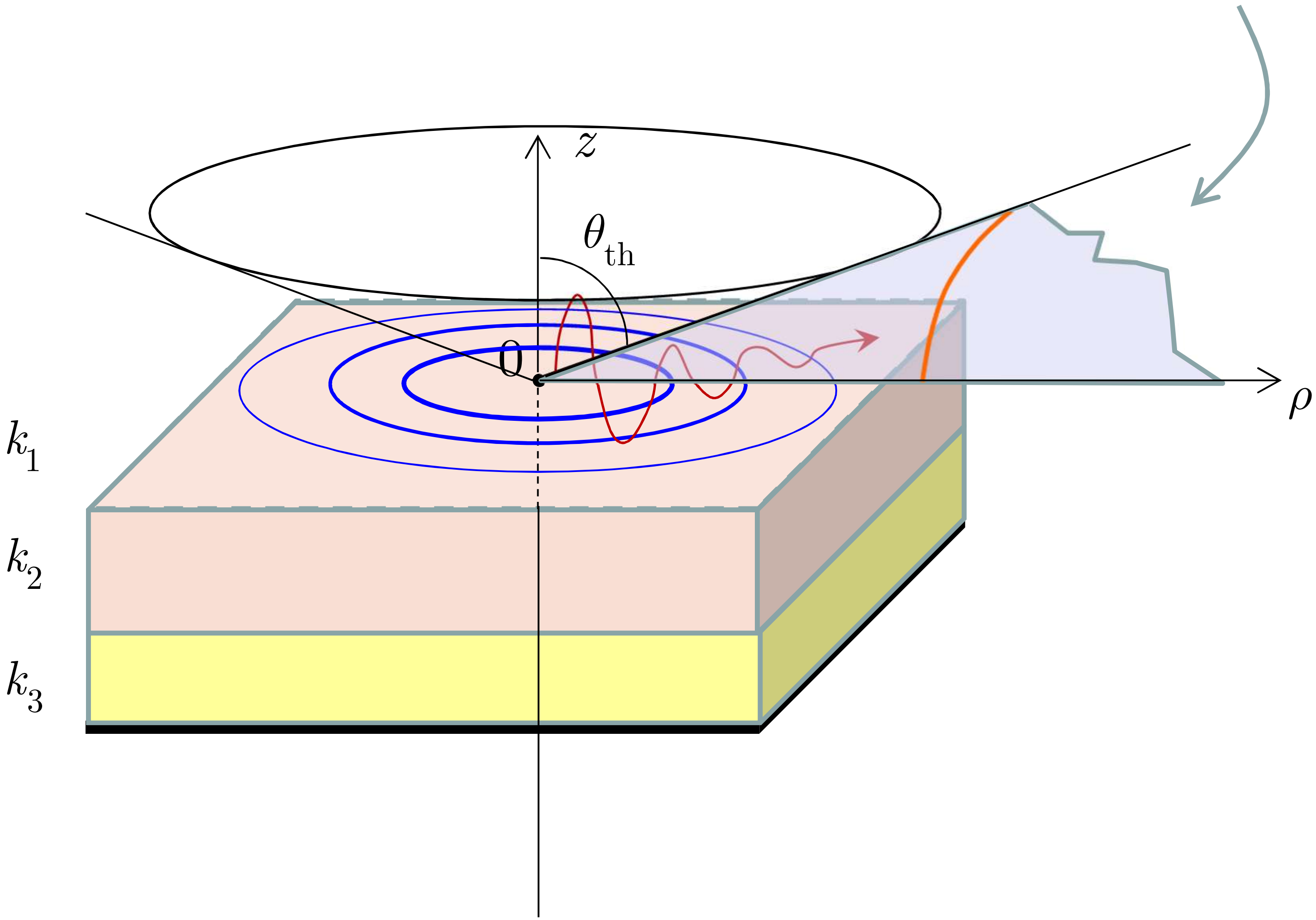
$$k_1 \sin\theta_{th} > \beta$$

If the attenuation constant is small, the leaky pole is close to the real axis, hence

$$k_1 \sin\theta_{th} \simeq \beta$$

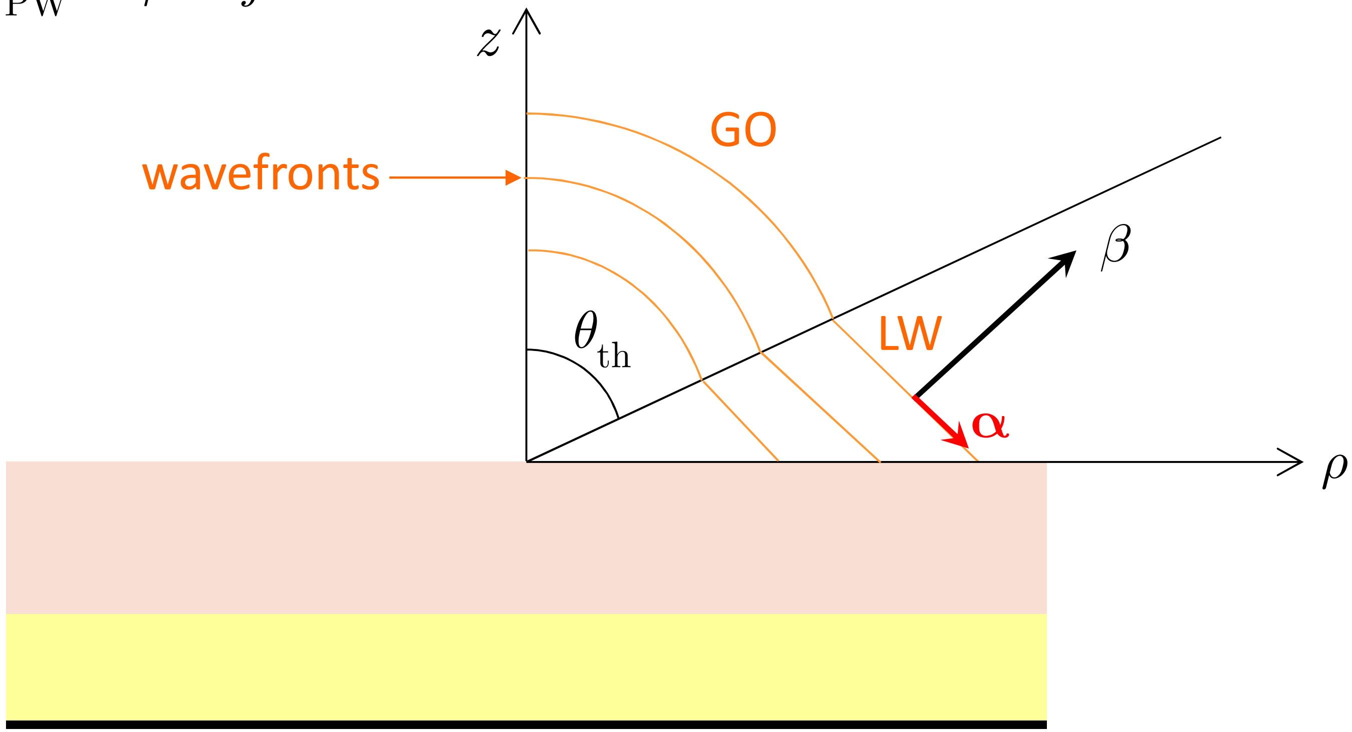
Leaky Waves: Angular Region of Definition

A leaky wave thus exists only inside the **bounded angular region** $\theta_{\text{th}} < \theta < \pi/2$:



Leaky Waves: Plane-Wave Constituents

The plane-wave constituents of a leaky wave inside the air region are **nonuniform**: $\mathbf{k}_{\mathrm{PW}} = \boldsymbol{\beta} - j\boldsymbol{\alpha}$

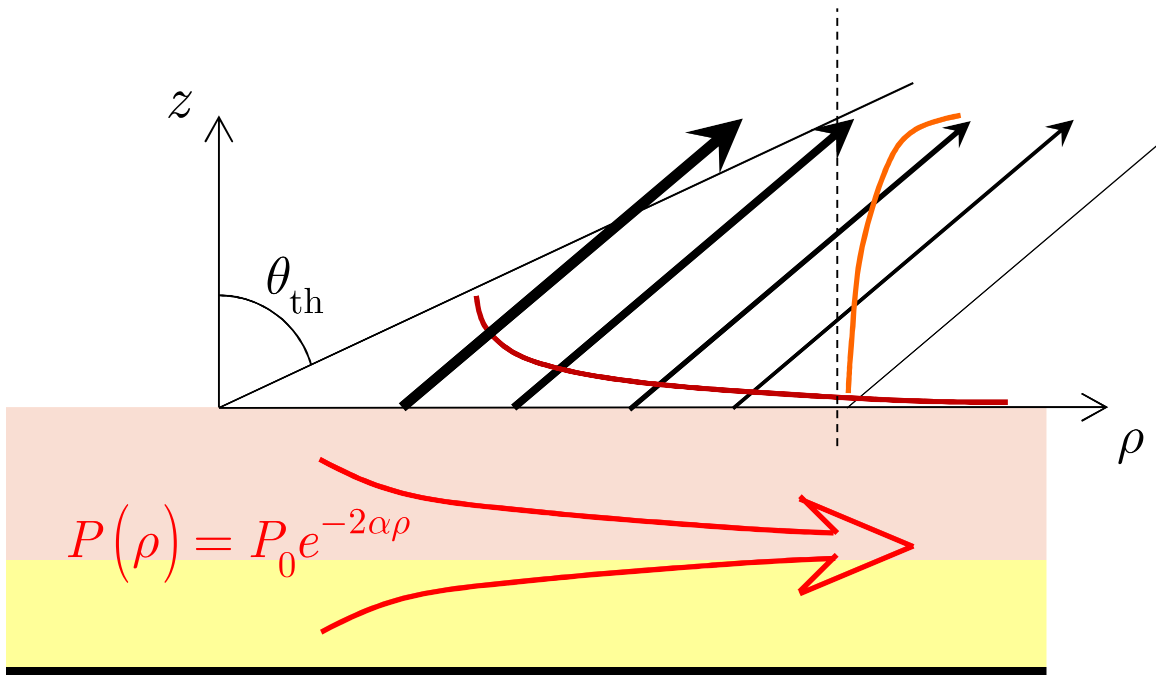


The component of the phase vector along the direction θ_{th} is equal to k_1 (exercise: prove this!), so we have **phase matching** between the LW and the spherical wave arising from the saddle point (GO field) along the cone $\theta=\theta_{\mathrm{th}}$ (i.e., on the LW *shadow boundary*).

Leaky Waves: Improper Nature

In uniform structures with common materials, leaky poles lie on *improper* Riemann sheets, therefore **the amplitude of the associated LW increases exponentially at infinity**. This *unphysical* feature of the LW puzzled the first who studied them in the 40s and 50s of the last century…

However, the exponential increase **actually occurs only inside their angular region of existence**, where it can be justified with a simple ray model:



Leaky Waves and Far-Field Patterns

Since a LW is exponentially attenuated in the radial direction within its angular region of validity, **is is always asymptotically negligible in the far field** w.r.t. the GO field (also termed the *space wave*), whose radial decay is algebraic ($1/\rho$).

However, if:

1- the LW pole is captured for some angles, i.e., $\beta < k_1$ (hence it is *radially fast*)
2- it is close to the real axis, i.e., $\alpha \ll \beta$ (the LW is *weakly attenuated*)
3- its residue is not negligible (i.e., the LW is *well excited* by the source)

then the LW pole may provide the dominant contribution to the integrand between $k_\rho = 0$ and $k_\rho = k_1$.

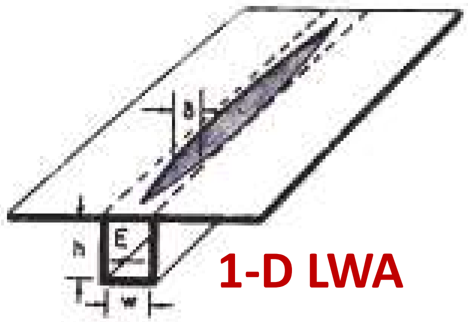
Since such an integrand, evaluated at $k_\rho = k_1 \sin\theta$, is the **far-field pattern** of the source, this means that the presence of a LW pole may determine the main features of such a pattern.

Near-Field Dominance and Far-Field Patterns

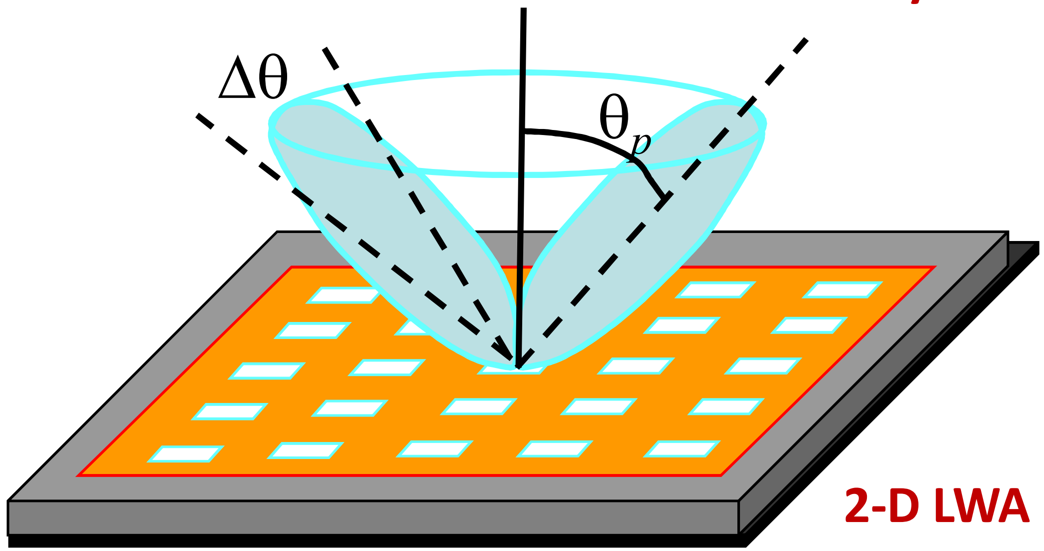
Alternatively, one may say that, under the above-listed conditions, the residue contribution of the LW pole **dominates the field at the air-dielectric interface** (i.e., the aperture field in the antenna jargon).

Since the far-field pattern is (proportional to) the Fourier transform of the aperture field, again we come to the conclusion that the pattern is essentially established by the excitation of the LW along the structure.

This is the operating principle of a wide class of radiators known as **Leaky-Wave Antennas (LWAs)...**



1-D LWA

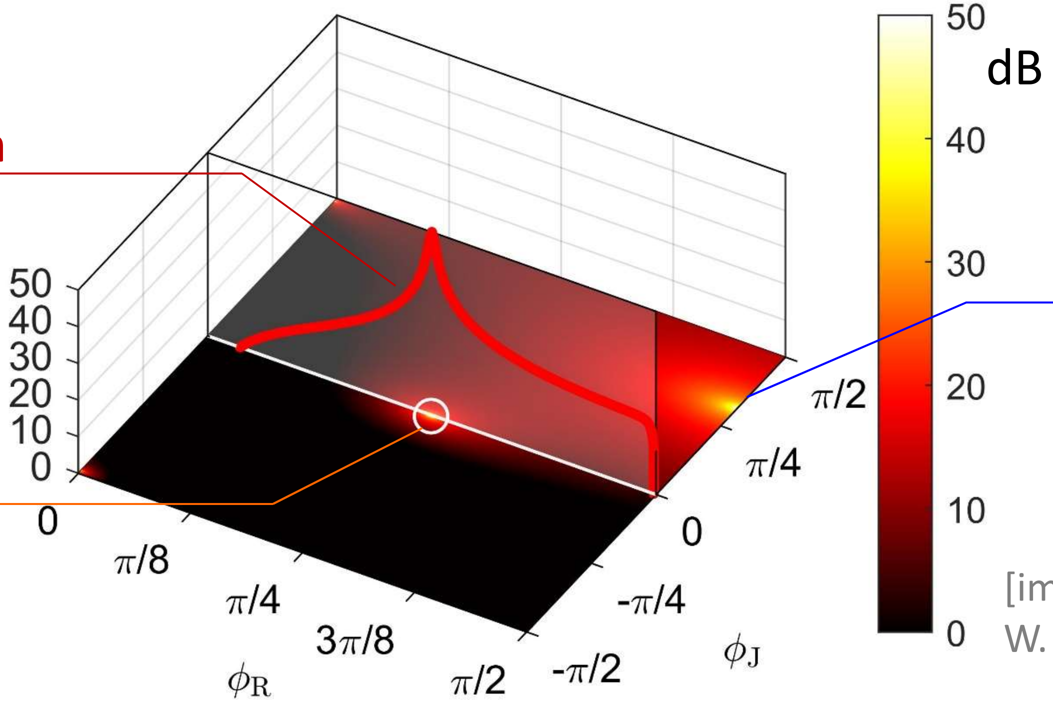


2-D LWA

Near-Field Dominance and Far-Field Patterns

This can clearly be appreciated by representing the absolute value of the pattern function $P(\theta)$, considering θ as a complex variable $\phi = \phi_{\mathrm{R}} + \mathrm{j}\phi_{\mathrm{J}}$:

Radiation pattern



Surface-wave pole

$$k_{\rho \mathrm{SW}} = k_0 \sin\left(\frac{\pi}{2} + j\phi_{\mathrm{J,SW}}\right) = k_0 \cosh\left(\phi_{\mathrm{J,SW}}\right) > k_0$$

Leaky-wave pole

$$k_{\rho \mathrm{LW}} = k_0 \sin\left(\phi_{\mathrm{R,LW}} + j\phi_{\mathrm{J,LW}}\right)$$

[image courtesy of W. Fuscaldo]

The main beam of the pattern is clearly due to the presence of a complex leaky pole in the vicinity of the real interval 0 ≤ ϕ_{R} ≤ π/2 (visible range).

References

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