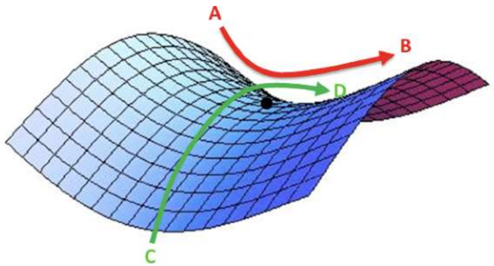
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 6

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Asymptotic Evaluation of Integrals:
The Method of Steepest Descent

Basic Idea

The method of steepest descent is a powerful approach for studying the large k asymptotics of integrals of the form

$$I(k) = \int_C f(z) e^{k\phi(z)} \mathrm{d}z$$

The **basic idea** of the method is to utilize the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\phi(z)$ has a constant imaginary part.

$$\phi(z) = u + jv$$
$$\mathrm{Im}\left[\phi(z)\right] = v = \mathrm{const}$$

$$\Rightarrow \quad I(k) = e^{jkv} \int_{C'} f(z) e^{ku} \mathrm{d}z$$

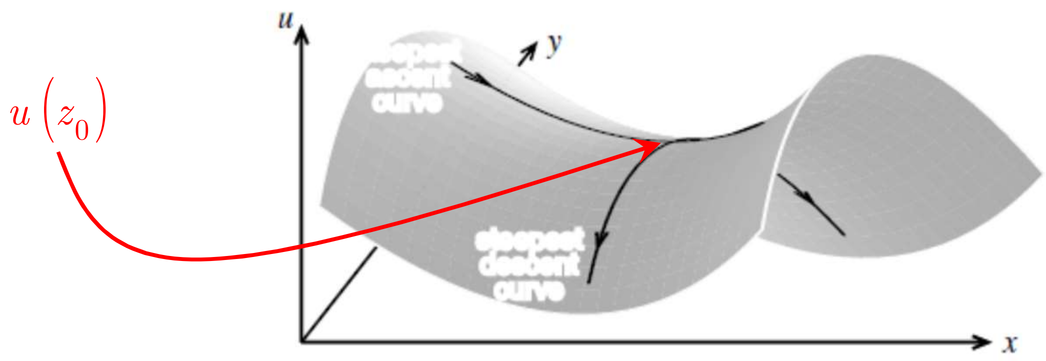
Although z is complex, u is real, hence the same approach used in the Laplace method can be used.

Steepest Descent Paths and Saddle Points

The paths on which v is constant are also paths for which either the decrease of u is maximal (paths of steepest descent) or the increase of u is maximal (paths of steepest ascent).

The asymptotic evaluation of $I(k)$ will make use of the former, hence the name **Steepest Descent Method**.

Usually the paths of steepest descent will go through a point z_0 for which $\phi'(z_0) = 0$. Such a point is a saddle point for the function u, hence the method is alternatively referred to as the **Saddle Point Method**.



Consequences of the Cauchy-Riemann Equations

$$\phi\left(z\right) = \phi\left(x+jy\right) = u\left(x,y\right) + jv\left(x,y\right)$$

If $\phi(z)$ is analytic at $z_0 = x_0 + jy_0$:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

At any point z where $\phi'\left(z\right) \neq 0$ it then results:

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

- the direction of maximum variation of u is orthogonal to the direction of maximum variation of v
- the curves of constant v are the steepest descent/ascent paths of u

Consequences of the Cauchy-Riemann Equations

Both u and v are *harmonic functions*:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

Maximum Modulus Theorem (Harmonic Functions)

Let $u(x,y)$ be an *harmonic function* in an *open connected* domain D.
If

$$\left| u\left(x_0, y_0\right) \right| \geq \left| u\left(x, y\right) \right|, \ \forall \left(x, y\right) \in D$$

then

$$u\left(x, y\right) = u\left(x_0, y_0\right), \ \forall \left(x, y\right) \in D$$

Hence the stationary points of u and v are **saddle points**.

Saddle Points: Local Behavior

In practice, we need to establish that the contour can be deformed onto the steepest descent curve, which passes through the saddle point.

A point z_0 is a **saddle point of order** N if, letting $n=N+1$,

$$\left.\frac{\partial^m \phi}{\partial z^m}\right|_{z=z_0} = 0, \quad m = 1, 2, \ldots, N$$

$$\left.\frac{\partial^n \phi}{\partial z^n}\right|_{z=z_0} = a e^{j\alpha}, \quad a > 0$$

Then, by letting $z - z_0 = \rho e^{j\theta}$, we can study the local behavior of ϕ through

$$\phi(z) - \phi(z_0) \simeq \frac{1}{n!}(z - z_0)^n \left.\frac{\partial^n \phi}{\partial z^n}\right|_{z=z_0} = \frac{1}{n!}\rho^n e^{jn\theta} a e^{j\alpha} = \frac{\rho^n a}{n!} e^{j(\alpha + n\theta)}$$

$$= \frac{\rho^n a}{n!}\left[\cos(\alpha + n\theta) + j\sin(\alpha + n\theta)\right]$$

Saddle Points: Local Behavior

Since the directions of steepest descent at $z = z_0$ are defined by $\mathrm{Im}[\phi(z) - \phi(z_0)] = 0$, it follows that $\sin(\alpha + n\theta) = 0$, and for u to decrease away from z_0, $\cos(\alpha + n\theta) < 0$.

Similarly, the directions of steepest ascent are given by $\sin(\alpha + n\theta) = 0$, $\cos(\alpha + n\theta) > 0$.

steepest descent directions $\theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n}$

steepest ascent directions $\theta = -\frac{\alpha}{n} + 2m\frac{\pi}{n}$

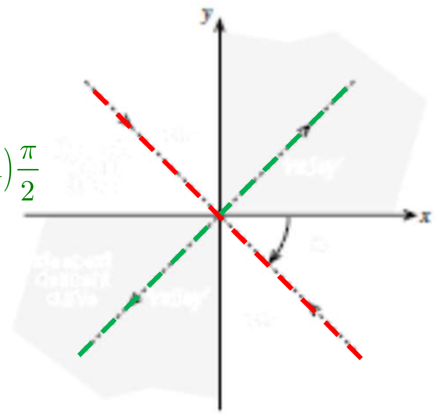
$(m = 0, 1, 2 \ldots)$

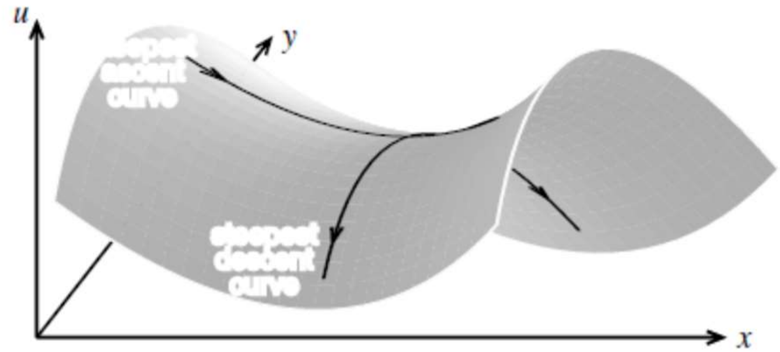
Example: N=1 (simple SP)

$\alpha = \frac{\pi}{2}$

$\theta_{\mathrm{SDP}} = -\frac{\pi}{4} + (2m+1)\frac{\pi}{2}$

$\theta_{\mathrm{SAP}} = -\frac{\pi}{4} + 2m\frac{\pi}{2}$





Examples

SPs and directions of steepest descent

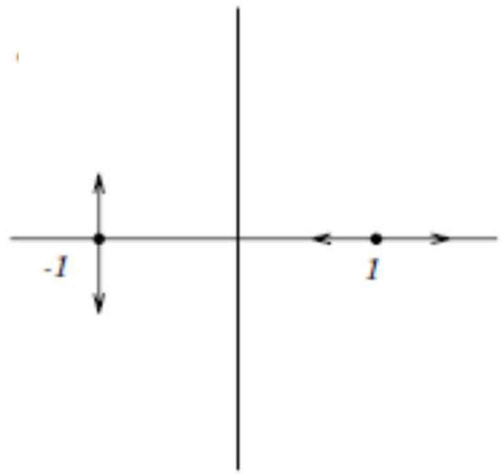
$$\phi(z) = z - \frac{z^3}{3}$$

$$\phi'(z) = 1 - z^2 = (1+z)(1-z)$$ $z_{\text{SP}} = \pm 1$

$$\phi''(1) = -2 = 2e^{j\pi} \rightarrow \alpha = \pi$$

$$\phi''(-1) = 2 = 2e^{j0} \rightarrow \alpha = 0$$

Two simple SPs



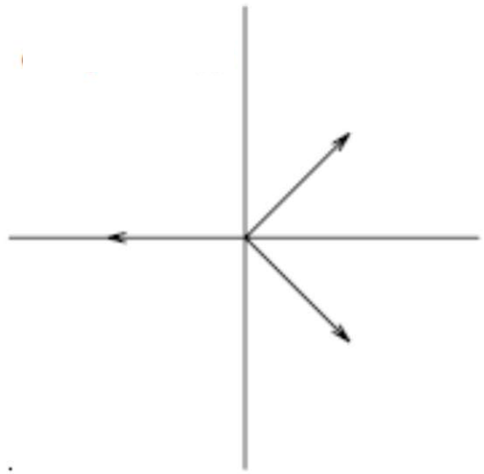
$$\phi(z) = \sinh z - z$$

$$\phi'(z) = \cosh z - 1$$ $z_{\text{SP}} = \cosh^{-1}(1) = 0 + j2m\pi$

$$\phi''(z_{\text{SP}}) = 0$$

$$\phi'''(z_{\text{SP}}) = 1 = 1e^{j0} \rightarrow \alpha = 0$$

An infinite number of 2nd-order SPs



Laplace Method for Complex Contours

In a nutshell:

- We need to **establish that the contour can be deformed onto the steepest descent curve**, which passes through the saddle point. This requires some global understanding of the geometry.
- Usually the contribution near the **saddle point** gives the *dominant contribution*.
- However, sometimes the contour *cannot* be deformed onto a curve passing through a saddle point. In this case, **endpoints or singularities** of the integrand yield the dominant contribution.
- Moreover, sometimes the deformation process introduces **poles** that can lead to *significant* (and *possibly dominant*) contributions to the asymptotic expansion.

Contribution of a Single Path of Steepest Descent

Let us consider a portion of the dominant contribution; that is, consider a single path of steepest descent originating from a saddle point z_0 of order n - 1.

Dominant Asymptotic Contribution (single SDP through a SP)

Let z_0 be a SP of order n - 1 and let $f(z)$ be of order $(z\text{-}z_0)^{\beta\text{-}1}$ near z_0, i.e.:

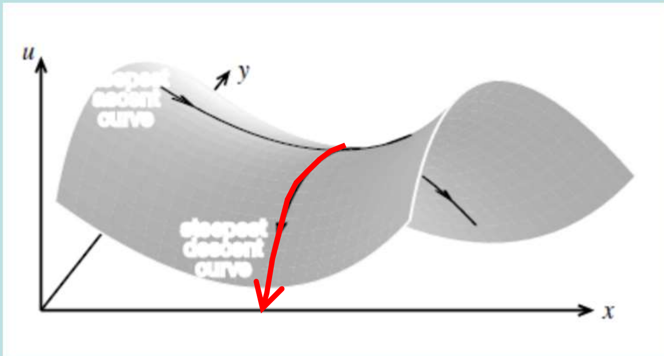
$$z \to z_0: \quad \phi(z) - \phi(z_0) \sim \frac{1}{n!}(z - z_0)^n \, \phi^{(n)}(z_0)$$

$$f(z) \sim f_0 (z - z_0)^{\beta-1} \quad (\mathrm{Re}\{\beta\} > 0)$$

$$\phi^{(n)}(z_0) = \left|\phi^{(n)}(z_0)\right| e^{j\alpha}$$

then

$$I(k) \sim \frac{f_0 (n!)^{\frac{\beta}{n}} e^{j\beta\theta}}{n} \frac{e^{k\phi(z_0)}\Gamma\left(\frac{\beta}{n}\right)}{\left(k\left|\phi^{(n)}(z_0)\right|\right)^{\frac{\beta}{n}}}$$



Derivation

Since we integrate along a SDP: $\phi(z)-\phi(z_0)=-t,\ \ t\in\Re^+$ $\quad[\rightarrow \phi'(z)\mathrm{d}z=-\mathrm{d}t]$

$$\Longrightarrow \quad I(k)\sim e^{k\phi(z_0)}\int_0^{+\infty}\left(-\frac{f(z)}{\phi'(z)}\right)e^{-kt}\mathrm{d}t$$

Near the SP:

$$\frac{1}{n!}(z-z_0)^n\,\phi^{(n)}(z_0)=-t \quad\Longrightarrow\quad |z-z_0|=t^{\frac{1}{n}}\frac{(n!)^{\frac{1}{n}}}{\left|\phi^{(n)}(z_0)\right|^{\frac{1}{n}}}$$

$$\frac{f(z)}{\phi'(z)}\sim\frac{f_0\,(z-z_0)^{\beta-1}}{\frac{1}{(n-1)!}(z-z_0)^{n-1}\,\phi^{(n)}(z_0)}=\frac{f_0\,(n!)^{\frac{\beta}{n}}\,e^{j\beta\theta-j(\theta n+\alpha)}}{n\left|\phi^{(n)}(z_0)\right|^{\frac{\beta}{n}}}t^{\frac{\beta}{n}-1}$$

Since on the SDP $\theta n+\alpha=(2m+1)\pi$, the result for $I(k)$ follows by recalling the definition of the Gamma function.

Higher-Order Asymptotic Terms

To recover the **full asymptotic expansion** for $I(k)$, one must solve for $z-z_0$ in terms of a power series in t ($z-z_0=\sum_{m=0}^{+\infty} c_m t^m$) from

$$-t=\phi(z)-\phi(z_0)=-(z-z_0)^n\hat{\phi}(z)$$

where $\hat{\phi}(z)=\sum_{m=0}^{+\infty}\hat{\phi}_m(z-z_0)^m$ where $\hat{\phi}(z)$ is analytic close to z_0.

This can be obtained by recursively solving $z-z_0=t^{\frac{1}{n}}/\left[\hat{\phi}(z)\right]^{\frac{1}{n}}$

This inversion can always be accomplished in principle, by the Implicit Function Theorem. In practice, the necessary devices to accomplish this are often direct and motivated by the particularities of the functions in question.

Asymptotic Nature of the Representation

The asymptotic nature of the above representations can be rigorously established by using Watson's Lemma (*proof by exercise*).

This shows the great advantage of the method of steepest descent: Because it is based on Watson's Lemma, it is possible both to justify it rigorously and to obtain the asymptotic expansion to all orders **using purely local information**.

Comparison with the Stationary Phase Method

We could consider deforming C into a path for which u rather than v is constant (so that v varies rapidly) and then apply an extension of the method of stationary phase. However, we expect intuitively that the self-canceling of oscillations is a **weaker decay mechanism** than the exponential decay of the exponential factor in the integrand.

In fact, without deformations to a Laplace type integral, in general **only the leading term** of the asymptotic expansion of a generalized Fourier integral can be found from purely local considerations.

Asymptotically Equivalent Contours

A difficulty encountered in practice is the deformation of the original contour of integration onto one or more of the paths of steepest descent.

However, the **local nature** of the method of steepest descent makes even this task relatively simple. This is because **quantitative** information about the deformed contours is needed only near the critical points [SPs, endpoints, singularities of $f(z)$ and $\phi(z)$]; away from these points **qualitative** information is sufficient.

A contour C_1 that coincides with a steepest descent contour C_s for some finite length near the critical point z_0 but that then continues merely as a descent contour is said to be **asymptotically equivalent** to C_s.

Asymptotic expansions derived from asymptotically equivalent contours differ only by an *exponentially small* quantity.

1st Worked Example

Find the complete asymptotic expansion of $I(k) = \int_0^1 \log t e^{jkt} \mathrm{d}t, \quad k \to +\infty$

- There is no stationary-phase point: $\phi(t) = t, \quad \to \quad \phi'(t) = 1 \neq 0$
- Integration by parts fails because $\frac{\mathrm{d}}{\mathrm{d}t} \log(t) = \frac{1}{t}$ is not integrable close to $t=0$

Let us then try with the SD method by replacing the real variable t by the complex variable $z=x+jy$:

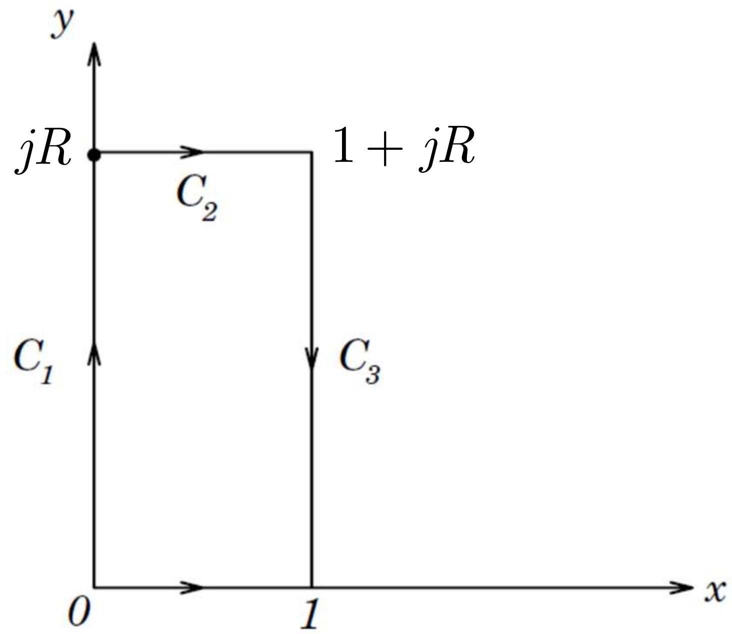
$$\phi(z) = jz = jx - y, \quad \to \quad \phi'(z) = j \neq 0 \quad \to \quad \text{no SP}$$

Steepest paths:

$$\mathrm{Im}\{\phi(z)\} = \text{const}, \quad \to \quad x = \text{const}; \quad y > 0: \ \text{SDP}; \ y < 0: \ \text{SAP}$$

1st Worked Example (cont'd)

We note that $\mathrm{Im}\ \phi(0) \neq \mathrm{Im}\ \phi(1)$; hence there is no continuous contour joining $z=0$ and $z=1$ on which $\mathrm{Im}\ \phi$ is constant.



By applying Cauchy Theorem to the path shown in figure:

$$I(k) = \int_{C_1+C_2+C_3} \log z e^{jkz} \mathrm{d}z$$

$$= j\int_0^R \log(jr) e^{-kr} \mathrm{d}r + \int_0^1 \log(x + jR) e^{jk(x+jR)} \mathrm{d}x - je^{jk}\int_0^R \log(1 + jr) e^{-kr} \mathrm{d}r$$

Letting R tend to infinity we have

$$I(k) = j\int_0^{+\infty} \log(jr) e^{-kr} \mathrm{d}r - je^{jk}\int_0^{+\infty} \log(1 + jr) e^{-kr} \mathrm{d}r$$

1st Worked Example (cont'd)

Using $s=kr$ the first integral becomes:

$$\frac{j}{k}\int_0^{+\infty}\left[\log\left(\frac{j}{k}\right)+\log s\right]e^{-s}\mathrm{d}s=-j\frac{\log k}{k}-\frac{\left(j\gamma+\frac{\pi}{2}\right)}{k}$$

$$\left(\gamma=-\int_0^{+\infty}\log s e^{-s}\mathrm{d}s=0.577216\ldots,\ \text{Euler-Mascheroni constant}\right)$$

In the second integral we use Watson Lemma with $\log\left(1+jr\right)=-\sum_{n=1}^{+\infty}\frac{\left(-jr\right)^n}{n}$

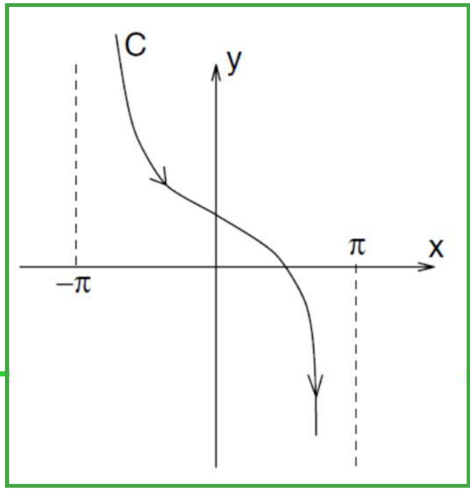
$$I\left(k\right)\sim -j\frac{\log k}{k}-\frac{\left(j\gamma+\frac{\pi}{2}\right)}{k}+je^{jk}\sum_{n=1}^{+\infty}\frac{\left(-j\right)^n\left(n-1\right)!}{k^{n+1}},\qquad k\to+\infty$$

2nd Worked Example: Hankel Function

Find the complete asymptotic expansion of

$$H_\nu^{(1)}\left(k\right) = \frac{1}{\pi}\int_C e^{jk\cos z} e^{j\nu\left(z-\frac{\pi}{2}\right)}\mathrm{d}z, \quad k \to +\infty$$

(*Hankel function of order* ν *and 1st kind*)

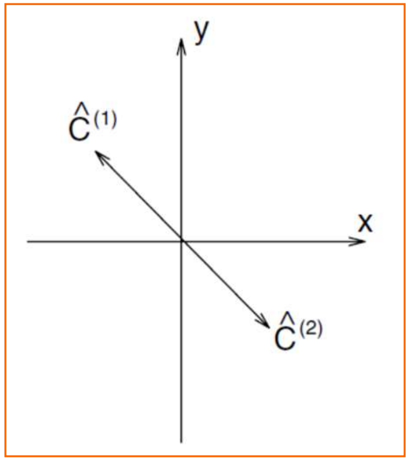


$$\phi\left(z\right) = j\cos z, \quad \to \quad \phi'\left(z\right) = -j\sin z \quad \to \quad z_{\mathrm{SP}} = 0\left(+2m\pi\right)$$

Simple SPs

$$\phi''\left(0\right) = -j = e^{-j\frac{\pi}{2}} \to \alpha = -\frac{\pi}{2}$$

$$\theta_{\mathrm{SDP}} = -\frac{\alpha}{2} + \left(2m+1\right)\frac{\pi}{2} = \frac{3\pi}{4}, \frac{7\pi}{4} \longrightarrow$$



2nd Worked Example: Hankel Function (cont'd)

Noting that $\int_{C} = \int_{\hat{C}_2} - \int_{\hat{C}_1}$ we subtract the formulas

$$I(k) \sim \frac{f_0 (n!)^{\frac{\beta}{n}} e^{j\beta\theta}}{n} \frac{e^{k\phi(z_0)} \Gamma\left(\frac{\beta}{n}\right)}{\left(k \left|\phi^{(n)}(z_0)\right|\right)^{\frac{\beta}{n}}}$$

using $\theta = \frac{3\pi}{4}\ (\hat{C}_1),\ \theta = \frac{7\pi}{4}\ (\hat{C}_2)$ and

$$z_0 = 0,\ \ n = 2,\ \ ,\phi(0) = j,\ \ \phi''(0) = -j,\ \ \beta = 1,\ \ f_0 = \frac{e^{-j\nu\frac{\pi}{2}}}{\pi}$$

$$\Rightarrow \quad \boxed{H_\nu^{(1)}(k) \sim \sqrt{\frac{2}{\pi k}} e^{j\left(k - \nu\frac{\pi}{2} - \frac{\pi}{4}\right)}, \quad k \to +\infty}$$

dominant asymptotic behavior

2nd Worked Example: Hankel Function (cont'd)

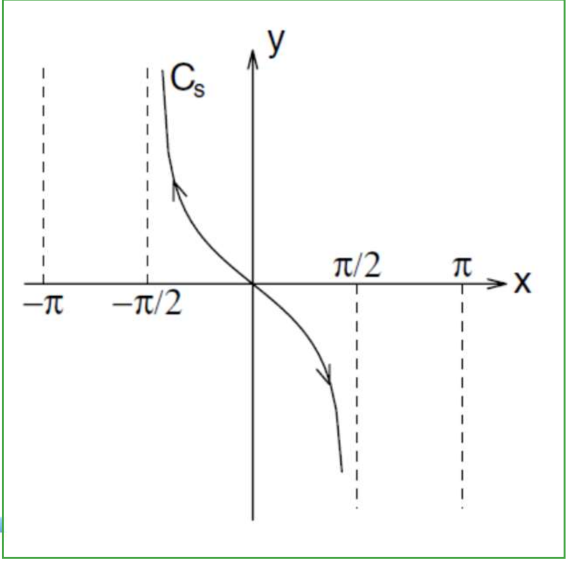
SDP/SAP through the SP $z=0$: $\quad \operatorname{Im}\{\phi(z)\} = \operatorname{Im}\{\phi(0)\} \rightarrow \cos x \cosh y = 1$

Near $z=0$: $\left(1 - \frac{x^2}{2} + \ldots\right)\left(1 + \frac{y^2}{2} + \ldots\right) = 1 \rightarrow y^2 - x^2 = (y + x)(y - x) = 0$

SDP $(y+x)$ SAP $(y-x)$

$$\left|y\right| \rightarrow +\infty \quad \Rightarrow \quad \cos x \rightarrow \frac{1}{2} e^{-|y|} \Rightarrow x \rightarrow \begin{cases} -\frac{\pi}{2}, & y \rightarrow +\infty \\ +\frac{\pi}{2}, & y \rightarrow -\infty \end{cases}$$

The integral converges when $\operatorname{Re}\phi = \sinh y \, \sin x < 0$; i.e., for $y \rightarrow +\infty$, $\pi < x < 0$, and for $y \rightarrow \infty$, $0 < x < \pi$.



We conclude that we may deform the original contour C to the SDP C_s through the SP $z=0$.

2nd Worked Example: Hankel Function (cont'd)

SDP transformation $\phi(z)-\phi(0)=-t,\ t>0\ (t\in\mathbb{R}):$ $\quad j(\cos z-1)=-t$

Near z=0, t=0: $\quad \dfrac{z^2}{2!}-\dfrac{z^4}{4!}+\ldots=e^{-j\frac{\pi}{2}}t$

We solve for z as a function of t iteratively: $\quad z=\sqrt{2}e^{-j\frac{\pi}{4}}t^{1/2}\left(1+\dfrac{z^2}{24}+\ldots\right)$

$$z=\sqrt{2}e^{-j\frac{\pi}{4}}t^{1/2}+\frac{\sqrt{2}}{12}e^{-j\frac{3\pi}{4}}t^{3/2}+\ldots$$

$$H_\nu^{(1)}(k)=\frac{1}{\pi}\int_{C_s}e^{jk}e^{-kt}e^{j\nu z(t)}e^{-j\nu\frac{\pi}{2}}\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t$$

$$\sim\frac{1}{\pi}e^{jk}e^{-j\nu\frac{\pi}{2}}\int_{C_s}e^{-kt}\left(1+j\nu z+\frac{(j\nu z)}{2!}+\ldots\right)\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t,\qquad k\to+\infty$$

2nd Worked Example: Hankel Function (cont'd)

Using the expansion of $z(t)$ above and splitting the steepest descent contour into two pieces, one each from the origin, after some algebra we have:

$$H_\nu^{(1)}(k) \sim \frac{2}{\pi} e^{jk} e^{-j\nu\frac{\pi}{2}} \int_0^{+\infty} e^{-kt}\left(c_0 t^{-1/2} + c_1 + c_2 t^{1/2} + c_3 t + \ldots\right) \mathrm{d}t, \quad k \to +\infty$$

By inserting the appropriate coefficients c_n we finally find:

$$\begin{aligned} H_\nu^{(1)}(k) &\sim \frac{2}{\pi} e^{jk} e^{-j\nu\frac{\pi}{2}} \left(c_0 \frac{\Gamma\left(\frac{1}{2}\right)}{k^{1/2}} + \frac{c_1}{k} + c_2 \frac{\Gamma\left(\frac{3}{2}\right)}{k^{3/2}} + \frac{c_3}{k^2} + \ldots \right) \\ &= \frac{2}{\pi} e^{jk} e^{-j\nu\frac{\pi}{2}} \left(\frac{\sqrt{2\pi}}{2} \frac{e^{-j\pi/4}}{k^{1/2}} + \frac{\nu}{k} + \frac{\sqrt{2\pi}}{2} \frac{\left(\frac{1}{4} - \nu^2\right) e^{-j3\pi/4}}{k^{3/2}} + \frac{j\nu\left(\nu^2 - 1\right)}{3k^2} + \ldots \right) \end{aligned}$$

Alternative Approach

A somewhat quicker procedure makes use of **a different change of variables**.

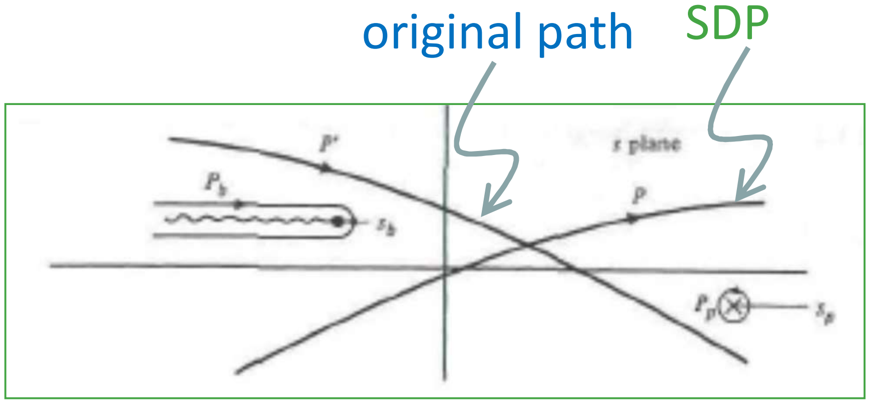
In particular, a suitable **polynomial** $\tau(s)$ is introduced: $\tau(s) = \phi(z)$

$$I(k) = \int_C f(z) e^{k\phi(z)} \mathrm{d}z = \int_{P'} G(s) e^{k\tau(s)} \mathrm{d}s$$

$$G(s) = f(z) \frac{\mathrm{d}z}{\mathrm{d}s}, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\tau'(s)}{\phi'(z)}$$

the vicinity of z_{s} in the complex z plane being chosen to correspond to the vicinity of the point 0 in the complex s plane.

$$I(k) = \int_{P'} = \int_{P} + \int_{P_{\mathrm{p}}} + \int_{P_{\mathrm{b}}}$$



Alternative Approach (cont'd)

Assuming $G(s)$ is regular close to $s=0$, one may write

$$\int_P G(s)e^{k\tau(s)}\mathrm{d}s \sim G(0)\int_P e^{k\tau(s)}\mathrm{d}s \qquad \text{(1st-order approximation)}$$

comparison integral

The transformation $\tau(s) = \phi(z)$ should be chosen so that:

- Re$\tau(s)$ should decrease most rapidly from $s=0$, hence the SP(s) of $\tau(s)$ should be at (or close to) $s=0$
- The mapping derivative $\mathrm{d}z/\mathrm{d}s$ should be finite near $s=0$, to ensure regularity of $G(s)$

Therefore, the derivative $\tau'(s)$ should have **the same order of zero** at the SP(s) in the s plane as the original $\phi'(z)$ at the SP(s) z_s in the z plane.

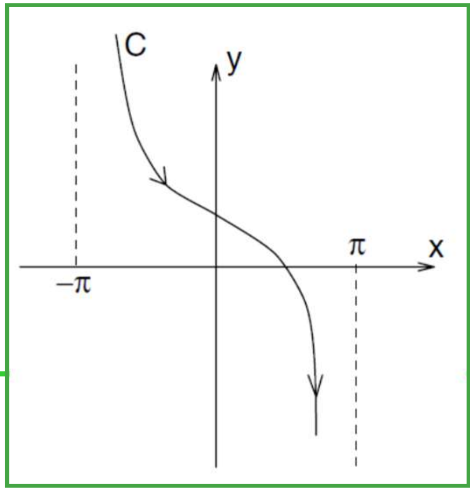
The simplest $\tau(s)$ satisfying these requirements will yield the simplest comparison integral.

Hankel Function (Alternative Approach)

Find the complete asymptotic expansion of

$$H_\nu^{(1)}(k) = \frac{1}{\pi}\int_C e^{jk\cos z} e^{j\nu\left(z-\frac{\pi}{2}\right)} \mathrm{d}z, \quad k \to +\infty$$

(*Hankel function of order ν and 1st kind*)



$$\phi(z) = j\cos z, \quad \to \quad \phi'(z) = -j\sin z \quad \to \quad z_{\text{SP}} = 0\,(+2m\pi)$$

Simple SPs

$$\phi''(0) = -j$$

$$\phi(z) = j\cos z = j - s^2 = \tau(s)$$

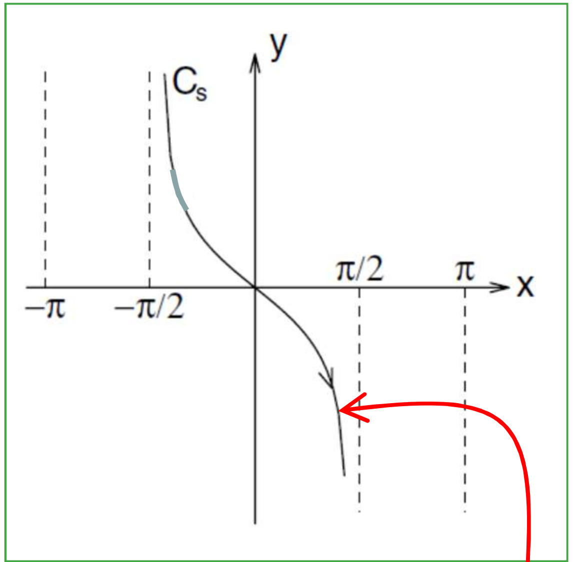
so that

$$\phi(z_{\text{SP}} = 0) = j = \tau(s = 0)$$

$$\tau'(s) = -2s, \tau''(0) = -2 \to s = 0 \text{ simple zero}$$

Hankel Function (Alternative Approach)

$$\frac{\mathrm{d}z}{\mathrm{d}s}=\frac{\tau'(s)}{\phi'(z)}\xrightarrow[\text{(de L'Hôpital)}]{s\to 0}\frac{\tau''(0)}{\phi''(z_{\mathrm{SP}})\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)_{s\to 0}}$$



n even: SDP

$$\Rightarrow \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)_{s\to 0}=\sqrt{\frac{\tau''(0)}{\phi''(z_{\mathrm{SP}})}}$$

$$=\sqrt{\frac{-2}{-j}}=\sqrt{-2j}=\sqrt{2e^{-j\frac{\pi}{2}}}=\sqrt{2}e^{j\left(-\frac{\pi}{4}+n\pi\right)}$$

$$G(s)=f(z)\frac{\mathrm{d}z}{\mathrm{d}s}\to G(0)=f(z_{\mathrm{SP}})\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)_{s=0}=\frac{1}{\pi}e^{-j\nu\frac{\pi}{2}}\sqrt{2}e^{-j\frac{\pi}{4}}$$

Hankel Function (Alternative Approach)

$$H_\nu^{(1)}(k) = \frac{1}{\pi}\int_C e^{jk\cos z} e^{j\nu\left(z-\frac{\pi}{2}\right)}\mathrm{d}z = \int_{-\infty}^{+\infty} G(s)\, e^{k\tau(s)}\mathrm{d}s$$

$$\sim G(0)\int_{-\infty}^{+\infty} e^{k\tau(s)}\mathrm{d}s = \frac{1}{\pi}e^{-j\nu\frac{\pi}{2}}\sqrt{2}e^{-j\frac{\pi}{4}}\int_{-\infty}^{+\infty} e^{k\left(j-s^2\right)}\mathrm{d}s$$

$$= \frac{1}{\pi}e^{-j\nu\frac{\pi}{2}}\sqrt{2}e^{-j\frac{\pi}{4}}e^{jk} \underbrace{\int_{-\infty}^{+\infty} e^{-ks^2}\mathrm{d}s}_{=\sqrt{\frac{\pi}{k}}\ \text{(Gauss integral)}}$$

$$= \sqrt{\frac{2}{\pi k}}e^{j\left(k-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)}$$

(dominant asymptotics, as already obtained)

Hankel Function (Alternative Approach)

$$H_\nu^{(1)}(k) = \frac{1}{\pi}\int_C e^{jk\cos z} e^{j\nu\left(z-\frac{\pi}{2}\right)} \mathrm{d}z = \int_{-\infty}^{+\infty} G(s)\, e^{k\tau(s)} \mathrm{d}s$$

$$\sim \int_{-\infty}^{+\infty} \sum_{n=0}^{+\infty} \frac{G^{(n)}(0)}{n!} s^n e^{k\tau(s)} \mathrm{d}s = \sum_{n=0}^{+\infty} \frac{G^{(n)}(0)}{n!} \int_{-\infty}^{+\infty} s^n e^{k\left(j-s^2\right)} \mathrm{d}s$$

$$= e^{jk} \sum_{n=0}^{+\infty} \frac{G^{(n)}(0)}{n!} \underbrace{\int_{-\infty}^{+\infty} s^n e^{-ks^2} \mathrm{d}s}$$

$$= \begin{cases} 0, & n \text{ odd} \\ (-1)^{n/2} \dfrac{\mathrm{d}^{n/2}}{\mathrm{d}k^{n/2}} \sqrt{\dfrac{\pi}{k}}, & n \text{ even} \end{cases}$$

(complete asymptotic expansion)

Comparison Integrals: Various Forms

As already said, the derivative τ '(s) should have **the same order of zero** at the SP(s) in the s plane as the original ϕ'(z) at the SP(s) z_s in the z plane.

$$G(s) = f(z)\frac{\mathrm{d}z}{\mathrm{d}s}, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\tau'(s)}{\phi'(z)}$$

If $f(z)$ has singularities near z_s, $G(s)$ has singularities near $s=0$; these must be **isolated in the simplest form** and require consideration of *a new class* of comparison integrals.

Comparison Integrals: Various Forms

CASE 1

single SP $z = z_s$ of order m, $f(z)$ regular close to z_s

$$\phi(z) = \tau(s) = \phi(z_s) - s^{m+1} \qquad z = z_s \to s = 0$$

$$I(k) = \int_0^{+\infty} G(s)\, e^{k\tau(s)} \mathrm{d}s \sim G(0)\, e^{k\phi(z_s)} \underbrace{\int_0^{+\infty} e^{-ks^{m+1}} \mathrm{d}s}_{= \frac{1}{m+1}\Gamma\left(\frac{1}{m+1}\right)\frac{1}{k^{\frac{1}{m+1}}}}$$

(dominant asymptotics)

integral along a SD path starting at the SP and ending in an appropriate 'valley' at infinity

canonical integral, evaluated analytically (**algebraic dependence** on k)

Comparison Integrals: Various Forms

CASE 2

double SPs $z=z_1$ and $z=z_2$ of order 1, *arbitrarily close to each other*; $f(z)$ regular close to $z_{1,2}$

$$\phi(z) = \tau(s) = a_0 + \sigma s - \frac{s^3}{3}$$

$$z = z_{1,2} \to s = s_{1,2} = \pm\sqrt{\sigma}$$

($\tau(s)$ has two simple zeros at $s_{1,2}$)

$$\Rightarrow \quad I(k) = \int_P G(s) e^{k\tau(s)} \mathrm{d}s \sim G(0) \int_P e^{k\left(a_0 + \sigma s - \frac{s^3}{3}\right)} \mathrm{d}s$$

(dominant asymptotics)

$$= \frac{1}{k^{1/3}} G(0) e^{k a_0} \int_P e^{\zeta t - \frac{t^3}{3}} \mathrm{d}t \Bigg|_{\zeta = k^{2/3}\sigma}$$

the 'rainbow integral'

new canonical integral expressible in terms of **the Airy function**

Comparison Integrals: Various Forms

CASE 3

Triple collinear SPs $z=z_1$, $z=z_2$, and $z=z_3$ of order 1, *arbitrarily close to each other*; $f(z)$ regular in the vicinity of $z_{1,2,3}$

$$\phi(z) = \tau(s) = a_0 - (a + s^2)^2$$

$$z = z_{1,2,3} \rightarrow s = s_{1,2,3} = -j\sqrt{a}, 0, +j\sqrt{a}$$

($\tau(s)$ has three simple zeros at $s_{1,2,3}$)

$$\Rightarrow \quad I(k) = \int_P G(s) e^{k\tau(s)} \mathrm{d}s \sim G(0) \int_P e^{k\left[a_0 - (a+s^2)^2\right]} \mathrm{d}s$$

(dominant asymptotics)

$$= \frac{1}{(2k)^{1/4}} G(0) e^{ka_0} \int_P e^{-\frac{1}{2}(t+p^2)^2} \mathrm{d}p \Bigg|_{t=a\sqrt{2k}}$$

new canonical integral expressible in terms of the **parabolic cylinder function** $D_{-1/2}\left(a\sqrt{2k}\right)$

Comparison Integrals: Various Forms

CASE 4

Single SPs $z = z_1$ of order *1*, $f(z)$ has a simple or multiple pole at z_p *arbitrarily close to the SP*

$$\phi(z) = \tau(s) = \phi(z_s) - s^2$$

$$G(s) = \frac{a}{s-b} + T(s)$$

regular close to $s=0$

$$I(k) \sim T(0) \int_P e^{k\tau(s)} \mathrm{d}s + a \int_P \frac{e^{k\tau(s)}}{s-b} \mathrm{d}s$$

same canonical integral as before $= e^{k\phi(z_s)} \sqrt{\frac{\pi}{k}}$

new canonical integral expressible in terms of the **error function** or **Fresnel integrals**

References

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