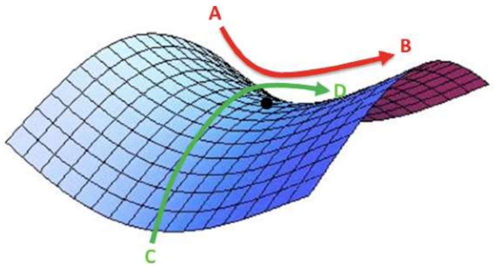
Ph.D. in Information and Communication Engineering

Ph.D. Course on

Analytical Techniques for Wave Phenomena



Lesson 3

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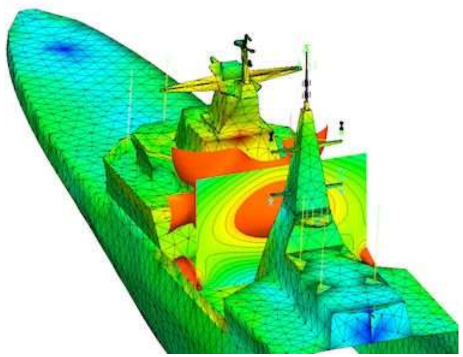
Wavefields: Numerical vs. Asymptotic Approaches

Numerical methods:

- may be *cumbersome* or even *not feasible* when dealing with objects with linear dimensions large with respect to the involved wavelength

- usually *do not* provide much physical insight into the wave processes relevant to the considered configuration.



Advantages of the Asymptotic Methods

Asymptotic representations (or expansions) of the wave fields have several advantages:

- *Simplicity of the resulting expressions*
- *High degree of accuracy with a small number of terms*
- *Physical insight into the involved wave phenomena*

In this and the next lessons we aim at providing basic information on asymptotic expansions, either derived directly from the wavefield equations (ray optics) or from integral representations of the wavefield.

Asymptotic Expansions:
Introduction and Behavioral Survey

A Behavioral Survey: the Error Function

Let us start by examining types of expansion which may be developed for the well-known **error function**:

$$\phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, \mathrm{d}u$$

This function is important in its own right, and of special interest in asymptotics through having provided one of the earliest examples of a *Stokes discontinuity*.

Error Function: a Convergent Expansion

Expansion of the exponential as a power series

$$e^{-u^2} = \sum_{s=0}^{+\infty} \frac{\left(-u^2\right)^s}{s\,!}$$

absolutely convergent series for any u, uniformly convergent on any closed and bounded subset of the complex plane

followed by term by term integration, leads to the series

$$\phi\left(z\right) = \frac{2z}{\sqrt{\pi}} \sum_{s=0}^{+\infty} \frac{\left(-z^2\right)^s}{s\,!\left(2s+1\right)}$$

which is **absolutely convergent** for any z.

Drawbacks of the Convergent Expansion

Though theoretically exact for any magnitude and phase of the variable z, such a convergent series can prove very inconvenient except for small values of $|z|$.

For instance, its individual terms do not begin to decrease until

$$s \simeq |z|^2$$

and their sum does not even begin to approximate the error function until about **three times** as many terms have been assembled.

More seriously, for large $|z|$ **the final sum is far smaller than the largest individual terms**, which therefore have to be calculated to many extra significant figures…

The Asymptotic Approach

Fortunately, the alternative 'asymptotic' approach produces a series in which, in contrast, ease of calculation to a prescribed accuracy **increases** with $|z|$.

Let us consider for instance the phase sector $|\arg z| < \frac{\pi}{2}$

To start with, the well known Gauss integral

$$\int_0^{+\infty} e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{2}$$

allows for writing

$$\phi(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-u^2}\,\mathrm{d}u$$

The Asymptotic Approach

In the new integral, e^{-u^2} is significant in magnitude only **near the lower limit** $u=z$, and can therefore be expanded about this point.

It is convenient to choose as expansion parameter $w = u^2 - z^2$ so that

$$\phi\left(z\right) = 1 - \frac{2}{z\sqrt{\pi}} e^{-z^2} \int_0^{+\infty} e^{-w} \left(1 + \frac{w}{z^2}\right)^{-1/2} \mathrm{d}w$$

We now make use of the Binomial Theorem, which provides for the following power-series representation about the point $w=0$:

$$\left(1 + \frac{w}{z^2}\right)^{-1/2} = \frac{1}{\sqrt{\pi}} \sum_{r=0}^{+\infty} \frac{\left(r - 1/2\right)!}{r!} \left(-\frac{w}{z^2}\right)^r$$

Detour on the Generalized Factorial via Gamma Function

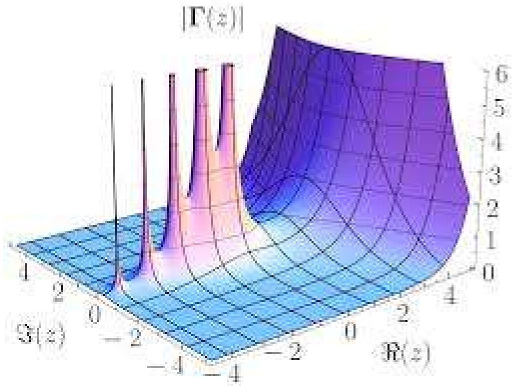
What is the meaning of the factorial for a **non-integer** number?

In the the aforementioned binomial expansion, it is understood that the factorial is defined via the so-called (Eulerian) **Gamma function**:

$$\left(z-1\right)! = \Gamma\left(z\right) = \int\limits_0^{+\infty} e^{-w} w^{z-1} \mathrm{d}w, \qquad \mathrm{Re}\, z > 0$$

The indicated condition guarantees convergence of the integral in a neighborhood of $w=0$.

The Gamma function can then be analytically extended to a meromorphic function that is holomorphic in the whole complex plane except the non-positive integers, where the function has simple poles



The Asymptotic Approach

Including the binomial expansion inside the integral and performing again term by term integration we obtain

$$\phi(z) = 1 - \frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r-1/2)!}{r!\left(-z^2\right)^r} \underbrace{\int_0^{+\infty} e^{-w} w^r \mathrm{d}w}_{=r!}$$

$$= 1 - \frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r-1/2)!}{\left(-z^2\right)^r}, \qquad |\arg z| < \frac{\pi}{2}$$

asymptotic expansion of the error function in the indicated phase range

The Asymptotic Approach

The terms in this 'asymptotic series' $\sum_{r=0}^{+\infty} \frac{\left(r-1/2\right)!}{\left(-z^2\right)^r}$ behave in a radically different way from those of the convergent series $\sum_{s=0}^{+\infty} \frac{\left(-z^2\right)^s}{s!\left(2s+1\right)}$:

For moderate or large $|z|$:

- the terms in the former first progressively decrease in magnitude, then reach a minimum around $r \approx |z|^2$ and thereafter increase;
- while those in the latter first increase, reach a maximum around $s \approx |z|^2$ and thereafter decrease.

Because of the ultimate progressive increase in magnitude of its terms, an asymptotic power series is **divergent**. Nevertheless, even if crudely broken off at its least term (thereby retaining its **first few terms**), it produces **remarkably accurate results**, especially for large values of z.

Inspiring Quotes

One remarkable fact of applied mathematics is the ubiquitous appearance of divergent series, hypocritically renamed asymptotic expansions. Isn't it a scandal that we teach convergent series to our sophomores and do not tell them that few, if any, of the series they meet will converge? The challenge of explaining what an asymptotic expansion is ranks among the outstanding taboo problems of mathematics.

Gian Carlo Rota, 1996

Inspiring Quotes

Divergent series are the devil, and it is a shame to base on them any demonstration whatsoever.

Niels Henrik Abel, 1828

The series is divergent, therefore we may be able to do something with it.

Oliver Heaviside, cited by M. Kline

Morris Kline, *Mathematical thought from ancient to modern times.* Oxford University Press USA, 1990.

Asymptotic Expansions in Physics and Engineering

Both in theoretical and applied sciences it is desirable to approach the solution of a problem by a method of successive approximation in which the very first term will provide at least a rough value for the answer.

The above comparative analysis of convergent and asymptotic series suggests that the most commonly applied methods are indeed asymptotic rather than convergent. Evidence shows this to be true for ubiquitous methods such as:

- *WKB methods*
- *stationary-point methods*
- *perturbative expansions*

extensively applied from oceanography to seismology through optics, applied electromagnetics, quantum mechanics and quantum field theory.

Asymptotic Expansions in Physics and Engineering

A simple elementary example of this is the **Stirling-Laplace asymptotic formula** for the factorial

$$n\,! \sim \left(2\pi\right)^{1/2} e^{-n} n^{n+1/2}$$

compared with the virtual non-appearance of the corresponding *convergent* expansion

$$n\,! = 1 - 0.5772n + \ldots$$

Taxonomy of Late Terms

The most frequent exception to the predilection towards asymptotics in physics is the **hypergeometric form**, which displays properties midway between those of asymptotic and exponential type:

$$\frac{(r+a)!}{(\text{variable})^r}$$

asymptotic
(divergent)

$$\frac{(r+a)!}{(r+b)!(\text{variable})^r}$$

hypergeometric type
(conditionally convergent)

$$\frac{1}{(r+b)!(\text{variable})^r}$$

exponential type
(convergent)

Its finite radius of convergence is often associated with some *transition point*; hence it appears, e.g., in **co-operative phenomena** (especially phase transitions), such as the Ising model, ferromagnetism, Fermi-Thomas electron screening, and **hydrodynamics** (boundary layers, flow discontinuities).

Stokes Discontinuities

The foregoing derivation of the asymptotic expansion for $\phi(z)$ was based on the understanding that $|\arg(z)| < \pi/2$.

Now let us suppose z to be **purely imaginary**, i.e., $z = \mathrm{j}y$; by letting $u = \mathrm{jv}$:

$$\phi\left(jy\right) = \frac{2j}{\sqrt{\pi}} \int_0^y e^{v^2} \mathrm{d}v$$

The integrand is large only close to the upper limit $v = y$, and can therefore be expanded about this point. By letting $f = y^2 - v^2$:

$$\phi\left(jy\right) = \frac{je^{y^2}}{y\sqrt{\pi}} \int_0^{y^2} e^{-f} \left(1 - \frac{f}{y^2}\right)^{-1/2} \mathrm{d}f$$

Stokes Discontinuities

Since the integrand is **real** throughout the specified range of integration from 0 to y^2, but would be **purely imaginary** over the range from y^2 to +∞, then

$$\phi(jy) = \frac{je^{y^2}}{y\sqrt{\pi}} \operatorname{Re} \int_0^{+\infty} e^{-f} \left(1 - \frac{f}{y^2}\right)^{-1/2} \mathrm{d}f$$

$$= \frac{je^{y^2}}{y\sqrt{\pi}} \operatorname{Re} \sum_{r=0}^{+\infty} \frac{(r - 1/2)!}{r!\, y^{2r}} \int_0^{+\infty} e^{-f} f^r \mathrm{d}f = \frac{je^{y^2}}{y\pi} \sum_{r=0}^{+\infty} \frac{(r-1/2)!}{y^{2r}}$$

An identical reasoning covers the case where z is a negative imaginary.

In both cases, therefore:

$$\phi(z) = -\frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r-1/2)!}{(-z^2)^r}, \qquad |\arg(z)| = \frac{\pi}{2}$$

Stokes Discontinuities

Let us now compare:

$$\phi(z) = 1 - \frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r - 1/2)!}{(-z^2)^r}, \qquad |\arg z| < \frac{\pi}{2}$$

$$\phi(z) = -\frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r - 1/2)!}{(-z^2)^r}, \qquad |\arg(z)| = \frac{\pi}{2}$$

The second lacks the unit term. This type of discontinuity in the asymptotic expansion of the (continuous!) error function was discovered by Stokes in 1864. Typically, at a certain phase drawn in the complex plane as a 'Stokes ray', an 'associated function' appears, disappears, or changes its numerical multiplier.

Stokes Discontinuities

Let us proceed to still larger phases $\pi/2 < |\arg(z)| \leq \pi$: by reversing the sign of u in the defining formula:

$$\phi(z) = -\frac{2}{\sqrt{\pi}} \int_0^{-z} e^{-u^2} \mathrm{d}u = -\phi(-z)$$

(a 'continuation formula')

we get

$$\phi(z) = -1 - \frac{e^{-z^2}}{z\pi} \sum_{r=0}^{+\infty} \frac{(r - 1/2)!}{(-z^2)^r}, \qquad \frac{\pi}{2} < |\arg z| \leq \pi$$

where now the 'associated function' has changed (previously it was the constant +1, now it is the constant -1).

Asymptotic Expansions:
Definitions and Fundamental Concepts

Example and First Definitions

Example: $$I\left[\varepsilon\right] = \int_0^{+\infty} \frac{e^{-t}}{1+\varepsilon t}\,\mathrm{d}t, \qquad \varepsilon > 0$$

Assume you wish to evaluate this for small positive values of ε.

Integrating by parts:

$$I\left[\varepsilon\right] = 1 - \varepsilon \int_0^{+\infty} \frac{e^{-t}}{\left(1+\varepsilon t\right)^2}\,\mathrm{d}t$$

Iterating:

$$I\left[\varepsilon\right] = 1 - \varepsilon + 2!\,\varepsilon^2 - 3!\,\varepsilon^3 + \ldots + \left(-1\right)^N N!\,\varepsilon^N + \\ + \left(-1\right)^{N+1}\left(N+1\right)!\,\varepsilon^{N+1} \int_0^{+\infty} \frac{e^{-t}}{\left(1+\varepsilon t\right)^{N+2}}\,\mathrm{d}t$$

Example and First Definitions

$$I[\varepsilon] = 1 - \varepsilon + 2!\,\varepsilon^2 - 3!\,\varepsilon^3 + \dots + (-1)^N N!\,\varepsilon^N +$$
$$+ (-1)^{N+1}(N+1)!\,\varepsilon^{N+1}\int_0^{+\infty}\frac{e^{-t}}{(1+\varepsilon t)^{N+2}}\,\mathrm{d}t$$

Terminology:

- $-\varepsilon$ is of order ε
- $2!\,\varepsilon^2$ is of smaller order than ε
- $I[\varepsilon] \cong 1 - \varepsilon + 2!\,\varepsilon^2$ is an approximation correct to order ε^2

(this assumes that ε is a ‘small’ quantity).

Let us now try to make these statements more precise…

Example and First Definitions

Definitions:

1) $$f(k) = O\left(g(k)\right)$$ *'$f(k)$ is of order $g(k)$ as $k \to k_0$'*

$$\leftrightarrow \exists M, \delta \text{ s.t. } \left|f(k)\right| \leq M \left|g(k)\right| \quad \forall k \in \left(k_0 - \delta, k_0 + \delta\right)$$

2) $$f(k) = o\left(g(k)\right)$$ *'$f(k)$ is much smaller than $g(k)$ as $k \to k_0$'*

or $f(k) \ll g(k)$

$$\leftrightarrow \lim_{k \to k_0} \left|\frac{f(k)}{g(k)}\right| = 0$$

3) *'$f(k)$ is an approximation to $I(k)$ valid to order $\delta(k)$ as $k \to k_0$'*

$$\leftrightarrow \lim_{k \to k_0} \frac{I(k) - f(k)}{\delta(k)} = 0$$

Asymptotic Sequences, Asymptotic Expansions

The ordered sequence $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ is characterized by the fact that its $(j+1)$-th term is much smaller than its j-th term.

This is the defining property of an **asymptotic sequence**.

The representation

$$I[\varepsilon] = 1 - \varepsilon + 2!\,\varepsilon^2 - 3!\,\varepsilon^3 + \ldots + (-1)^N N!\,\varepsilon^N + \\ + (-1)^{N+1} (N+1)!\,\varepsilon^{N+1} \int_0^{+\infty} \frac{e^{-t}}{(1+\varepsilon t)^{N+2}}\,\mathrm{d}t$$

provides an **asymptotic expansion** with respect to the aforementioned asymptotic sequence

$$\left\{\varepsilon^j\right\}_{j=0}^{+\infty}$$

Asymptotic Sequences, Asymptotic Expansions

Definitions:

1) The ordered sequence of functions $\left\{\delta_j\left(k\right)\right\}, \quad j=1,2,\ldots$ is an **asmptotic sequence** as $k \to k_0$ if

$$\delta_{j+1}\left(k\right) \ll \delta_j\left(k\right)$$

2) Let $I\left(k\right)$ be continuous and let $\left\{\delta_j\left(k\right)\right\}$ be an asymptotic sequence as $k \to k_0$. Then the formal series

$$\sum_{j=1}^{N} a_j \delta_j\left(k\right)$$

is an **asymptotic expansion** of $I\left(k\right)$ as $k \to k_0$ valid to order $\delta_N\left(k\right)$ if

$$I\left(k\right) = \sum_{j=1}^{m} a_j \delta_j\left(k\right) + O\left(\delta_{m+1}\left(k\right)\right), \quad m = 1,2,\ldots N$$

Asymptotic Sequences, Asymptotic Expansions

We then write:

$$I(k) \sim \sum_{j=1}^{N} a_j \delta_j(k), \quad k \to k_0 \qquad (*)$$

Note: in general, $I(k) \sim \eta(k), \quad k \to k_0$ means $\lim_{k \to k_0} \left| \frac{I(k)}{\eta(k)} \right| = 1$

Here, (*) means

$$a_n = \lim_{k \to k_0} \frac{I(k) - \sum_{j=1}^{n-1} a_j \delta_j(k)}{\delta_n(k)}, \quad n = 1, 2, \ldots N$$

When an arbitrary number of terms can be calculated we write:

$$I(k) \sim \sum_{j=1}^{+\infty} a_j \delta_j(k), \quad k \to k_0$$

Asymptotic does not mean Convergent

Coming back to our example, it is important to realize that:

$$I[\varepsilon] \sim \sum_{j=0}^{+\infty} (-1)^j \, j!\, \varepsilon^j, \quad \varepsilon \to 0$$

DOES NOT imply that the series $\sum_{j=0}^{+\infty} (-1)^j \, j!\, \varepsilon^j$ is convergent.

In fact, this series is not convergent for any value of ε.

Indeed, for ε fixed the term $\left|(-1)^j \, j!\, \varepsilon^j\right|$ tends to infinity as $j \to \infty$.

But for fixed j this term vanishes as $\varepsilon \to 0$, and this is the reason that the above expansion provides a good approximation to $I(\varepsilon)$ as $\varepsilon \to 0$.

A Second Example

Example: Asymptotic expansion of $J(k) = \int_0^{+\infty} \frac{e^{-kt}}{1+t}\,\mathrm{d}t, \qquad k \to +\infty$

$$t' = kt, \varepsilon = \frac{1}{k} \quad \Rightarrow \quad J = \varepsilon \int_0^{+\infty} \frac{e^{-t'}}{1+\varepsilon t'}\,\mathrm{d}t', \qquad \varepsilon \to 0$$

$$\Rightarrow \quad J(k) = \frac{1}{k} - \frac{1}{k^2} + \frac{2!}{k^3} + \ldots + (-1)^{N-1}\frac{(N-1)!}{k^N} + R_N(k)$$

$$R_N(k) = (-1)^N \frac{N!}{k^{N+1}} \int_0^{+\infty} \frac{e^{-t}}{(1+t/k)^{N+1}}\,\mathrm{d}t$$

$$|R_N(k)| \le \frac{N!}{k^{N+1}} \ll \frac{1}{k^N} \quad \Rightarrow \quad J(k) \sim \sum_{j=1}^{+\infty} (-1)^{j-1} \frac{(j-1)!}{k^j}$$

A Third Example

Example: Asymptotic expansion of $I(k)=\int_k^{+\infty}\frac{e^{-t}}{t}\mathrm{d}t,\qquad k\to+\infty$

Integrating by parts:

$$I(k)=e^{-k}\left[\frac{1}{k}-\frac{1}{k^2}+\frac{2!}{k^3}+\ldots+(-1)^{N-1}\frac{(N-1)!}{k^N}\right]+R_N(k)$$

$$R_N(k)=(-1)^N N!\int_k^{+\infty}\frac{e^{-t}}{t^{N+1}}\mathrm{d}t$$

$$\left|R_N(k)\right|\le\frac{N!}{k^{N+1}}\int_k^{+\infty}e^{-t}\mathrm{d}t=\frac{N!e^{-k}}{k^{N+1}}\ll\frac{e^{-k}}{k^N}$$

$$\Rightarrow\quad I(k)\sim e^{-k}\sum_{j=0}^{+\infty}(-1)^{j-1}\frac{(j-1)!}{k^j}$$

Accuracy of Asymptotic Expansions

Asymptotic series frequently give remarkably good approximations.

For example, when ***k* = 10** and ***N* = 2**, the error between the exact answer and the first two terms of the series, $R_2(10)$, satisfies

$$R_2(10) < 2 \cdot 10^{-13} \qquad \left[I(10) \cong 4.6 \cdot 10^{-6} \right]$$

which is clearly very small.
In fact, even when ***k* = 3** and ***N* = 2**, we have

$$R_2(3) < 3.7 \cdot 10^{-3} \qquad \left[I(3) \cong 1.3 \cdot 10^{-2} \right]$$

However, we **cannot** take too many terms in the series, because the remainder, which decreases for a while, *eventually increases* as *N* increases.

In principle, one can find the "optimal" value of *N* for fixed *k* for which the remainder is smallest (best approximation). In most applications, obtaining the first few terms of the asymptotic expansion is sufficient.

Uniqueness Property

Given an asymptotic sequence $\left\{\delta_j\left(k\right)\right\}$, the asymptotic expansion of a function $f\left(k\right)$ is **unique**.

In fact, assuming that $f\left(k\right) \sim \sum_{j=1}^{+\infty} a_j \delta_j\left(k\right)$ $\quad f\left(k\right) \sim \sum_{j=1}^{+\infty} b_j \delta_j\left(k\right)$

by letting $c_j = a_j - b_j$ one finds

$$0 \sim \sum_{j=1}^{+\infty} c_j \delta_j\left(k\right)$$

Now, dividing by $\delta_1\left(k\right)$ and taking the asymptotic limit one finds $c_1 = 0$

Repeating with $\delta_2\left(k\right), \delta_3\left(k\right), \ldots$ it results $c_j = 0, \quad \forall j$, hence the claimed uniqueness.

Extension to Complex Parameters

So far the expansion parameter (k) was real. However, the previous definitions can be extended to **complex** values of the parameter k, which we will now call z.

Consider a function $f(z)$ that is *analytic everywhere outside a circle* $|z| = R$. Then we know that $f(z)$ has a convergent Taylor series at infinity of the form

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots = \sum_{j=0}^{+\infty} \frac{a_j}{z^j}$$

In this case *the convergent Taylor series is equivalent to a convergent asymptotic series* with an asymptotic sequence

$$\left\{ \frac{1}{z^j} \right\}_{j=0}^{+\infty}$$

If $f(z)$ is *not analytic at infinity*, it **cannot** possess an asymptotic expansion valid for **all** $\arg(z)$ as $z \to \infty$. Typically asymptotic expansions are found to be valid only within some sector of the complex plane; that is, the expansion is constrained by some bounds on $\arg(z)$.

Asymptotic Power Series in the Complex Plane

A function f is said to have an **asymptotic power series** in a sector of the z plane as $z \to \infty$ if

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots$$ (the series is generally *not* convergent.)

Let another function $g(z)$ have an asymptotic power series representation in the same sector of the form

$$g(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots$$

Then the arithmetic combinations $f + g$ (sum) and $f \cdot g$ (product) also have asymptotic power series representations that are obtained by adding or multiplying the series termwise.

Asymptotic Power Series in the Complex Plane

An asymptotic power series can be integrated or differentiated termwise to yield an asymptotic expansion:

$$f'(z) \sim -\frac{a_1}{z^2} - 2\frac{a_2}{z^3} + \dots$$

$$\int_z^\infty \left[f(\zeta) - a_0 - \frac{a_1}{\zeta} \right] \sim \frac{a_2}{z} + \frac{a_3}{2z^2} + \dots$$

Note that more general asymptotic series (as opposed to asymptotic power series) can be integrated termwise but it is, in general, not permissible to differentiate termwise in order to obtain an asymptotic expansion.

Terms 'Beyond All Orders'

A given asymptotic expansion can represent two entirely different functions.

Suppose as $z \to \infty$ for $\mathrm{Re}[z] > 0$ (i.e., $-\pi/2 < \arg z < \pi/2$), the function $f(z)$ is given by the asymptotic power series expansion

$$f(z) \sim \sum_{j=0}^{+\infty} \frac{a_j}{z^j}$$

Then the same expansion also represents $f(z) + e^{-z}$ in this sector.

The reason is that the asymptotic power series representation of e^{-z} for $\mathrm{Re}[z] > 0$ is zero, that is

$$e^{-z} \sim \sum_{j=0}^{+\infty} \frac{b_j}{z^j} \Rightarrow b_j = 0, \forall j \qquad \lim_{z\to\infty} z^n e^{-z} = 0, \forall n$$

The term e^{-z} is *transcendentally small*, or said to be "*beyond all orders*", with respect to the asymptotic power series. An asymptotic power series contains no information about terms beyond all orders.

Remarks

The previous property can be considered as a drawback of the adopted definition of an asymptotic expansion.

The adopted definition was proposed by H. Poincaré in 1886 and furnished the departure for most subsequent theoretical developments.

A different definition of a *complete asymptotic expansion* was proposed by R. B. Dingle, who showed that a latently exact meaning can be ascribed to such expansions (i.e., that they encode precise information about the function they are associated to).

R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*. London and New York: Academic Press, 1973.

Exponentially small terms may be crucial in physics, typically being associated to Non Perturbative (NP) effects not described by the usual Perturbative (P) asymptotic expansions…

J. P. Boyd, "The devil's invention: asymptotic, superasymptotic and hyperasymptotic series." *Acta Appl. Math.* **56**.1 (1999): 1-98.

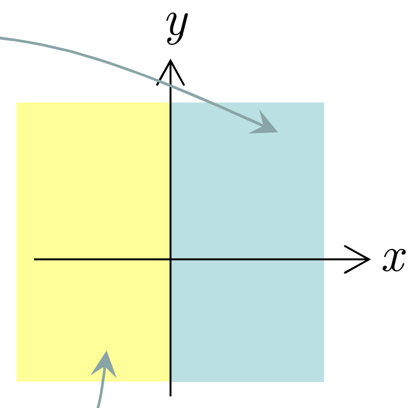
D. Dorigoni, "An introduction to resurgence, trans-series and alien calculus," *Ann. Phys.* **409** (2019): 167914.

Discontinuities in Complex Asymptotic Expansions

When $f(z)$ has an asymptotic representation in a sector of the complex plane, it can happen that an *entirely different* representation is valid in an adjacent sector.

In fact, even when $f(z)$ is analytic for large but finite values of z, the asymptotic representation can change discontinuously as the sector is crossed:

$$f(z) = \sinh\left(\frac{1}{z}\right) \quad \Rightarrow \quad f(z) \underset{z\to 0}{\sim} \frac{1}{2}\begin{cases} e^{1/z}, & \left|\arg\{z\}\right| < \dfrac{\pi}{2} \\ e^{-1/z}, & \dfrac{\pi}{2} < \arg\{z\} < \dfrac{3\pi}{2} \end{cases}$$



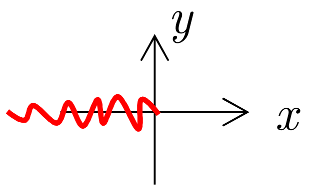
Discontinuities in Complex Asymptotic Expansions

- If $f(z)$ is ***analytic*** at infinity, its asymptotic expansion coincides with its power series, and the series converges. In this case, the asymptotic expansion of $f(z)$ changes continuously at infinity.

example: $$f(z) = \frac{1}{z+1} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{z^{n+1}}, \quad |z| > 1$$

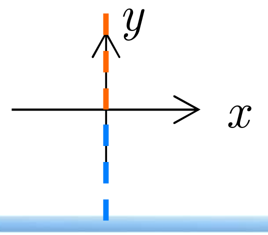
- If the point at infinity is a ***branch point***, the asymptotic expansion changes discontinuously across the branch cut.

example: $$f(z) = \frac{1}{\sqrt{z}+1}$$



- If the point at infinity is a local ***essential*** singular point, and even though $f(z)$ is single valued in the neighborhood of infinity, the asymptotic expansion has lines (rays) across which it changes discontinuously. This is usually referred to as the **Stokes phenomenon**, after George Stokes who first discovered this in 1864.

example: $$f(z) = \sinh(z)$$



References

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