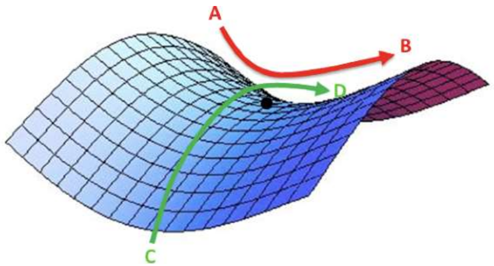
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 4

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Asymptotic Expansions:
Ray Optics

Motivation: Discontinuities of the Electromagnetic Field

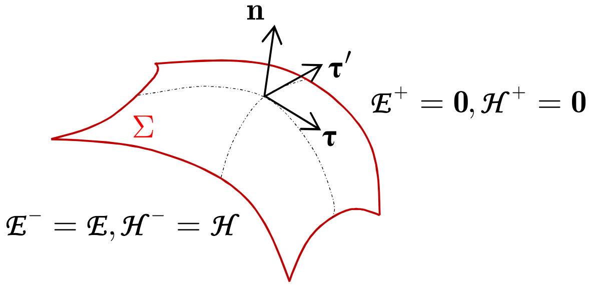
Let us consider the electromagnetic field in a free space filled with a **non-conductive**, **homogeneous**, **isotropic**, **non-dispersive** medium:

$$\mathcal{D} = \varepsilon \mathcal{E}$$

$$\mathcal{B} = \mu \mathcal{H}$$

$$\mathcal{J} = \mathbf{0}$$

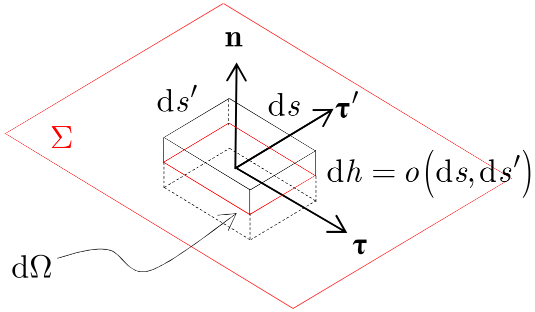
Let Σ be a surface where the time-domain electromagnetic field $\mathcal{E}$, $\mathcal{H}$, has a **discontinuity**; for instance, it is *non-zero* on one side of Σ and zero on the other side:



Motivation: Discontinuities of the Electromagnetic Field

As in the usual derivation of the boundary conditions across interfaces between different media, let us enforce the magnetic Gauss law to an elementary parallelepiped dΩ:

$$\oint_{\partial d\Omega} \boldsymbol{\nu} \cdot \mathcal{B} \mathrm{d}S = 0$$



$$\mathbf{n} \cdot \mathcal{B}^{+} \mathrm{d}s \mathrm{d}s' - \mathbf{n} \cdot \mathcal{B}^{-} \mathrm{d}s \mathrm{d}s' = 0$$

$$\Rightarrow \quad 0 = \mathcal{B}_{\mathrm{n}}^{+} = \mathcal{B}_{\mathrm{n}}^{-} = \mu \mathcal{H}_{\mathrm{n}}$$

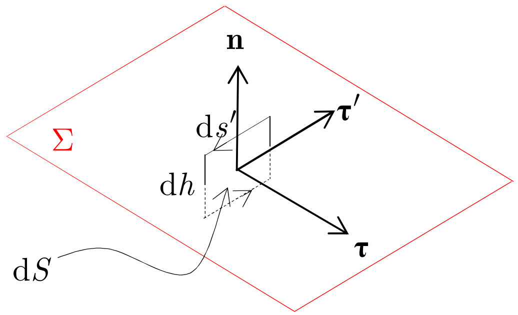
The magnetic field $\mathcal{H}$ is *purely tangential* on Σ.

Dually, also the electric field $\mathcal{E}$ is *purely tangential* on Σ.

Motivation: Discontinuities of the Electromagnetic Field

Let us now enforce the Faraday-Neumann-Lenz law to an elementary circuit d*S* across the surface Σ:

$$\oint_{\partial \mathrm{d}\sigma} \mathcal{E} \cdot \mathbf{ds} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathrm{d}\sigma} \boldsymbol{\tau} \cdot \mathcal{B} \mathrm{d}S$$



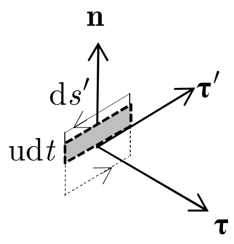
The LHS gives: $-\boldsymbol{\tau}' \cdot \mathcal{E}^{+} \mathrm{d}s' + \boldsymbol{\tau}' \cdot \mathcal{E}^{-} \mathrm{d}s'$

If the surface Σ were static, the RHS would be zero and one would then conclude that also the tangential electric field is zero. Hence the total field would be zero, against the assumption of a non-zero field on Σ^{-}.

Therefore, the surface Σ *must* move, i.e., **the discontinuity of the e.m. field propagates**.

Motivation: Discontinuities of the Electromagnetic Field

By letting then *u* be the velocity of Σ along ***n*** one has



$$-\boldsymbol{\tau}' \cdot \mathcal{E}^{+}\mathrm{d}s' + \boldsymbol{\tau}' \cdot \mathcal{E}^{-}\mathrm{d}s' = \boldsymbol{\tau}' \cdot \mathcal{E}\,\mathrm{d}s' = -\boldsymbol{\tau} \cdot \mathcal{B}\,\mathrm{d}s'u = -\mu u\boldsymbol{\tau} \cdot \mathcal{H}\,\mathrm{d}s'$$

Since

$$\boldsymbol{\tau}' \cdot \mathcal{E} = \mathbf{n} \times \boldsymbol{\tau} \cdot \mathcal{E} = -\boldsymbol{\tau} \times \mathbf{n} \cdot \mathcal{E} = -\boldsymbol{\tau} \cdot \mathbf{n} \times \mathcal{E}$$

this can be written as

$$\boldsymbol{\tau} \cdot \mathbf{n} \times \mathcal{E} = \mu u \boldsymbol{\tau} \cdot \mathcal{H}$$

Motivation: Discontinuities of the Electromagnetic Field

The arbitrariness of τ in the tangent plane allows for deducing

$$\boxed{\begin{aligned} &\mathbf{n} \times \mathcal{E} = \mu u \mathcal{H} \\ &\mathbf{n} \times \mathcal{H} = -\varepsilon u \mathcal{E} \end{aligned}}$$

and dually: (second relation above)

This set of relations in turn implies (*proof by exercise*):

$$u = \frac{1}{\sqrt{\mu\varepsilon}}$$

The surface of discontinuity moves with the *velocity of light* in vacuum

$$\mathcal{H} = \frac{1}{\zeta}\mathbf{n} \times \mathcal{E}$$

The fields, purely tangential to S, are linked as in *uniform plane waves*

$$\left(\zeta = \sqrt{\frac{\mu}{\varepsilon}}\right)$$

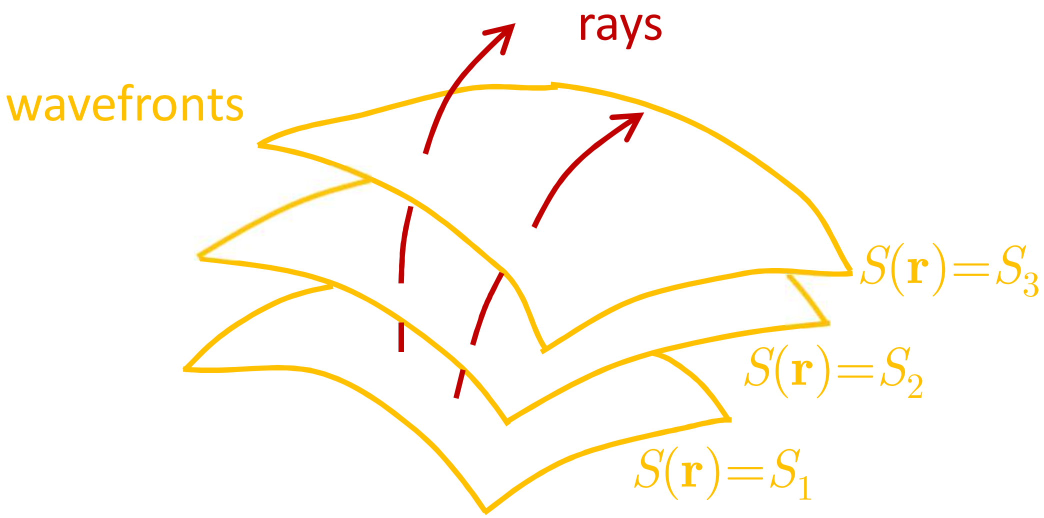
Therefore, the time-domain electromagnetic field in the vicinity of a surface of discontinuity (*wavefront*) **behaves locally as a uniform plane wave**.

Wavefronts, Iconal, Rays

As time flows, Σ moves and thus **a family of surfaces** arise, whose equation can always be cast in the form

$$S(\mathbf{r}) = ct \qquad (c \text{ velocity of light in vacuum})$$

in terms of a suitable (real) function $S(\mathbf{r})$ of the coordinates known as **iconal**.



The lines orthogonal to the wavefronts are termed **rays**.

The Iconal Equation

Differentiating the equation of the wavefronts:

$$\mathrm{d}S\left(\mathbf{r}\right) = c\,\mathrm{d}t$$

On the other hand, if $\mathrm{d}\mathbf{r} \parallel \nabla S$ then

$$\mathrm{d}S\left(\mathbf{r}\right) = \nabla S \cdot \mathrm{d}\mathbf{r} = \left|\nabla S\right|\left|\mathrm{d}\mathbf{r}\right| = \left|\nabla S\right| u\,\mathrm{d}t$$

$$\Rightarrow \quad \left|\nabla S\right| = \frac{c}{u} = n\left(\mathbf{r}\right)$$

($n\left(\mathbf{r}\right) = \sqrt{\mu_{\mathrm{r}}\varepsilon_{\mathrm{r}}}$ refractive index)

$$\Rightarrow \quad \boxed{\left|\nabla S\right|^2 = n^2\left(\mathbf{r}\right)}$$

iconal equation

Characteristic Equation of the Wave Equation

The iconal equation is a *non-linear, first-order* PDE.

By letting $\Phi(\mathbf{r},t) = S(\mathbf{r}) - ct$, from the iconal equation it follows that

$$\left|\nabla\Phi\right|^2 - \frac{n^2(\mathbf{r})}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2 = 0$$

The **characteristic equation** of a *linear, second-order* PDE is obtained by replacing the second derivatives with the square of the first derivatives. Hence we see that we have obtained the characteristic equation of the **wave equation**

$$\nabla^2\Psi - \frac{n^2(\mathbf{r})}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

We have thus deduced the well-known fact that Φ (*solution of the characteristic equation*) is constant on the surfaces of discontinuity of ψ (*solution of the wave equation*).

High-Frequency Ansatz

Since time discontinuities correspond to high-frequency components of the Fourier spectrum, the previous discussion should motivate the following *ansatz* for **approximate, high-frequency solutions** of the time-harmonic Maxwell equations:

$$\mathbf{E}(\mathbf{r}) \cong \mathbf{E}_0(\mathbf{r})\, e^{-jk_0 S(\mathbf{r})}$$

where $\mathbf{E}_0(\mathbf{r}), S(\mathbf{r})$ are a *slowly varying* functions (i.e., their characteristic length scale of variation is large w.r.t. the wavelength in the medium) ***independent*** of k_0 (i.e., of frequency).

Such an ansatz says that, in the high-frequency limit, the field *behaves locally* as a *uniform plane wave*.

High-Frequency Solution of Maxwell Equations

By inserting the ansatz in the source-free, time-harmonic Maxwell Equations one readily finds:

$$\nabla S \times \mathbf{E}_0 - \zeta_0 \mathbf{H}_0 + \left(\frac{j}{k_0}\right) \nabla \times \mathbf{E}_0 = \mathbf{0} \qquad \nabla S \cdot \mathbf{H}_0 + \left(\frac{j}{k_0}\right) \nabla \cdot \mathbf{H}_0 = 0$$

$$\nabla S \times \mathbf{H}_0 + \left(\frac{n^2}{\zeta_0}\right) \mathbf{E}_0 + \left(\frac{j}{k_0}\right) \nabla \times \mathbf{H}_0 = \mathbf{0} \qquad \nabla S \cdot \mathbf{E}_0 + \left(\frac{j}{k_0 n^2}\right) \nabla \cdot \left(n^2 \mathbf{E}_0\right) = 0$$

We now let $k_0 \to \infty$, so terms in $1/k_0$ are neglected. After left-multiplying vectorially by ∇S one has (*derivation by exercise*)

$$\nabla S \times \left(\nabla S \times \mathbf{E}_0\right) - n^2 \mathbf{E}_0 = \nabla S \left(\underbrace{\nabla S \cdot \mathbf{E}_0}_{=0}\right) - \mathbf{E}_0 \left|\nabla S\right|^2 + n^2 \mathbf{E}_0 = \left[n^2 - \left|\nabla S\right|^2\right] \mathbf{E}_0 = \mathbf{0}$$

$$\Rightarrow \quad \boxed{\left|\nabla S\right|^2 = n^2(\mathbf{r})} \quad \text{iconal equation}$$

Luneburg-Kline Asymptotic Series

However, for finite frequencies we are not able to appreciate the error committed by representing the field through the above procedure. An answer to this problem can be found by resorting to a **full asymptotic expansion** of the field.

Let us do that first using the **scalar approximation** (much used in optics), i.e. by considering a generic component $u(\mathbf{r}, k_0)$ of the e.m. field, solution of the scalar wave equation (valid for slowly varying refraction indices):

$$\nabla^2 u(\mathbf{r}, k_0) + k_0^2 n^2(\mathbf{r}) u(\mathbf{r}, k_0) = 0$$

In particular, let us assume that the following asymptotic expansion holds:

$$\boxed{u(\mathbf{r}, k_0) \sim e^{-jk_0 S(\mathbf{r})} \sum_{m=0}^{+\infty} \frac{A_m(\mathbf{r})}{(-jk_0)^m}}$$

Asymptotic series
of Luneburg and Kline (LK)

Luneburg-Kline Asymptotic Series

By introducing the LK series into the scalar wave equation one has

$$\sum_{m=0}^{N} \frac{Q_m(\mathbf{r})}{(-jk_0)^{m-2}} = o\left(k_0^{-N}\right)$$

where

$$Q_0(\mathbf{r}) = |\nabla S|^2 - n^2(\mathbf{r})$$
$$Q_1(\mathbf{r}) = \left(\nabla^2 S + 2\nabla S \cdot \nabla\right) A_0$$
$$\ldots.$$
$$Q_m(\mathbf{r}) = \left(\nabla^2 S + 2\nabla S \cdot \nabla\right) A_{m-1} + \nabla^2 A_{m-2}$$

This is satisfied for any N if and only if $Q_m(\mathbf{r}) = 0 \;\; (m = 0, 1, 2 \ldots)$, which yield a set of equations for S and the A_m's.

In particular, for *m*=0 one finds again

$$\boxed{|\nabla S|^2 = n^2(\mathbf{r})}$$

iconal equation

The Ray Equation

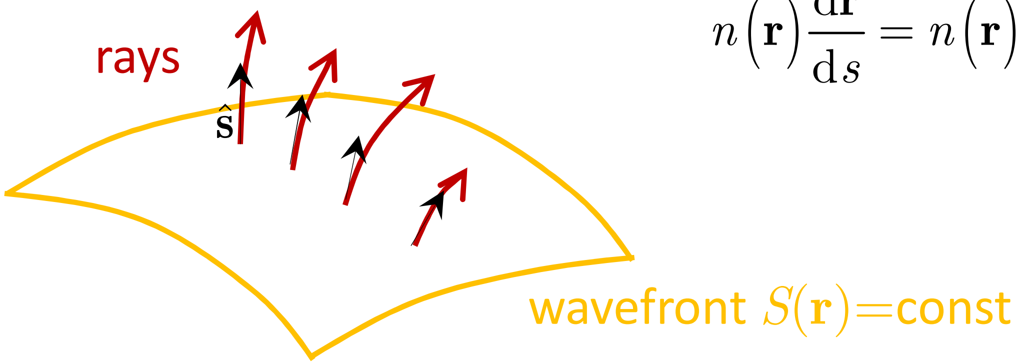
Let us consider a single-valued eikonal $S(\mathbf{r})$ and let us define the unit vector

$$\hat{\mathbf{s}}\left(\mathbf{r}\right) = \nabla S \,/\, \left|\nabla S\right| = \nabla S \,/\, n\left(\mathbf{r}\right)$$

and let us consider the trajectories (**rays**) $\mathbf{r}\left(s\right)$ tangent to $\hat{\mathbf{s}}\left(\mathbf{r}\right)$ at each **r**.

(Whenever S is not single-valued, the region will be spanned by a multiplicity of ray-families.)

Assuming s is the curvilinear abscissa along the ray,



$$n\left(\mathbf{r}\right)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = n\left(\mathbf{r}\right)\hat{\mathbf{s}}\left(\mathbf{r}\right) = \nabla S = \frac{1}{k_0}\mathbf{k}$$

($\mathbf{k}$ local wavevector)

The Ray Equation

Let us further derive the latter equation w.r.t the curvilinear abscissa s:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}s}}_{=\hat{\mathbf{s}}(\mathbf{r})\cdot\nabla}\left(n(\mathbf{r})\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right)=\underbrace{\hat{\mathbf{s}}(\mathbf{r})}_{=\nabla S/n(\mathbf{r})}\cdot\nabla\left(\underbrace{n(\mathbf{r})\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}}_{=\nabla S}\right)=\frac{1}{n(\mathbf{r})}\nabla S\cdot\nabla\nabla S=\frac{1}{n(\mathbf{r})}\frac{1}{2}\underbrace{\nabla\left(\overbrace{\nabla S\cdot\nabla S}^{=n^2(\mathbf{r})}\right)}_{=2\nabla S\cdot\cancel{\nabla\times\nabla S}+2\nabla S\cdot\nabla\nabla S}$$

$$=\frac{1}{n(\mathbf{r})}\frac{1}{2}\underbrace{\nabla n^2}_{=2n\nabla n}=\nabla n$$

$$\Rightarrow\quad\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left(n(\mathbf{r})\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right)=\nabla n}$$

Ray Equation

In a *homogeneous* medium: $\left(n(\mathbf{r})=n=\mathrm{const}\right)$

$$n\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2}=\mathbf{0}\quad\Rightarrow\quad r(s)=s\hat{\mathbf{s}}+\mathbf{r}_0$$

The rays are straight lines

The Ray Equation

Instead, in a *non-homogeneous* medium the rays are **curved lines**:

Frenet equations:

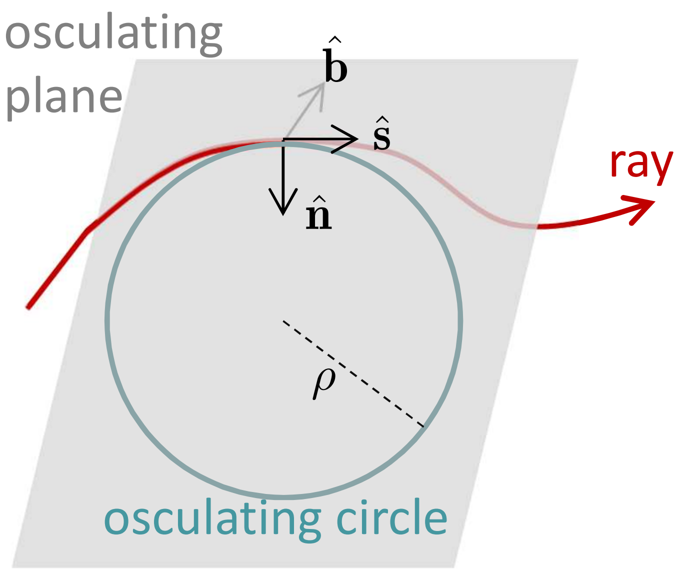
curvature radius

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}s} = \frac{1}{\rho}\hat{\mathbf{n}}$$

$$\frac{\mathrm{d}\hat{\mathbf{n}}}{\mathrm{d}s} = -\frac{1}{\rho}\hat{\mathbf{s}} + \frac{1}{\tau}\hat{\mathbf{b}}$$

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}s} = -\frac{1}{\tau}\hat{\mathbf{n}}$$

torsion



Hence, from the ray equation:

$$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\left(\mathbf{r}\right)\hat{\mathbf{s}}\right) = \frac{\mathrm{d}n}{\mathrm{d}s}\hat{\mathbf{s}} + n\left(\mathbf{r}\right)\frac{1}{\rho}\hat{\mathbf{n}} \quad \rightarrow \quad \hat{\mathbf{n}}\cdot\nabla n = n\left(\mathbf{r}\right)\frac{1}{\rho} > 0$$

The ray lies locally in the plane formed by ∇n and $\hat{\mathbf{s}}$ and bends towards ∇n .

The Malus-Dupin Theorem

The ray equation expresses a necessary condition for a ray bundle to be orthogonal to a family of wavefronts, but it does not always imply the existence of an eikonal.

The eikonal exists only if

$$\boxed{\nabla \times \left(n\left(\mathbf{r}\right)\hat{\mathbf{s}}\right) = \mathbf{0}}$$

This is equivalent to assuming the dyadic $\nabla\left(n\left(\mathbf{r}\right)\hat{\mathbf{s}}\right)$ to be symmetric.

It can be shown that if this condition is verified at one point, then it holds everywhere, also if the refractive index is discontinuous (e.g., on reflecting or refracting surfaces).

This result is known as the **Malus-Dupin Theorem**.

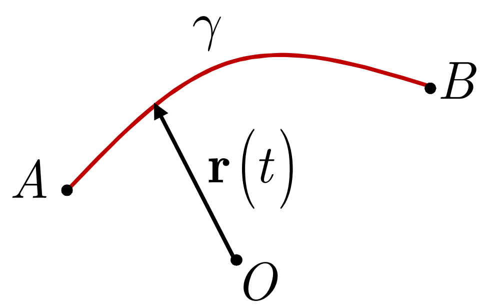
Variational Properties of the Rays

Definition

Optical path along the curve γ:

$$I\left[\mathbf{r}\left(t\right)\right] = \int_{\gamma} n\left[\mathbf{r}\left(t\right)\right]\mathrm{d}s = \int_{t_A}^{t_B} n\left[\mathbf{r}\left(t\right)\right]\left|\mathbf{r}'\left(t\right)\right|\mathrm{d}t$$

$$\left(\mathrm{d}s = \left|\mathbf{r}'\right|\mathrm{d}t\right)$$



The optical path is a *functional* of the kind

$$\boxed{I = \int_{a}^{b} F\left(x, \mathbf{y}, \mathbf{y}'\right)\mathrm{d}x}$$

where $\begin{cases} x \rightarrow t \\ \mathbf{y}\left(x\right) \rightarrow \mathbf{r}\left(t\right) \\ a, b \rightarrow t_A, t_B \end{cases}$

Let us look for the conditions under which such a functional is *extremal*…

Variational Properties of the Rays

Variation: $\mathbf{y}=\mathbf{f}\left(x\right)\rightarrow\mathbf{y}=\mathbf{f}\left(x\right)+\alpha\mathbf{g}\left(x\right)$

$$\mathbf{g}(a)=\mathbf{g}(b)=\mathbf{0}$$

$$\Rightarrow \quad I\left(\alpha\right)=\int_a^b F\left[x,\mathbf{y},\mathbf{y}'\right]_{\mathbf{y}=\mathbf{f}\left(x\right)+\alpha\mathbf{g}\left(x\right)}\mathrm{d}x$$

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha}=\int_a^b\nabla_{\mathbf{y}}F\cdot\mathbf{g}\left(x\right)+\nabla_{\mathbf{y}'}F\cdot\mathbf{g}'\left(x\right)\mathrm{d}x=$$

(integrating by parts the second addend)

$$=\int_a^b\left[\nabla_{\mathbf{y}}F-\frac{\mathrm{d}}{\mathrm{d}x}\nabla_{\mathbf{y}'}F\right]\cdot\mathbf{g}\left(x\right)\mathrm{d}x=0$$

The arbitrariness of $\mathbf{g}\left(x\right)$ allows for concluding

$$\boxed{\nabla_{\mathbf{y}}F-\frac{\mathrm{d}}{\mathrm{d}x}\left(\nabla_{\mathbf{y}'}F\right)=\mathbf{0}}$$

Euler-Lagrange system
of ordinary differential equations

Variational Properties of the Rays

Let us apply this to the optical path, where $F\left(t,\mathbf{r},\mathbf{r}'\right)=n\left(\mathbf{r}\right)\left|\mathbf{r}'\right|$

$$\Rightarrow \qquad \nabla_{\mathbf{r}}F=\left|\mathbf{r}'\right|\nabla_{\mathbf{r}}n, \qquad \nabla_{\mathbf{r}'}F=n\left(\mathbf{r}\right)\frac{1}{\left|\mathbf{r}'\right|}\mathbf{r}'$$

The Euler-Lagrange equations then read

$$\nabla_{\mathbf{r}}F-\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\mathbf{r}'}F=\mathbf{0}\Rightarrow\left|\mathbf{r}'\right|\nabla_{\mathbf{r}}n-\frac{\mathrm{d}}{\mathrm{d}t}\left[n\left(\mathbf{r}\right)\frac{1}{\left|\mathbf{r}'\right|}\mathbf{r}'\right]=\mathbf{0}$$

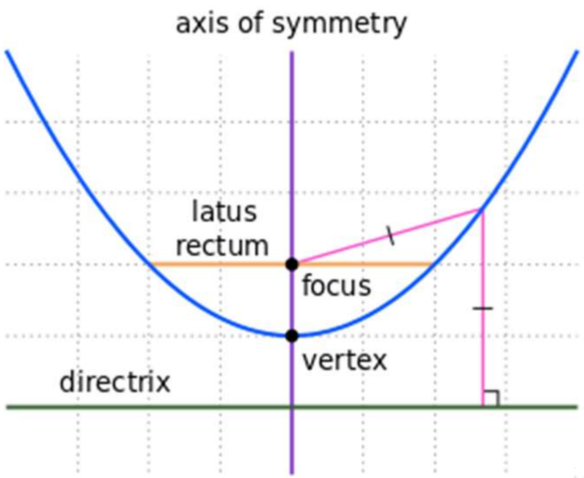
$$\Rightarrow \qquad \boxed{\frac{\mathrm{d}}{\mathrm{d}s}\left[n\left(\mathbf{r}\right)\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right]=\nabla_{\mathbf{r}}n}$$

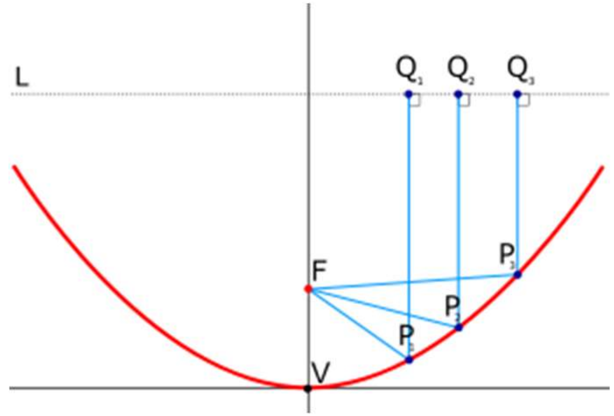
i.e., the Ray Equation!

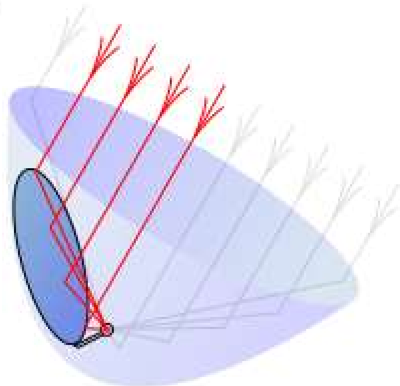
Fermat Principle: the optical path is stationary along a ray

Consequences of the Fermat Principle

The metric properties of conical sections give rise to corresponding ray-optical properties. For instance, we have the focusing property of a **parabolic** mirror:



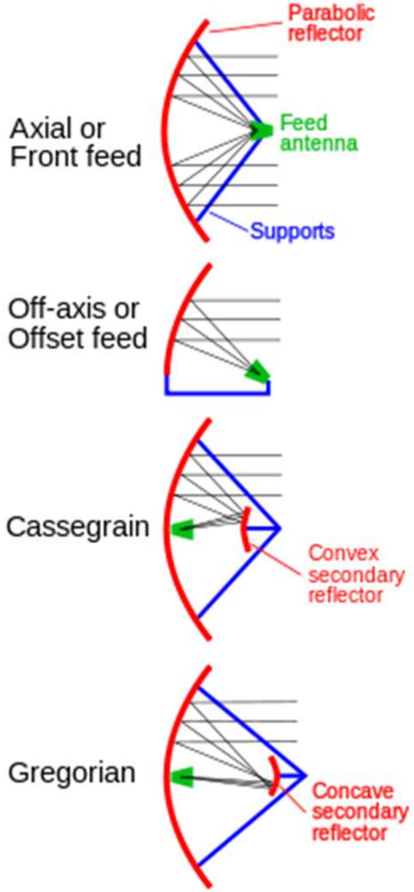




off-axis parabolic reflector

Consequences of the Fermat Principle

These can be combined with the metric properties of the **ellipse** and the **hyperbola** to provide composite reflector-antenna systems…



Cassegrain configuration:
parabolic reflector, hyperbolic sub-reflector

Field-Transport Equation for A_0

Recall: $$\sum_{m=0}^{N} \frac{Q_m(\mathbf{r})}{\left(-jk_0\right)^{m-2}} = o\left(k_0^{-N}\right)$$ LK expansion

$Q_0(\mathbf{r}) = \left|\nabla S\right|^2 - n^2(\mathbf{r}) = 0$ eikonal equation

$Q_1(\mathbf{r}) = \left(\nabla^2 S + 2\nabla S \cdot \nabla\right) A_0 = 0$ field-transport equation for A_0

....

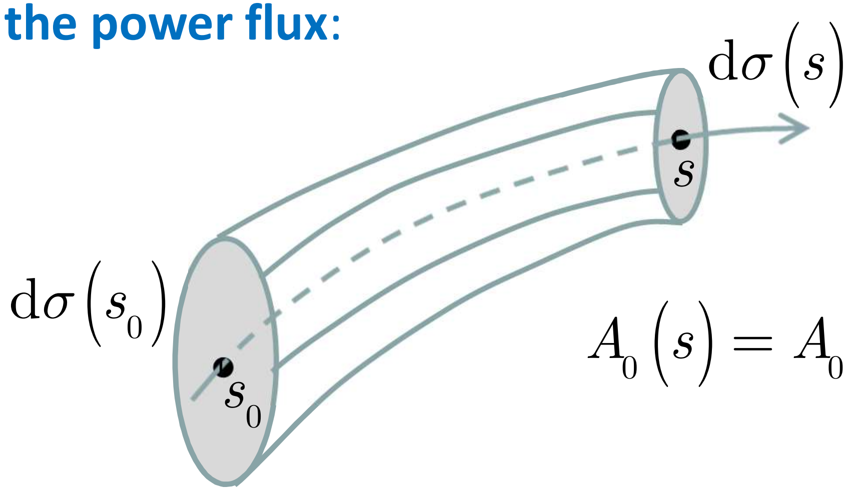
$Q_m(\mathbf{r}) = \left(\nabla^2 S + 2\nabla S \cdot \nabla\right) A_{m-1} + \nabla^2 A_{m-2} = 0, \quad m = 2, 3, \ldots$ field-transport equation for the higher-order terms A_{m}

Let us now examine the first of the so-called transport equations:

$$A_0 \nabla^2 S + 2\nabla S \cdot \nabla A_0 = 0 \quad \longrightarrow \quad A_0^2 \nabla \cdot \underbrace{\nabla S}_{n\hat{\mathbf{s}}} + \underbrace{\nabla S}_{n\hat{\mathbf{s}}} \cdot \underbrace{2A_0 \nabla A_0}_{\nabla\left(A_0^2\right)} = \nabla \cdot \left(A_0^2 n\hat{\mathbf{s}}\right) = 0$$

Field-Transport Equation for A_0

The quantity $A_0^2 n\,\hat{\mathbf{s}}$ can be considered the analogous of the Poynting vector for the field $A_0(\mathrm{r})\exp\left(-jk_0 S(\mathbf{r})\right)$. The field-transport equation then expresses the **conservation of the power flux**:



$$A_0(s) = A_0(s_0)\left(\frac{n(s_0)\,\mathrm{d}\sigma(s_0)}{n(s)\,\mathrm{d}\sigma(s)}\right)^{1/2}$$

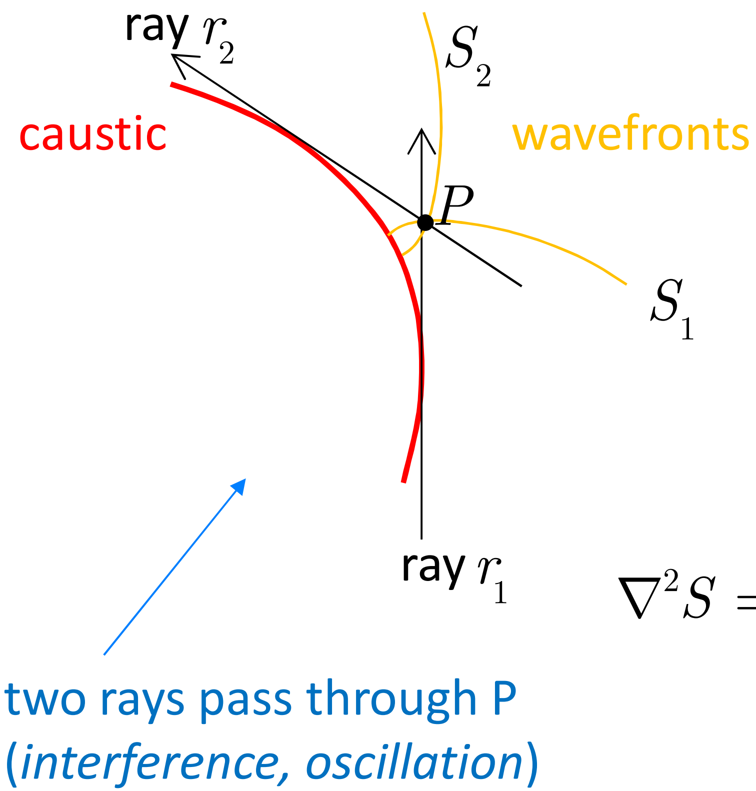
transport equation

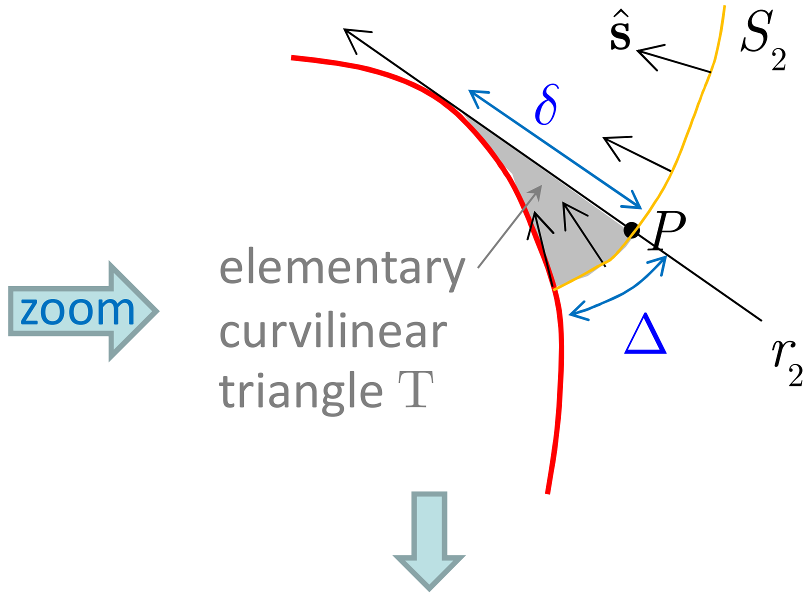
$$A_0\nabla^2 S + 2\underbrace{\underbrace{\nabla S}_{n\hat{\mathbf{s}}}\cdot\nabla}_{n\,\mathrm{d}/\mathrm{d}s} A_0 = 0 \quad\rightarrow\quad \nabla^2 S + 2n\frac{\mathrm{d}}{\mathrm{d}s}\ln A_0 = 0$$

$$\rightarrow\quad A_0(s) = A_0(s_0)\exp\left(-\int_{s_0}^{s}\frac{\nabla^2 S}{n(s')}\mathrm{d}s'\right) \quad\Rightarrow\quad \mathrm{d}\sigma\rightarrow 0 \leftrightarrow \nabla^2 S\rightarrow -\infty$$

Caustics as Envelopes of Ray Budles

The field amplitude predicted by ray optics becomes infinite on certain surfaces called **caustics**, where $\nabla^2 S \to -\infty$. These are the **envelopes** of the ray bundles:



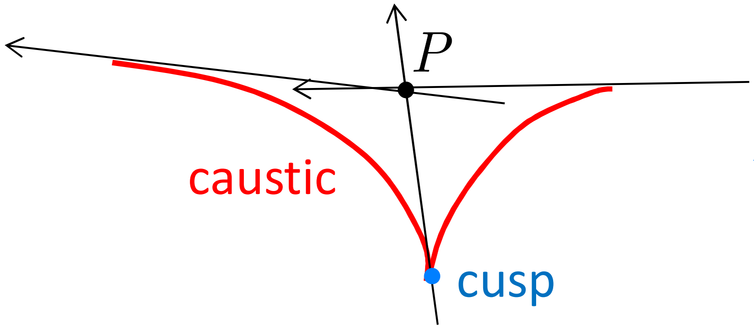


$$\nabla^2 S = \nabla \cdot \nabla S = \nabla \cdot \left(n\,\hat{\mathbf{s}}\right) \cong \frac{1}{\mathrm{area}\mathrm{T}} \oint_{\partial \mathrm{T}} n\,\hat{\mathbf{s}} \cdot \left(-\hat{\mathbf{s}}\right) \mathrm{d}s$$

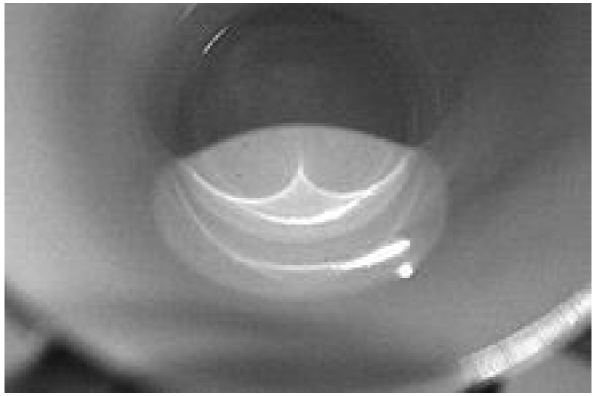
$$\propto -\frac{1}{\delta\Delta} \cdot n\Delta = O\left(-\frac{1}{\delta}\right), \quad \delta \to 0$$

Caustics: Cusps

Furthermore, at special points (cusps) of a caustic, it diverges in a manner *different* than at all other points of the caustic.



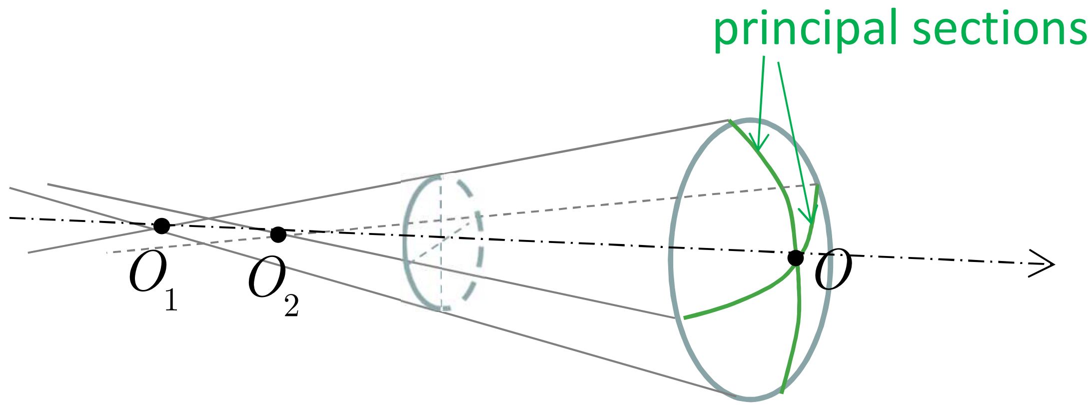
three rays pass through P
(*complex interference figure*)



Nephroid caustic at bottom of tea cup
(*note the presence of a cusp*)

Caustics in a Homogeneous Medium

In a homogeneous medium, the caustic is the locus of the **principal centers of curvature** (*foci*) of the wavefront (hence, it is a two-sheeted surface in general):



$$\nabla^2 S = \mathrm{Tr}\nabla\nabla S = \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$\rho_1 = \overline{O_1 O}$$

$$\rho_2 = \overline{O_2 O}$$

Example: Omnidirectional Cylindrical Wave

Let us consider the ray-optics approximation for an omnidirectional cylindrical wave in a homogeneous medium, i.e., a solution of the 2D Helmholtz equation

$$\nabla^2 u(\rho, k_0) + k_0^2 n^2 u(\rho, k_0) = 0$$

in cylindrical coordinates (ρ,ϕ,z): $\quad u(\rho, k_0) \sim e^{-jk_0 S(\rho)} \sum_{m=0}^{+\infty} \frac{A_m(\rho)}{(-jk_0)^m}$

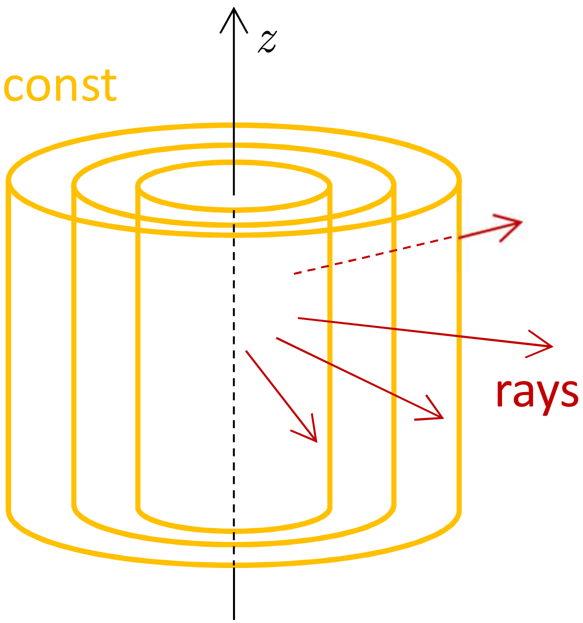
$$|\nabla S|^2 = n^2 \quad \rightarrow \quad \left|\frac{\mathrm{d}S}{\mathrm{d}\rho}\right| = n$$

$$\rightarrow \quad \boxed{S(\rho) = \pm n\rho + S_0} \text{ eikonal}$$

$$\boxed{\mathbf{r}(s) = s\hat{\boldsymbol{\rho}} + \mathbf{r}_0} \text{ rays}$$

wavefronts $n\rho$=const

(outgoing wave)



Here the caustics are the z axis and a circle at infinity.

Example: Omnidirectional Cylindrical Wave

Let us now calculate the amplitude A_0:

$$\nabla^2 S = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}S}{\mathrm{d}\rho}\right) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right)\left(n\rho + S_0\right) = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(n\rho\right) = \frac{n}{\rho}$$

$$\Rightarrow A_0\left(s\right) = A_0\left(s_0\right)\exp\left[-\int_{s_0}^{s}\frac{\nabla^2 S}{n\left(s'\right)}\mathrm{d}s'\right] = A_0\left(s_0\right)\exp\left[-\frac{1}{2}\int_{\rho_0}^{\rho}\frac{1}{\rho'}\mathrm{d}\rho'\right]$$

$$= A_0\left(s_0\right)\frac{\rho_0^{1/2}}{\rho^{1/2}} = \frac{\text{const}}{\rho^{1/2}}$$

(i.e., the expected amplitude spreading for a cylindrical wave)

Therefore, to the lowest asymptotic order we have $\boxed{u\left(\rho, k_0\right) = \text{const}\,\frac{e^{-jnk_0\rho}}{\rho^{1/2}}}$

Compare with the asymptotic behavior of the **exact** 2D omnidirectional outgoing cylindrical wave:

$$H_0^{(2)}\left(nk_0\rho\right) \sim \sqrt{\frac{2j}{\pi n k_0 \rho}}e^{-jnk_0\rho} = \text{const}\,\frac{e^{-jnk_0\rho}}{\rho^{1/2}}, \quad k_0\rho \to \infty$$

Example: Omnidirectional Spherical Wave

Let us now consider the ray-optics approximation for an omnidirectional spherical wave in a homogeneous medium, i.e., a solution of the 3D Helmholtz equation

$$\nabla^2 u\left(r,k_0\right) + k_0^2 n^2 u\left(r,k_0\right) = 0$$

in spherical coordinates (ρ,θ,ϕ): $\quad u\left(r,k_0\right) \sim e^{-jk_0 S(r)} \sum_{m=0}^{+\infty} \frac{A_m\left(r\right)}{\left(-jk_0\right)^m}$

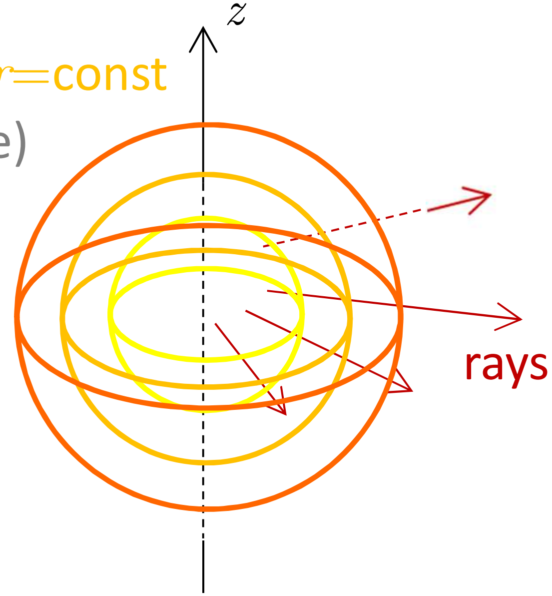
$$\left|\nabla S\right|^2 = n^2 \;\rightarrow\; \left|\frac{\mathrm{d}S}{\mathrm{d}r}\right| = n$$

$$\rightarrow \quad \boxed{S\left(\rho\right) = \pm nr + S_0} \text{ eikonal}$$

$$\boxed{\mathbf{r}(s) = s\hat{\mathbf{r}} + \mathbf{r}_0} \text{ rays}$$

wavefronts nr=const

(outgoing wave)



Here the caustics both degenerate to a point (the centre of the spherical wavefronts).

Example: Omnidirectional Spherical Wave

Let us now calculate the amplitude A_0:

$$\nabla^2 S = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}S}{\mathrm{d}r}\right) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(nr + S_0\right) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(nr^2\right) = \frac{2n}{r}$$

$$\Rightarrow \quad A_0(s) = A_0(s_0)\exp\left(-\int_{s_0}^{s}\frac{\nabla^2 S}{n(s')}\mathrm{d}s'\right) = A_0(s_0)\exp\left(-\frac{1}{2}\int_{\rho_0}^{\rho}\frac{2}{r'}\mathrm{d}r'\right)$$

$$= A_0(s_0)\frac{r_0}{r} = \frac{\text{const}}{r}$$

(i.e., the expected amplitude spreading for a spherical wave)

Therefore, to the lowest asymptotic order we have

$$\boxed{u(r, k_0) = \text{const}\,\frac{e^{-jnk_0 r}}{r}}$$

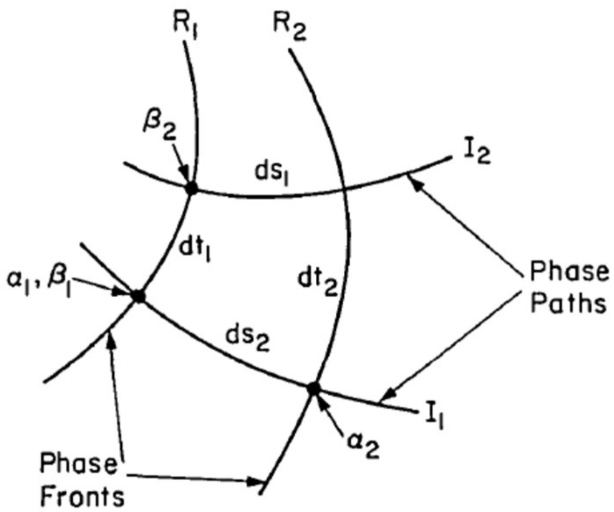
Compare with the **exact** 3D omnidirectional outgoing spherical wave:

$$\frac{e^{-jnk_0 r}}{4\pi r}$$

Evanescent Waves and Complex Eikonals

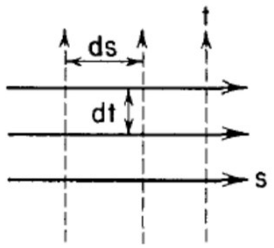
The asymptotic L-K approach has been generalized (by L. B. Felsen) assuming a high-frequency representation in terms of **locally evanescent** plane waves, by introducing a **complex eikonal**:

$$S\left(\mathbf{r}\right) = R\left(\mathbf{r}\right) + jI\left(\mathbf{r}\right)$$

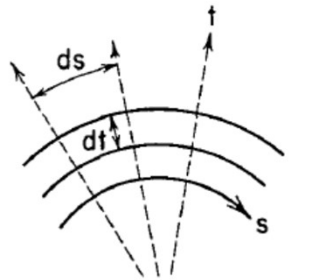


Evanescent Waves and Complex Eikonals

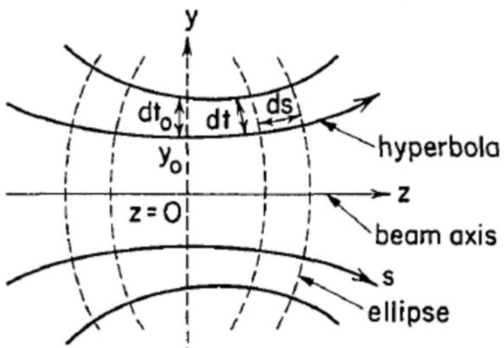
Examples of evanescent wave phenomena:



(a) evanescent field exterior to plane dielectric layer: $K = \hat{K} = 0$



(b) weakly evanescent field exterior to curved dielectric layer: $\hat{K} = 0,\ K \neq 0$



(c) Gaussian beam

Ray Optics of Maxwell Vector Fields: Electric Field

The electric field $\mathbf{E}$ in a dielectric inhomogeneous medium satisfies the **vector wave equation** (*proof by exercise*):

$$\nabla^2\mathbf{E} + k_0^2 n^2 \mathbf{E} + 2\nabla\left[\mathbf{E}\cdot\nabla\left(\ln n\right)\right] = \mathbf{0}$$

We look for a ray optical representation in the form

$$\boxed{\mathbf{E}\left(\mathbf{r}, k_0\right) \sim e^{-jk_0 S\left(\mathbf{r}\right)} \sum_{m=0}^{+\infty} \frac{\mathbf{E}_m\left(\mathbf{r}\right)}{\left(-jk_0\right)^m}}$$

LK asymptotic series (vector form)

Ray Optics of Maxwell Vector Fields

Inserting the LK representation into the vector wave equation and equating to zero the coefficients of each power of k_0 one obtains (*proof by exercise*):

$$\left|\nabla S\right|^2 - n^2 = 0$$ eikonal equation

$$\left(\nabla^2 S + 2\nabla S\cdot\nabla\right)\mathbf{E}_0 + 2\left[\mathbf{E}_0\cdot\nabla\left(\ln n\right)\right]\nabla S = \mathbf{0}$$ field-transport equation for $\mathbf{E}_0$

$$\left(\nabla^2 S + 2\nabla S\cdot\nabla + 2\nabla S\nabla\left(\ln n\right)\right)\cdot\mathbf{E}_m + \nabla^2\mathbf{E}_{m-1} + 2\nabla\left[\mathbf{E}_{m-1}\cdot\nabla\left(\ln n\right)\right] = \mathbf{0},\quad m > 0$$

field-transport equation for the higher-order terms $\mathbf{E}_m$

Comparing with the scalar treatment, we see that the vector theory reduces to the scalar one only if the $\mathbf{E}_m$ are perpendicular to ∇n. In general, E_x, E_y, and E_z mix because of the terms containing $\nabla\left(\ln n\right)$; hence, an initially linearly polarized field does not maintain its polarization during propagation.

Lowest-Order Term: Polarization

In order to study the lowest-order term of the LK series, it is convenient to replace $\mathbf{E}_0(\mathbf{r})$ with $\mathbf{E}'(\mathbf{r})$:

$$\mathbf{E}_0\left(\mathbf{r}\right) = \mathbf{E}'\left(\mathbf{r}\right)\exp\left[-\frac{1}{2}\int_0^s \frac{\nabla^2 S}{n\left(s'\right)}\mathrm{d}s'\right]$$

The lowest-order transport equation then reduces to (*proof by exercise*)

$$n\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}'\left(\mathbf{r}\right) + \hat{\mathbf{s}}\left(\mathbf{E}'\cdot\nabla n\right) = \mathbf{0}$$

Multiplying scalarly by $\hat{\mathbf{s}}$ and using the ray equation this becomes

$$n\,\hat{\mathbf{s}}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}'\left(\mathbf{r}\right) + \mathbf{E}'\cdot\underbrace{\nabla n}_{=\mathrm{d}/\mathrm{d}s\left(n\,\hat{\mathbf{s}}\right)} = n\,\hat{\mathbf{s}}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}'\left(\mathbf{r}\right) + \mathbf{E}'\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left(n\,\hat{\mathbf{s}}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\,\hat{\mathbf{s}}\cdot\mathbf{E}'\right) = 0$$

Therefore, $n\,\hat{\mathbf{s}}\cdot\mathbf{E}'$ is constant along the ray.

Lowest-Order Term: Polarization (cont'd)

In particular, if $\mathbf{E}'$ is perpendicular to $\hat{\mathbf{s}}$ at one point, it remains perpendicular along the whole ray path.

In this case, multiplying the transport equation by $\mathbf{E}'$:

$$n\mathbf{E}' \cdot \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}' + \underbrace{\mathbf{E}' \cdot \hat{\mathbf{s}}}_{=0}\left(\mathbf{E}' \cdot \nabla n\right) = \frac{1}{2} n \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathbf{E}' \cdot \mathbf{E}'\right) = 0$$

Therefore, $\mathbf{E}' \cdot \mathbf{E}'$ is also constant along the ray.

In conclusion, $\mathbf{E}'$ is a constant-amplitude vector orthogonal to $\hat{\mathbf{s}}$ along a ray if it is such at one point of the ray.

Asymptotic Expansion of the Magnetic Field

Inserting the LK representation of the electric field into the first Maxwell equation one obtains

$$\mathbf{H}\left(\mathbf{r},k_0\right) \sim \frac{1}{\zeta_0} e^{-jk_0 S(\mathbf{r})} \sum_{m=0}^{+\infty} \frac{n\hat{\mathbf{s}}\times\mathbf{E}_m + \nabla\times\mathbf{E}_{m-1}}{\left(-jk_0\right)^m} = e^{-jk_0 S(\mathbf{r})} \sum_{m=0}^{+\infty} \frac{\mathbf{H}_m\left(\mathbf{r}\right)}{\left(-jk_0\right)^m}$$

$$\mathbf{H}_m\left(\mathbf{r}\right) = \begin{cases} \dfrac{1}{\zeta}\hat{\mathbf{s}}\times\mathbf{E}_0, & m=0 \\ \dfrac{1}{\zeta}\hat{\mathbf{s}}\times\mathbf{E}_m + \nabla\times\mathbf{E}_{m-1}, & m>0 \end{cases}$$

$$\zeta = \frac{\zeta_0}{n}$$

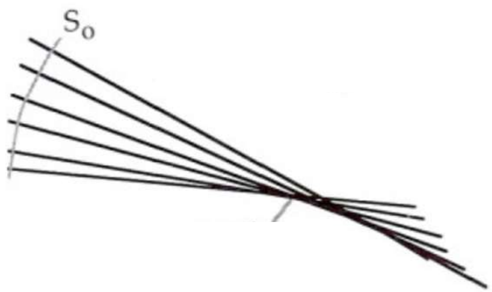
characteristic impedance of the medium

In particular, $\zeta\mathbf{H}_0 = \hat{\mathbf{s}}\times\mathbf{E}_0$; hence, if $\mathbf{E}_0$ is orthogonal to $\hat{\mathbf{s}}$, then $\hat{\mathbf{s}}$, $\mathbf{E}_0$, and $\mathbf{H}_0$ are mutually orthogonal and **the field is a TEM wave to the zeroth order in $1/k_0$**.

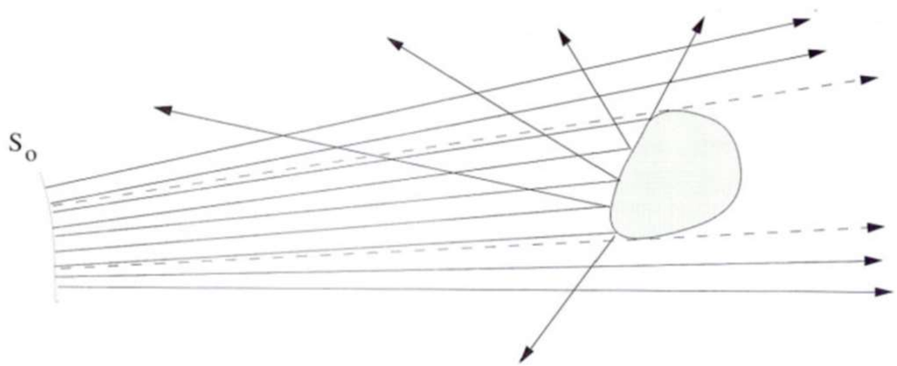
The Limits of Ray Optics

Some features of the GO field are clearly **unphysical**:

- Divergence of the amplitude on the caustics



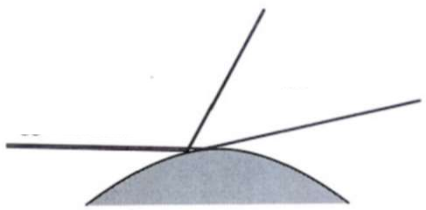
- Abrupt transition between shadow and illuminated regions



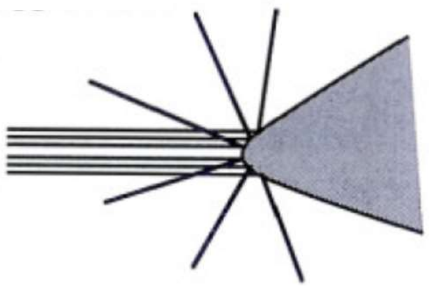
In such regions the assumption of slow variability of the amplitude (w.r.t. the wavelength) is not satisfied.

The Limits of Ray Optics

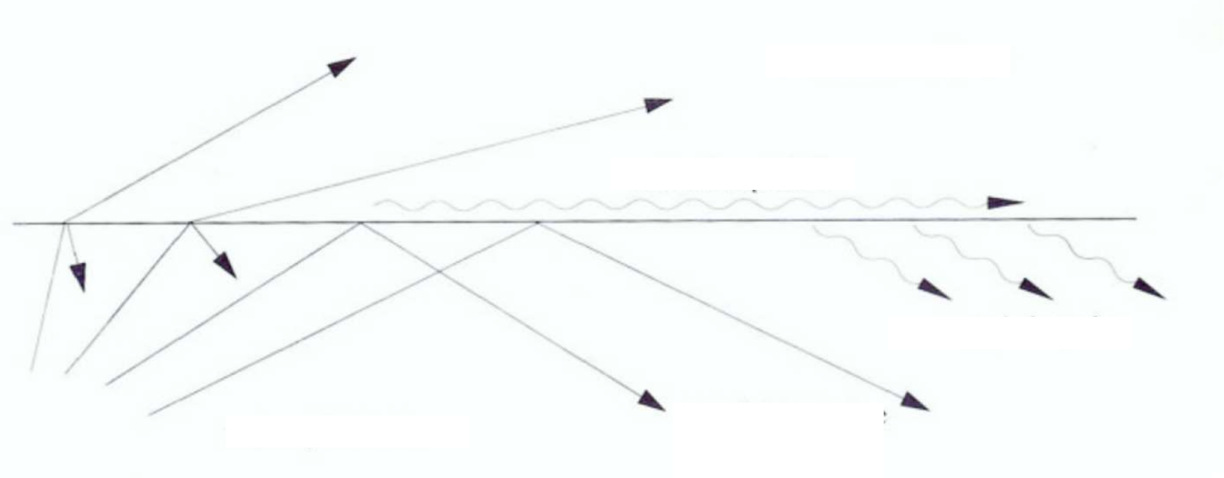
This also occurs in other cases:



grazing incidence on curved surfaces



incidence on sharp wedges or tips



excitation of surface, lateral waves, or leaky waves

The study of wave phenomena beyond GO is the domain of **diffraction theories**…

References

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S. Solimeno, B. Crosignani, and P. Di Porto, *Guiding, diffraction and confinement of optical radiation*. Cambridge, MA: Academic Press, 1986.