$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \approx 2{,}72\ldots$$

Si definisce $\underline{\log x} = \log_e x = \ln x$ $\underline{\text{logaritmo naturale}}$

Conseguenze della def. di e.

1) $\lim_{n\to+\infty} \underbrace{n}_{\downarrow +\infty}\, \underbrace{\log\left(1+\frac{1}{n}\right)}_{\downarrow 0} = \lim_{n\to+\infty} \log\left(1+\frac{1}{n}\right)^n = \log e = 1$

Attenzione: vera solo se la base è e.

2) $\boxed{\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^n = (1^{+\infty}) = \frac{1}{e}}$

$$\left(1-\frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \frac{1}{\left(\frac{n-1+1}{n-1}\right)^n} = \frac{1}{\underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{\downarrow e}\underbrace{\left(1+\frac{1}{n-1}\right)}_{\downarrow 1}}$$

$$\longrightarrow \frac{1}{e\cdot 1} = \frac{1}{e}$$

3) Se $b_n \to \pm\infty$. Allora

$$\lim_{n\to+\infty}\left(1+\frac{1}{b_n}\right)^{b_n} = (1^{+\infty}) = e\ ; \quad \lim_{n\to+\infty}\left(1-\frac{1}{b_n}\right)^{b_n} = \frac{1}{e}$$

Pb b_n potrebbe non essere intero.

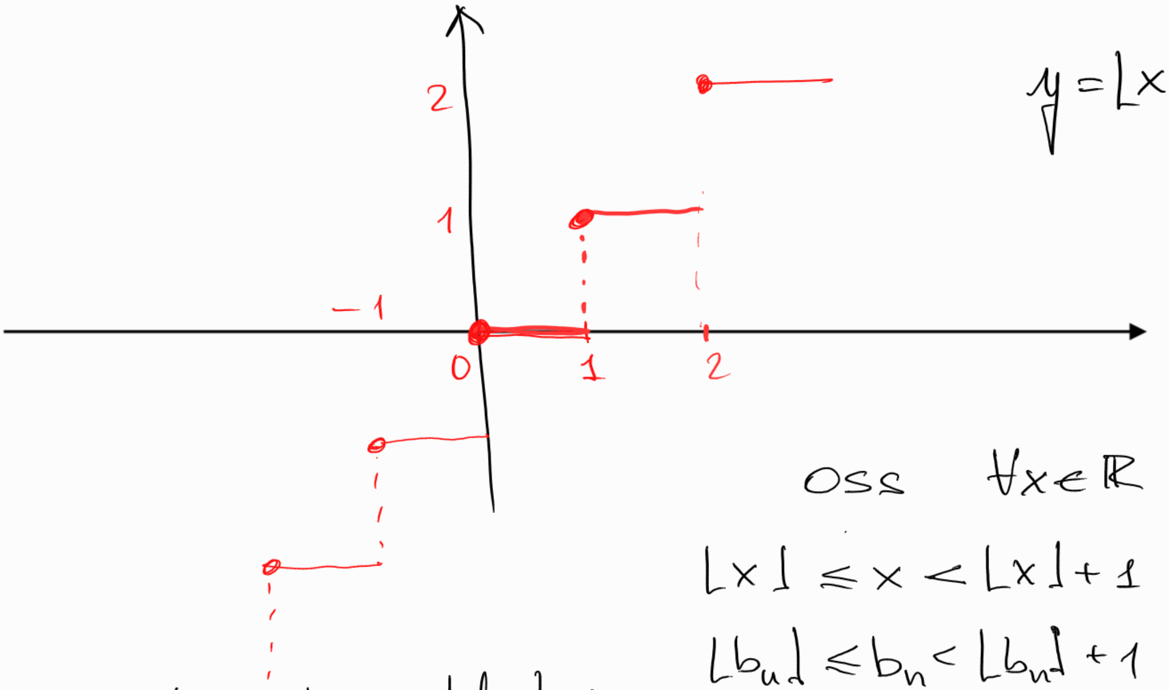
Definiamo $\lfloor x \rfloor$ = parte intera di $x = \max\{k\in\mathbb{Z} : k \le x\}$

$\lfloor 2 \rfloor = 2$

$\lfloor \pi \rfloor = 3$

$\lfloor -4 \rfloor = -4$

$\lfloor -\pi \rfloor = -4$



$y = \lfloor x \rfloor$

OSS $\forall x \in \mathbb{R}$

$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$

$\lfloor b_n \rfloor \le b_n < \lfloor b_n \rfloor + 1$

$b_n \to +\infty$ posso supporre def $b_n > 0$

$$\left(1+\frac{1}{b_n}\right)^{b_n} \le \left(1+\frac{1}{b_n}\right)^{\lfloor b_n \rfloor + 1} \le \left(1+\frac{1}{\lfloor b_n \rfloor}\right)^{\lfloor b_n \rfloor + 1} =$$

$$= \underbrace{\boxed{\left(1+\frac{1}{\lfloor b_n \rfloor}\right)^{\lfloor b_n \rfloor}}}_{\downarrow \; e} \underbrace{\boxed{\left(1+\frac{1}{\lfloor b_n \rfloor}\right)}}_{\downarrow \; 1} \to e.$$

OSS $\lfloor b_n \rfloor \to +\infty$

e sono interi

Cerco l'altro carabiniere

$$\left(1+\frac{1}{b_n}\right)^{b_n} \ge \left(1+\frac{1}{b_n}\right)^{\lfloor b_n \rfloor} \ge \frac{\boxed{\left(1+\frac{1}{\lfloor b_n \rfloor + 1}\right)^{\lfloor b_n \rfloor + 1}} \to e}{\left(1+\frac{1}{\lfloor b_n \rfloor + 1}\right) \to 1} \to e$$

carabinieri

$$\Rightarrow \left(1+\frac{1}{b_n}\right)^{b_n} \to e$$

Se $b_n \to -\infty$, definisco $c_n = -b_n \to +\infty$

$$\left(1+\frac{1}{b_n}\right)^{b_n} = \left(1-\frac{1}{c_n}\right)^{-c_n} = \left(\frac{c_n - 1}{c_n}\right)^{-c_n} = \frac{1}{\left(\frac{c_n-1}{c_n}\right)^{c_n}} =$$

$$= \left(\frac{c_n - 1 + 1}{c_n - 1}\right)^{c_n} = \left(1 + \frac{1}{c_n - 1}\right)^{c_n} = \underbrace{\left(1 + \frac{1}{c_n - 1}\right)^{c_n - 1}}_{\to e} \underbrace{\left(1 + \frac{1}{c_n - 1}\right)}_{\to 1} \to e$$

$$\lim_{n\to+\infty} \left(1 + \frac{5}{n}\right)^n = (1^{+\infty}) =$$

$$= \lim_{n\to+\infty} \Big[\underbrace{\Big(1 + \frac{1}{n/5}\Big)^{n/5}}_{\to e}\Big]^5 = e^5 \qquad b_n \to +\infty$$

Facendo gli stessi passaggi si dimostra che

4) $$\lim_{n\to+\infty} \left(1 + \frac{a}{n}\right)^n = \lim_{n\to+\infty} \left[\left(1 + \frac{1}{n/a}\right)^{n/a}\right]^a = e^a \qquad \forall a \in \mathbb{R}$$

$n/a \to +\infty$ se $a>0$, $-\infty$ se $a<0$

Vale anche se $a=0$, banalmente:

se $a = 0$ $\quad \left(1 + \frac{a}{n}\right)^n = 1 \to 1 = e^0$

$$\lim_{n\to+\infty} \underbrace{\left(\frac{n-3}{n+2}\right)}_{\to 1}^{n-4} = (1^{+\infty}) = \lim_{n\to+\infty} \Bigg[\underbrace{\Bigg(1 - \frac{1}{\left(\frac{n+2}{5}\right)}\Bigg)^{\frac{n+2}{5}}}_{\to \frac{1}{e}}\Bigg]^{\overbrace{\frac{(n-4)5}{n+2}}^{\to 5}} = e^{-5}$$

$$\frac{n-3}{n+2} = \frac{n+2-5}{n+2} = 1 - \frac{5}{n+2} = 1 - \frac{1}{\left(\frac{n+2}{5}\right)} \qquad \frac{n+2}{5} \to +\infty$$

$$\lim_{n\to+\infty} \left(\frac{n^2-3}{n^2}\right)^{3n+1} = (1^{+\infty}) = \lim_{n\to+\infty} \Bigg[\underbrace{\Bigg(1 - \frac{1}{\left(\frac{n^2}{3}\right)}\Bigg)^{n^2/3}}_{\to \frac{1}{e}}\Bigg]^{\overbrace{\frac{(3n+1)3}{n^2}}^{\to 0}} =$$

$$= \left(\frac{1}{e}\right)^0 = 1$$

5) Sia $b_n \to \pm\infty$

Allora

$$\lim_{n\to+\infty} \underbrace{b_n}_{\to \pm\infty} \underbrace{\log\left(1+\frac{1}{b_n}\right)}_{0} = (\pm\infty \cdot 0) = \lim_{n\to+\infty} \log\left(\left(1+\frac{1}{b_n}\right)^{b_n}\right) =$$

$$= \log e = 1 .$$

$$\lim_{n\to+\infty} b_n \log\left(1-\frac{1}{b_n}\right) = \log\left(\frac{1}{e}\right) = -1$$

$$\lim_{n\to+\infty} \underbrace{(n^2-4)}_{\to+\infty} \underbrace{\log\left(1+\frac{1}{n+1}\right)}_{\to 0} = (+\infty \cdot 0) = +\infty$$

$$\boxed{\frac{(n^2-4)}{(n+1)}}_{\to+\infty} \underbrace{\boxed{(n+1)\log\left(1+\frac{1}{n+1}\right)}}_{\to 1} \to +\infty$$

$$\lim_{n\to+\infty} \frac{\log(n^3)}{\log n + 1} = \left(\frac{+\infty}{+\infty}\right) = \lim_{n\to+\infty} \frac{3\log n}{\log n + 1} =$$

$$= \lim_{n\to+\infty} \frac{3 \cancel{\log n}}{\cancel{\log n}\left(1+\frac{1}{\log n}\right)} = 3$$

$$\lim_{n\to+\infty} \frac{\log(e^n+5)}{3n+7} = \left(\frac{+\infty}{+\infty}\right) =$$

$$= \lim_{n\to+\infty} \frac{\log\left(e^n \cdot \left(1+\frac{5}{e^n}\right)\right)}{n\,(3+o(1))} = \lim_{n\to+\infty} \frac{\overbrace{\log(e^n)}^{=n} + \overbrace{\log\left(1+\frac{5}{e^n}\right)}^{=o(1)}}{n\,(3+o(1))} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n}\left(1+\frac{o(1)}{n}\right)^{= o(1)}}{\cancel{n}\,(3+o(1))} = \frac{1}{3}$$

$$\lim_{n\to+\infty} \underbrace{(n^2-7)}_{+\infty} \underbrace{\log_2\left(\frac{n^2+5}{n^2}\right)}_{\log_2 1 = 0} = (+\infty\cdot 0) =$$

$$\lim_{n\to+\infty} \log_2\left(\left(1+\frac{5}{n^2}\right)^{n^2-7}\right) = \log_2 e^5 = 5\log_2 e.$$

$$\left(1+\frac{5}{n^2}\right)^{n^2-7} = (1^{+\infty}) = \underbrace{\left[\left(1+\frac{5}{n^2}\right)^{\frac{n^2}{5}}\right]}_{\to e}^{\boxed{\frac{(n^2-7)5}{n^2}} \to 5} \to e^5$$

In alternativa. $\log_2 x = \dfrac{\log x}{\log 2}$

Formula di cambio di base del log

$$(n^2-7)\log_2\left(1+\frac{5}{n^2}\right) = \frac{1}{\log 2}\underbrace{\frac{(n^2-7)5}{n^2}}_{\to 1}\cdot\underbrace{\frac{n^2}{5}\log\left(1+\frac{5}{n^2}\right)}_{\to 1} \Longrightarrow$$

$$\to \frac{5}{\log 2}$$

Ricordo che dire $a_n = o(1)$ significa $\lim\limits_{n\to+\infty} a_n = 0$

$a_n = 5 + o(1)$ " $\lim\limits_{n\to+\infty} a_n = 5$

$$\frac{3+o(1)}{2+o(1)} = \frac{3}{2} + o(1)$$

$$\frac{5+o(1)}{n^2} = o(1) \quad \text{vuol dire:}$$

se $a_n \to 5$, allora $\frac{a_n}{n^2} \to 0$

$$o(1)+o(1) = o(1) \quad \text{significa:}$$

"se $a_n \to 0$ e $b_n \to 0$, allora $a_n + b_n \to 0$"

$$o(1) \cdot o(1) = o(1) \quad \text{significa:}$$

"se $a_n \to 0$ e $b_n \to 0$, allora $a_n \cdot b_n \to 0$"

$$c\, o(1) = o(1) \quad \forall c \in \mathbb{R}. \quad \text{significa:}$$

"se $a_n \to 0$, e $c \in \mathbb{R}$, allora $c a_n \to 0$"

$$\frac{1}{1+o(1)} = 1 + o(1)$$

$$(n+o(1))^2 = n^2(1+o(1)) \qquad \text{E' vera?}$$

Significa:

Se $a_n \to 0$, allora $\frac{(n+a_n)^2}{n^2} \xrightarrow{?} 1$

$$\| $$

$$\frac{\left(n\left(1+\frac{a_n}{n}\right)\right)^2}{n^2} = \frac{\cancel{n^2}\left(1+\frac{a_n}{n}\right)^2}{\cancel{n^2}} \to 1$$

0

E' vero che $(n+o(1))^2 = n^2 + o(1)$?

Ci stiamo chiedendo:

"se $a_n \to 0$, allora $(n+a_n)^2 - n^2 \xrightarrow{?} 0$?"

Non è vero, in quanto se prendo $a_n = \frac{1}{n} \to 0$

$$\left(n+\frac{1}{n}\right)^2 - n^2 = \cancel{n^2} + \frac{1}{n^2} + 2 - \cancel{n^2} \overset{?}{\longrightarrow} 0$$ NO, tende a 2.

DEF Siano a_n, b_n due succ^{ni} def^{te} diverse da zero.
Scriveremo $a_n \sim b_n$ per $n \to +\infty$ (a_n asintoticamente equivalente a b_n)
se $a_n = b_n(1+o(1))$ per $n \to +\infty$
o equivalentemente $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$

Esempio: $3n^5 - n^4 + 7 \sim 3n^5$

$$\frac{1}{2n^2+1} \sim \frac{1}{2n^2}$$

OSS due successioni asintoticamente equivalenti (se hanno limite) hanno lo stesso limite.
Ma non è detto il viceversa. Se a_n e b_n hanno lo stesso limite, non è detto che siano asintoticamente equivalente.

Per es. $\lim_{n\to+\infty} n^2 = \lim_{n\to+\infty} n = +\infty$ ma non sono asint. equivalenti

Abbiamo già usato questo concetto per es. facendo i limiti di polinomi.

$$\lim_{n\to+\infty} \frac{3n^5+2n^3-7}{n^5-n^2+1} = \lim_{n\to+\infty} \frac{3n^5(1+o(1))}{n^5(1+o(1))} = 3.$$