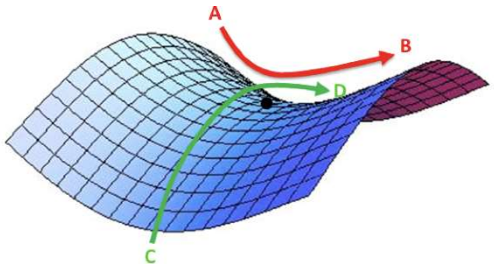
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 5

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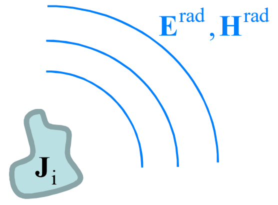
Integral Representation of Wave Fields

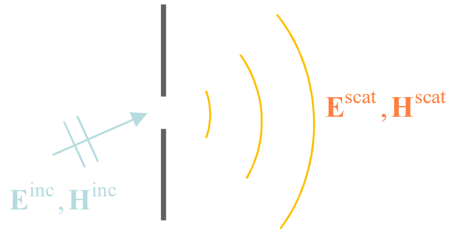
Integral Representations of Wave Fields

A wave (e.g., electromagnetic) field can very often be expressed through **integrals**.

This is the case of, e.g., the standard **radiation integrals** in free space (or in particular bounded environments), which express the wave field in terms of *primary* sources (e.g., impressed currents) as a superposition integral having a suitable Green's function as its kernel.



Similar integrals involving *secondary* sources are found also in **diffraction/scattering** problems, where the field is typically expressed in terms of its values on suitable reference surfaces via Huygens principle.



Asymptotic Evaluation of the Integral Representations

Such radiation or diffraction integrals can be evaluated analytically only in a limited number of cases; alternatively, they can be **evaluated numerically** or approximated through **asymptotic expansions**.

In this and the next lesson we aim at providing basic information on the principal techniques for the **asymptotic evaluation of integrals**.

Asymptotic Evaluation of Integrals: Laplace Type Integrals

Laplace-type Integrals

Prototype: the **Laplace transform** $$F(s) = \int_0^{+\infty} f(t) e^{-st} \mathrm{d}t$$

Let us consider **generalized Laplace-type integrals** of the kind

$$I(k) = \int_a^b f(t) e^{-k\phi(t)} \mathrm{d}t$$

where $f(t)$, $\phi(t)$ are *real differentiable* functions and k is a *real* parameter.

We wish to obtain an asymptotic expansion of $I(k)$ in the limit $k \to +\infty$.

Asymptotics for Laplace-type Integrals: Intuition

Example:

$$I(k) = \int_0^b f(t) e^{-kt} \mathrm{d}t$$

As $k \to \infty$, the integrand becomes exponentially small for all t except for t near 0, because as $t \to 0$ and $k \to \infty$, kt could remain finite.

The global asymptotic behavior is thus related to the local behavior of the integrand as $t \to 0$

Example:

$$J(k) = \int_0^{+\infty} (1+t) e^{-kt} \mathrm{d}t \overset{\text{by parts}}{=} \frac{1}{k} + \frac{1}{k^2}$$

The asymptotic behavior is the same as for

$$\int_0^R e^{-kt} \mathrm{d}t = \frac{1 - e^{-kR}}{k} \sim \frac{1}{k}$$

Asymptotics for Laplace-type Integrals: Intuition

Similarly, we expect that the asymptotic behavior of

$$I\left(k\right)=\int_{a}^{b} f\left(t\right)e^{-k\phi\left(t\right)}\mathrm{d}t$$

be determined by the **local** behavior of the integrand in the neighborhood of the point $t=c$ where the function $\phi(t)$ has its **minimum** in the interval $a \leq t \leq b$.

Asymptotics for Laplace-type Integrals: Intuition

The minimum can occur either at the boundaries or at an interior point, which in the latter case necessarily means $\phi'(t)=0$. It follows that one only needs to carefully study such (**critical**) points.

We will separately consider:

- *The case that $\phi(t)$ is **monotonic** (hence the major contribution to the asymptotics of $I(k)$ comes from the boundaries)*
- *The case that $\phi(t)$ has a **local minimum** in $[a,\ b]$*

$\phi(t)$ Monotonic: Case of $f(t)$ Regular

In this case if $f(t)$ is sufficiently smooth the **integration by parts** approach provides the full asymptotic expansion.

Example:

$$I(k) = \int_0^{+\infty} \left(1+t^2\right)^{-2} e^{-kt} \mathrm{d}t$$

$$I(k) = \left[\left(1+t^2\right)^{-2} \frac{e^{-kt}}{-k}\right]_0^{+\infty} + \frac{1}{k} \int_0^{+\infty} -4t\left(1+t^2\right)^{-3} e^{-kt} \mathrm{d}t$$

$$= \frac{1}{k} + O\left(\frac{1}{k^2}\right)$$

This approach can be made rigorous by proving the following:

$\phi(t)$ Monotonic: $f(t)$ Regular: Integration by Parts

Theorem

Suppose that $f(t)$ has $N+1$ continuous derivatives while $f^{(N+2)}(t)$ is piecewise continuous on $a \leq t \leq b$. Then

$$I(k) = \int_a^b f(t) e^{-kt} \mathrm{d}t \sim \sum_{n=0}^{N} \frac{e^{-ka}}{k^{n+1}} f^{(n)}(a), \qquad k \to +\infty$$

Two generalizations:

1) If $b = +\infty$, then the above result is also valid provided that as $t \to \infty$ $f(t) = \mathrm{O}(e^{\alpha t})$, α real constant, so that $I(k)$ exists for k sufficiently large.

2) If $\phi(t)$ is *monotonic* in $[a, b]$, then the integral can be transformed to the above form by the change of variables $\tau = \phi(t)$. → $\int_a^b f(t) e^{-k\phi(t)} \mathrm{d}t$

$\phi(t)$ Monotonic: Case of $f(t)$ Singular

If $f(t)$ is not sufficiently smooth at $t = a$, then the integration by parts approach may not work.

Example:

$$I(k) = \int_0^5 \left(t^2 + 2t\right)^{-1/2} e^{-kt} \mathrm{d}t$$

By parts:

$$I(k) = \left[\frac{\left(t^2 + 2t\right)^{-1/2}}{-k} e^{-kt}\right]_0^5 + \frac{1}{k}\int_0^5 e^{-kt} \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(t^2 + 2t\right)^{-1/2}\right] \mathrm{d}t$$

but this is **singular** at $t = 0$... (in fact, $f(t) = \mathrm{O}(t^{-1/2})$ as $t \to 0$)

$\phi(t)$ Monotonic: Case of $f(t)$ Singular

Owing to the rapid decay of $\exp(-kt)$, $I(k)$ should be asymptotically equivalent to

$$\int_0^R \left(t^2 + 2t\right)^{-1/2} e^{-kt} \mathrm{d}t$$

for any R. If $R<2$ we may expand via Taylor series:

$$\left(t^2 + 2t\right)^{-1/2} = \left(2t\right)^{-1/2} \left(1 + \frac{t}{2}\right)^{-1/2} \sim \left(2t\right)^{-1/2} \left(1 - \frac{t}{4}\right) = \left(2t\right)^{-1/2} - \frac{\left(2t\right)^{1/2}}{8}$$

Hence

$$I\left(k\right) \sim \int_0^R \left(2t\right)^{-1/2} e^{-kt} \mathrm{d}t - \int_0^R \frac{\left(2t\right)^{1/2}}{8} e^{-kt} \mathrm{d}t$$

$\phi(t)$ Monotonic: Case of $f(t)$ Singular

To evaluate the above integrals in terms of known functions, we replace R by ∞. Again we expect that this introduces only an exponentially small error (i.e., terms beyond all orders) as $k \to \infty$. Thus

$$I(k) \sim \int_0^{+\infty} (2t)^{-1/2} e^{-kt} \mathrm{d}t - \int_0^{+\infty} \frac{(2t)^{1/2}}{8} e^{-kt} \mathrm{d}t$$

$$= \frac{1}{(2k)^{1/2}} \int_0^{+\infty} t^{-1/2} e^{-t} \mathrm{d}t - \frac{1}{2(2k)^{3/2}} \int_0^{+\infty} t^{1/2} e^{-t} \mathrm{d}t = \frac{\Gamma(1/2)}{(2k)^{1/2}} - \frac{\Gamma(3/2)}{2(2k)^{3/2}}$$

where

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t, \ \ \mathrm{Re}\{z\} > 0$$

(Gamma function)

$\phi(t)$ Monotonic, $f(t)$ Singular: Watson's Lemma

Theorem (Watson's Lemma)

If $f(t)$ is integrable in $[0,b]$, is $\mathrm{O}(e^{Ct})$ as $t \to +\infty$, and has the asymptotic expansion

$$f(t) \sim \sum_{n=0}^{+\infty} a_n t^{\lambda_n - 1}, \qquad t \to 0^+$$

with $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$ then

$$I(k) = \int_0^b f(t) e^{-kt} \mathrm{d}t \sim \sum_{n=0}^{+\infty} a_n \frac{\Gamma(\lambda_n)}{k^{\lambda_n}}, \qquad k \to +\infty$$

Actually, the same assumptions guarantee that

$$I(z) = \int_0^b f(t) e^{-zt} \mathrm{d}t \sim \sum_{n=0}^{+\infty} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}, \qquad \begin{cases} |z| \to +\infty \\ \left|\arg\{z\}\right| \le \dfrac{\pi}{2} - \delta < \dfrac{\pi}{2} \end{cases}$$

for some δ such that $0 < \delta < \pi/2$.

$\phi(t)$ with a Local Minimum

Now we consider the case that $\phi(t)$ is **not monotonic**.

We suppose, for simplicity, that the local minimum occurs at an **interior point** c, $a < c < b$, $\phi'(c) = 0$, $\phi''(c) > 0$.

Further, we assume that $\phi'(t) \neq 0$ in $[a, b]$ except at $t = c$ and that f and ϕ are sufficiently smooth.

By expanding both f and ϕ in the neighborhood of c, we expect that for large k

$$I(k) = \int_a^b f(t) e^{-k\phi(t)} \mathrm{d}t \sim \int_{c-R}^{c+R} f(c) e^{-k\left[\phi(c) + \frac{\phi''(c)}{2}(t-c)^2\right]} \mathrm{d}t$$

$\phi(t)$ with a Local Minimum

By letting $\tau = \sqrt{\frac{k}{2}\phi''(c)}\,(t-c)$ we have

$$I(k) \sim \frac{f(c)\,e^{-k\phi(c)}}{\sqrt{\frac{k}{2}\phi''(c)}} \int_{-R\sqrt{\frac{k}{2}\phi''(c)}}^{+R\sqrt{\frac{k}{2}\phi''(c)}} e^{-\tau^2}\,\mathrm{d}\tau$$

When k tends to infinity the latter integral becomes the Gauss integral and thus converges *exponentially* to $\sqrt{\pi}$. Hence

$$I(k) \sim f(c)\,e^{-k\phi(c)}\sqrt{\frac{2\pi}{k\phi''(c)}}$$

$\phi(t)$ with a Local Minimum: Laplace's Method

Theorem (Laplace's Method)

Assume that $\phi'(c) = 0$, $\phi''(c) > 0$ for some point c in the interval $[a, b]$; $\phi'(t) \neq 0$ in $[a, b]$ except at $t = c$; $\phi \in \mathrm{C}^4[a, b]$; and $f \in \mathrm{C}^2\,[a, b]$.

Then if c is an **interior point,**

$$I(k) = \int_a^b f(t)\, e^{-k\phi(t)}\,\mathrm{d}t \sim f(c)\, e^{-k\phi(c)} \sqrt{\frac{2\pi}{k\phi''(c)}}$$

with an error $\mathrm{O}(\mathrm{e}^{-k\phi(c)}/k^{3/2})$.

If c is an **endpoint,**

$$I(k) = \frac{1}{2} f(c)\, e^{-k\phi(c)} \sqrt{\frac{2\pi}{k\phi''(c)}}$$

with an error $\mathrm{O}(\mathrm{e}^{-k\phi(c)}/k)$.

Remark: Complete Asymptotic Expansions

The main idea of the proof of the Laplace method is to split $[a,\mathrm{b}]$ in two half-open intervals $[a,\mathrm{c})$ and $(c,\mathrm{b}]$, in each of which $\phi(t)$ is monotonic so that, using a change of variables, Watson's lemma can be applied:

E.g.,
$$\int_c^b f(t)e^{-k\phi(t)}\mathrm{d}t = e^{-k\phi(c)}\int_0^{\phi(b)-\phi(c)} \left.\frac{f(t)}{\phi'(\tau)}\right|_{t=t(\tau)} e^{-k\tau}\mathrm{d}\tau$$

where $\tau = \tau(t) = \phi(t) - \phi(c)$.

Since Watson's lemma can, in principle, provide infinite (i.e., **complete**) asymptotic expansions, also the Laplace method can, in principle, give complete asymptotic expansions. This fact, will be utilized further in connection with the *steepest descent method*.

Remark: Vanishing or Singular $f(t)$

Laplace's method can also be used when $f(t)$ either **vanishes** algebraically or becomes **infinite** at an algebraic rate.

Examples:

1) $$\int_0^5 \sin(s) e^{-k\sinh^4 s} ds \sim \int_0^R s e^{-ks^4} ds \sim \frac{1}{2}\int_0^{R^{1/2}} e^{-kt^2} dt \sim \frac{1}{4k^{1/2}}\int_0^{+\infty} e^{-\tau}\tau^{-1/2} d\tau$$
$$= \frac{1}{4k^{1/2}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\frac{\pi}{k}}$$

2) $$\int_0^{+\infty} \frac{e^{-kt^2}}{\sqrt{\sinh t}} dt = \frac{1}{2k^{1/4}}\int_0^{+\infty} e^{-\tau}\tau^{\frac{1}{4}-1} d\tau = \frac{\Gamma\left(\frac{1}{4}\right)}{2k^{1/4}}$$

Example: Asymptotic Expansion of the Gamma Function

Evaluate the Gamma function for **large real** values of its argument:

$$\Gamma\left(k+1\right) = \int_0^{+\infty} t^k e^{-t} \mathrm{d}t, \quad k \to +\infty$$

By writing $t^k = e^{k \ln t}$ we could let

$$f\left(t\right) = e^{-t}$$

$$\phi\left(t\right) = \ln t$$

However, the maximum of $\phi(t)$ occurs at infinity, where $f(t)$ decays *exponentially* to zero…

Example: Asymptotic Expansion of the Gamma Function

By writing $t^k e^{-t} = e^{k\ln t - t}$ we see that the true maximum of the exponent occurs when

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(k\ln t - t\right) = 0$$

hence at $t{=}k$ (since it depends on k, it is a *movable* maximum).

This suggests to make a change of variable: $t = ks$, such that

$$\Gamma\left(k+1\right) = k^{k+1}\int_0^{+\infty} e^{-k\left(s-\ln s\right)}\mathrm{d}s = k^{k+1}\int_0^{+\infty} f\left(s\right)e^{-k\phi\left(s\right)}\mathrm{d}s$$

where $\phi\left(s\right) = s - \ln s$ and $f\left(s\right) = 1$.

Example: Asymptotic Expansion of the Gamma Function

The function $\phi(s)$ now has a **fixed** minimum at $s{=}1$, where $\phi(s{=}1){=}1$. Furthermore,

$$\phi''(s) = \frac{1}{s^2} \quad \Rightarrow \quad \phi''(s=1) = 1$$

Therefore, using the standard formula of Laplace's method we find

$$\Gamma(k+1) \sim k^{k+1} f(c) e^{-k\phi(c)} \sqrt{\frac{2\pi}{k\phi''(c)}} = k^{k+1} e^{-k} \sqrt{\frac{2\pi}{k}} = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$$

known as the **Stirling-Laplace approximation** of the Gamma function (or the factorial: $\Gamma(n{+}1){=}n!$).

Asymptotic Evaluation of Integrals: Fourier Type Integrals

Fourier-type Integrals

Prototype: the **Fourier transform**

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{j\omega t} \mathrm{d}t$$

Let us consider **generalized Fourier-type integrals** of the kind

$$I(k) = \int_{a}^{b} f(t) e^{jk\phi(t)} \mathrm{d}t$$

where $f(t)$, $\phi(t)$ are *real continuous* functions and k is a *real* parameter.

We wish to obtain an asymptotic expansion of $I(k)$ in the limit $k \to +\infty$

Riemann-Lebesgue Lemma

The value of a Fourier-type integral **tends to zero** as $k \to \infty$. This is a consequence of the fact that as $k \to \infty$, the exponential factor oscillates rapidly and these oscillations are self-canceling.

Theorem (Riemann-Lebesgue Lemma)

$I(k) \to 0$ as $k \to \infty$, provided that:

- $\int_a^b |f(t)| \, dt$ exists

  $\phi(t)$ is continuously differentiable in $[a, b]$

  $\phi(t)$ is not constant on any subinterval of $[a, b]$

Asymptotics for Fourier-type Integrals: Intuition

Suppose that $t=c$ is a point in $[a,b]$ for which $\phi'(t)$ **does not vanish**. If Ω_c is a small neighborhood of c, then we expect that $I(k)$ can be approximated by

$$I_c(k) = f(c)\int_{\Omega_c} e^{jk\phi(t)}\mathrm{d}t$$

As $k\rightarrow\infty$, the rapid oscillation of $\exp(jk\phi)$ produces cancelations that, in turn, tend to decrease the value of $I_c(k)$.

But if we assume that $\phi'(t)$ **vanishes** at $t=c$, then, even for large k, there exists a small neighborhood of c throughout which $k\phi$ does **not** change so rapidly. In this region, $\exp(jk\phi)$ oscillates less rapidly and less cancelation occurs.

The Method of Stationary Phase

Thus we expect that the value of $I(k)$ for large k depends primarily on the behavior of f and ϕ near points for which $\phi'(t)=0$. Such points are called, in calculus, **stationary points**.

Furthermore, in many applications, ϕ has the physical interpretation of a **phase**.

Thus the asymptotic method based on the above arguments is usually referred to as the **method of stationary phase**.

As for Laplace-type integrals, we will separately consider:

- *The case that $\phi(t)$ is **monotonic** (hence the major contribution to the asymptotics of $I(k)$ comes from the boundaries)*
- *The case that $\phi(t)$ has a **stationary point** in $[a,b]$*

$\phi(t)$ Monotonic, $f(t)$ Regular: Integration by Parts

Theorem

Suppose that $f(t)$ has $N+1$ continuous derivatives while $f^{(N+2)}(\mathrm{t})$ is piecewise continuous on $a \le t \le b$ with $a,\ b$ finite. Then

$$I\left(k\right) = \int_a^b f\left(t\right) e^{jkt} \mathrm{d}t \sim \sum_{n=0}^{N} \frac{\left(-1\right)^n}{\left(jk\right)^{n+1}} \left[f^{\left(n\right)}\left(b\right) e^{jkb} - f^{\left(n\right)}\left(a\right) e^{jka} \right],$$

$$k \to +\infty$$

Generalization:

If $\phi(t)$ is *monotonic* in $[a, b]$, then the integral $\int_a^b f\left(t\right) e^{jk\phi\left(t\right)} \mathrm{d}t$

can be transformed to the above form by the change of variables $\tau = \phi(t)$.

$\phi(t)$ Monotonic, $f(t)$ Regular: Integration by Parts

Example

Evaluate $$I(k) = \int_0^1 \frac{e^{jkt}}{1+t}\,\mathrm{d}t, \quad k \to +\infty$$

Using integration by parts we find:

$$I(k) \sim e^{jk}\left[\frac{1}{2(jk)} + \frac{1}{2^2(jk)^2} + \dots + \frac{(n-1)!}{2^n(jk)^n}\right] - \left[\frac{1}{(jk)} + \frac{1}{(jk)^2} + \dots + \frac{(n-1)!}{(jk)^n}\right], \quad k \to +\infty$$

$\phi(t)$ Monotonic, $f(t)$ Singular: Generalized Watson's Lemma

Theorem (Generalized Watson's Lemma)

If $f(t)$ is zero with all its derivatives at $t=b$, $f(t)$ exists with all its derivatives in $(0,b]$, and

$$f\left(t\right)=t^{\gamma}+o\left(t^{\gamma}\right),\qquad t\to 0^{+}$$

then

$$I\left(k\right)=\int_{0}^{b}f\left(t\right)e^{jk\mu t}\mathrm{d}t\sim\left(\frac{1}{k}\right)^{\gamma+1}\Gamma\left(\gamma+1\right)e^{j\frac{\pi}{2}\left(\gamma+1\right)\mu}+o\left(k^{-\gamma+1}\right),$$

$$k\to+\infty$$

$\phi(t)$ with a Stationary Point

Now we consider the case that $\phi(t)$ is **not monotonic**.

We suppose, for simplicity, that the local minimum occurs at an **interior point** c, $a < c < b$, $\phi'(c) = 0$, $\phi''(c) > 0$.

Further, we assume that $\phi'(t) \neq 0$ in $[a, b]$ except at $t = c$ and that f and ϕ are sufficiently smooth.

By expanding both f and ϕ in the neighborhood of c, we expect that for large k

$$I(k) = \int_a^b f(t) e^{jk\phi(t)} \mathrm{d}t \sim \int_{c-R}^{c+R} f(c) e^{jk\left[\phi(c) + \frac{\phi''(c)}{2}(t-c)^2\right]} \mathrm{d}t$$

$\phi(t)$ with a Stationary Point

To evaluate this integral, we let

$$\mu\tau^2 = k\frac{\phi''(c)}{2}(t-c)^2 \quad \leftrightarrow \quad \tau = (t-c)\sqrt{\frac{|\phi''(c)|k}{2}}$$

where $\mu = \mathrm{sgn}\{\phi''(c)\}$:

$$\Rightarrow \quad I(k) \sim f(c)e^{jk\phi(c)}\sqrt{\frac{2}{k|\phi''(c)|}}\int_{-R\sqrt{k|\phi''(c)|/2}}^{+R\sqrt{k|\phi''(c)|/2}} e^{j\mu\tau^2}\,\mathrm{d}\tau$$

As $k \to \infty$ the last integral reduces to $\int_{-\infty}^{+\infty} e^{j\mu\tau^2}\,\mathrm{d}\tau = \sqrt{\pi}e^{\frac{j\pi\mu}{4}}$, hence

$$I(k) = \int_a^b f(t)e^{jk\phi(t)}\,\mathrm{d}t \sim e^{jk\phi(c)}f(c)\sqrt{\frac{2\pi}{k|\phi''(c)|}}e^{\frac{j\pi\mu}{4}}$$

$\phi(t)$ with a Stationary Point: Stationary Phase Method

Theorem (Stationary Phase Method)

If

- $t=c$ is the only point in $[a,\ b]$ where $\phi(t)$ vanishes;
- $f(t)$ vanishes infinitely smoothly at the two end points $t=a$ and $t=b$;
- both f and ϕ are infinitely differentiable on the half-open intervals $[a,\ c)$ and $(c,\ b]$;
- $\phi(t)-\phi(c)\sim\alpha(t-c)^2+o\left((t-c)^2\right),\quad f(t)\sim\beta(t-c)^\gamma+o\left((t-c)^\gamma\right)$
  $t\to$ c; $\gamma>-1$;

then:

$$I(k)=\int_a^b f(t)e^{jk\phi(t)}\mathrm{d}t\sim e^{jk\phi(c)}\beta\,\Gamma\left(\frac{\gamma+1}{2}\right)e^{j\frac{\pi(\gamma+1)}{4}\mu}\left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}}$$

$$+o\left(k^{-\frac{\gamma+1}{2}}\right),\quad k\to+\infty \qquad \mu=\operatorname{sgn}\alpha$$

$\phi(t)$ with a Stationary Point: Stationary Phase Method

This result can be **generalized** substantially. It is possible to allow $\phi(t)$ and $f(t)$ to have different asymptotic behaviors as $t \to c^+$ and $t \to c^-$.

For instance, if

$$\phi(t) - \phi(c) \sim \alpha_+ (t-c)^\nu, \quad f(t) \sim \beta_+ (t-c)^\gamma + o\left((t-c)^\gamma\right), \quad t \to c^+, \gamma > -1$$

then

$$\int_c^b f(t) e^{jk\phi(t)} \mathrm{d}t \sim \frac{1}{\nu} e^{jk\phi(c)} \beta_+ \Gamma\left(\frac{\gamma+1}{\nu}\right) e^{j\frac{\pi(\gamma+1)}{2\nu}\mu} \left(\frac{1}{k|\alpha_+|}\right)^{\frac{\gamma+1}{\nu}}, \quad k \to +\infty$$

$$\mu = \operatorname{sgn} \alpha_+$$

$\phi(t)$ with a Stationary Point: Stationary Phase Method

Example

Evaluate the leading behavior of the Bessel function of the first kind $J_n(n)$ as $n \to \infty$.

Using the integral representation $J_n(n) = \frac{1}{\pi}\int_0^\pi \cos\left(n\sin t - nt\right)\mathrm{d}t$

$$J_n(n) = \frac{1}{\pi}\mathrm{Re}\int_0^\pi e^{jn\left(\sin t - t\right)}\mathrm{d}t$$

$$\sin t \sim t - \frac{t^3}{6} + o\left(t^3\right),\quad t \to 0 \quad \text{(Taylor)}$$

$$\sim \frac{1}{\pi}\mathrm{Re}\int_0^\pi e^{-jn\frac{t^3}{6}}\mathrm{d}t = \frac{1}{3\pi}\cos\left(\frac{\pi}{6}\right)\Gamma\left(\frac{1}{3}\right)\left(\frac{6}{n}\right)^{\frac{1}{3}},\qquad n \to +\infty$$

where the previous formula has been applied with $\alpha_+ = -1/6$, $\nu = 3$, $\beta_+ = 1$, $\gamma = 0$.

Fourier vs. Laplace

There is an **important difference** between Fourier and Laplace type integrals.

- For Fourier type integrals, although the **stationary points** give the **dominant contribution**, we must **also** consider the **endpoints** if more than the leading term is needed. The endpoint contribution is only algebraically smaller than the stationary point contribution.

- In contrast, we recall that for the Laplace type integrals we have considered so far, the **entire** asymptotic expansion depends only on the behavior of the integrand in a small neighborhood of the **global minimum** of ϕ; the points away from the minimum are exponentially small in comparison.

References

M. J. Ablowitz and A. S. Fokas, *Complex variables. Introduction and applications*. Cambridge, UK: Cambridge University Press, 2003.